EXPECTATION, COVARIANCE & CORRELATION FOR JOINTLY DISTRIBUTED RANDOM VARIABLES

Defn: X, Y random variables with joint distribution $f(x,y)$. Let $u(X,Y)$ be a function of the r.v. X, Y. Then the expected value of $u(X,Y)$ is

Cont.: $$E\left[u(X,Y)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(x,y)\,f(x,y)\,dx\,dy$$

discrete: $$E\left[u(X,Y)\right] = \sum_x \sum_y u(x,y)\,f(x,y)$$

Defn: If $u(x,y) = x$ then $E[u(X,Y)] = E[X] = \mu_X$

If $u(x,y) = y$ " " $= \mu_Y$

$$\left.\begin{aligned} &\text{mean of } X \quad \mu_X = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x\,f(x,y)\,dx\,dy \\ &\text{mean of } Y \quad \mu_Y = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y\,f(x,y)\,dx\,dy \end{aligned}\right\}\text{Cont. case.}$$

Note: $$\mu_X = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x\,f(x,y)\,dx\,dy$$

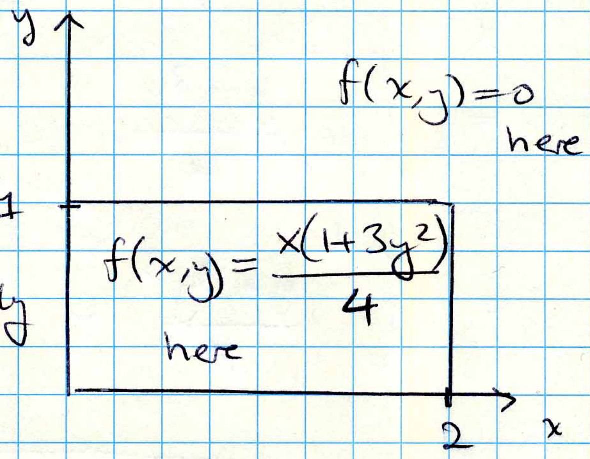
$$= \int_{-\infty}^{+\infty} x \underbrace{\left[\int_{-\infty}^{+\infty} f(x,y)\,dy\right]}_{\text{Marginal } g(x)} dx$$

$$= \int_{-\infty}^{+\infty} x\,g(x)\,dx$$

Example: Let X and Y denote voltages at two points in a circuit. The joint density function is given as

$$f(x,y) = \begin{cases} \dfrac{x(1+3y^2)}{4}, & 0<x<2 \text{ and } 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

a) Compute the mean for X and the mean for Y



$$\mu_X = \int_0^1 \int_0^2 x \cdot \frac{x(1+3y^2)}{4}\, dx\, dy$$

$$= \int_0^1 \frac{1+3y^2}{4} \int_0^2 x^2\, dx\, dy$$

$$= \int_0^1 \frac{1+3y^2}{4} \underbrace{\left(\frac{x^3}{3} \Big|_0^2 \right)}_{8/3} dy = \int_0^1 \frac{2}{3}(1+3y^2)\, dy$$

$$= \frac{2}{3}\left(y\Big|_0^1 + y^3\Big|_0^1 \right) = 4/3$$

$$\mu_Y = \int_0^1 \int_0^2 y \cdot \frac{x(1+3y^2)}{4}\, dx\, dy = \int_0^1 \frac{y+3y^3}{4} \int_0^2 x\, dx\, dy$$

$$= \int_0^1 \frac{y+3y^3}{4} \underbrace{\left(\frac{x^2}{2} \Big|_0^2 \right)}_{2} dy = \int_0^1 \frac{y+3y^3}{2}\, dy$$

$$= \frac{y^2}{4}\Big|_0^1 + \frac{3y^4}{8}\Big|_0^1 = \frac{1}{4} + \frac{3}{8} = 5/8$$

b) let $u(x,y) = y/x$ (the ratio of the voltages)

Compute $E[u(x,y)]$.

$$E[u(x,y)] = E[Y/X] = \int_0^1 \int_0^2 \frac{y}{x} f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^2 \frac{y}{\cancel{x}} \frac{\cancel{x}(1+3y^2)}{4}\,dx\,dy$$

$$= \int_0^1 \frac{y+3y^3}{4} \underbrace{\left(\int_0^2 dx\right)}_{2} dy$$

$$= \int_0^1 \frac{y+3y^3}{2} = \frac{y^2}{4}\Big|_0^1 + \frac{3y^4}{8}\Big|_0^1 = 5/8$$

Defn: The covariance of X and Y is

Differs from book notation → $\sigma_{XY}^2 = E\left[(X-\mu_X)(Y-\mu_Y)\right]$

$$\sigma_{XY}^2 = \sum_x \sum_y (x-\mu_x)(y-\mu_y) f(x,y) \quad \text{DISCRETE}$$

$$\sigma_{XY}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_y) f(x,y) \quad \text{CONTINUOUS}$$

Note: If large values of X result in large values of Y then σ_{XY} will be positive.

If small values of X result in large values of Y then σ_{XY} will be negative.

If X and Y are not related σ_{XY} will be close to 0. If X, Y independent it will be 0.

<u>Note</u>: A faster way to compute σ_{XY}

$$\sigma_{XY}^2 = E[XY] - \mu_X \mu_Y$$

<u>Proof</u>:

$$\sigma_{XY}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy - \mu_X y - \mu_Y x + \mu_X\mu_Y) f(x,y)\,dx\,dy$$

$$= \underbrace{\iint xy f(x,y)\,dx\,dy}_{E[XY]} - \mu_X \underbrace{\iint y f(x,y)\,dx\,dy}_{\mu_Y}$$

$$- \mu_Y \underbrace{\iint x f(x,y)\,dx\,dy}_{\mu_X} + \mu_X\mu_Y \underbrace{\iint f(x,y)\,dx\,dy}_{1}$$

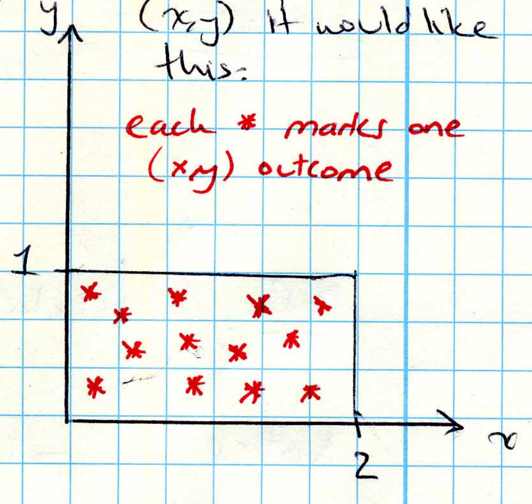
$$= E[XY] - \mu_X\mu_Y - \mu_Y\mu_X + \mu_X\mu_Y$$

$$= E[XY] - \mu_X\mu_Y$$

<u>Example</u>:

$$f(x,y) = \begin{cases} 1/2, & 0 \le x \le 2 \text{ and } 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

(Scatter plot)
If we plot the outcomes (x,y) it would like this:



$\mu_X = 1 \quad \mu_Y = 0.5$

$$\sigma_{XY}^2 = E[XY] - \mu_X\mu_Y$$

$$= \int_0^1\int_0^2 \frac{xy}{2}\,dx\,dy - 1 \times 0.5$$

$$E[XY] = \int_0^1 y \int_0^2 x/2 \, dx = \int_0^1 y \left(\frac{x^2}{4} \Big|_0^2 \right) dy = \int_0^1 y \, dy$$

$$= \frac{y^2}{2} \Big|_0^1 = 1/2$$

So $\sigma_{XY}^2 = E[XY] - \mu_X \mu_Y = \frac{1}{2} - 1 \times \frac{1}{2} = 0$

If X, Y independent then $\sigma_{XY}^2 = 0$.

Proof:

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x,y) \, dx \, dy$$

using independence

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \, g(x) h(y) \, dx \, dy$$

$$= \int_{-\infty}^{\infty} y h(y) \underbrace{\left[\int_{-\infty}^{\infty} x g(x) \, dx \right]}_{\mu_X} dy$$

$$= \mu_X \int_{-\infty}^{\infty} y h(y) \, dy = \mu_X \mu_Y$$

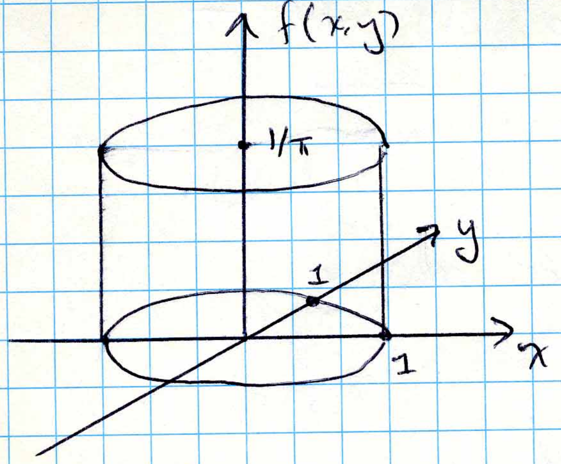
So $\sigma_{XY}^2 = E[XY] - \mu_X \mu_Y = \mu_X \mu_Y - \mu_X \mu_Y = \underline{0}$ ✓

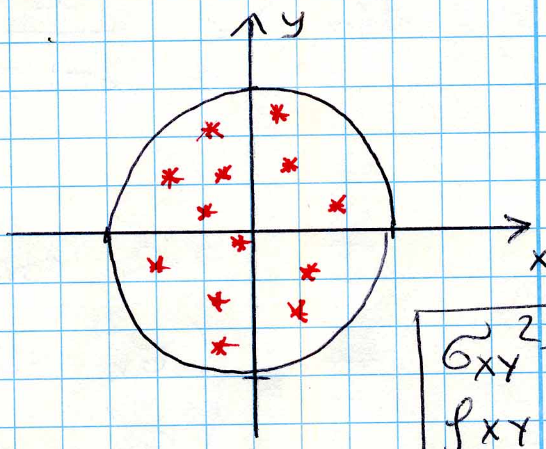
__Important__: The relationship doesn't hold in reverse. In other words, it is possible to have X, Y which are NOT independent but yet have $\sigma_{XY}^2 = 0$.

__Defn__: Correlation coefficient: $\rho_{XY} = \dfrac{\sigma_{XY}^2}{\sigma_X \sigma_Y}$

Notice $-1 \le \rho_{XY} \le 1$

Example: $f(x,y) = \begin{cases} 1/\pi , & x^2+y^2 \le 1 \\ 0 , & \text{elsewhere} \end{cases}$





$\boxed{\begin{matrix} \sigma_{XY}^2 = 0 \\ \rho_{XY} = 0 \end{matrix}}$

Scatter Plot

Earlier we showed that X, Y in this case are **NOT** independent.

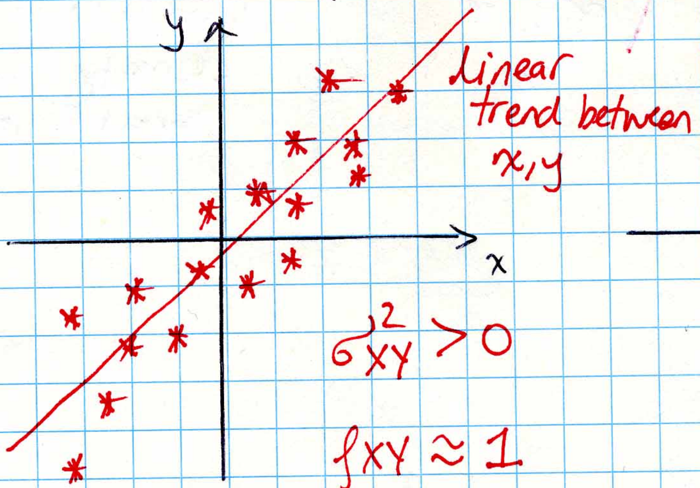
No functional relationship between Y, X.
Could show $\sigma_{XY}^2 = 0$ but complicated integrals

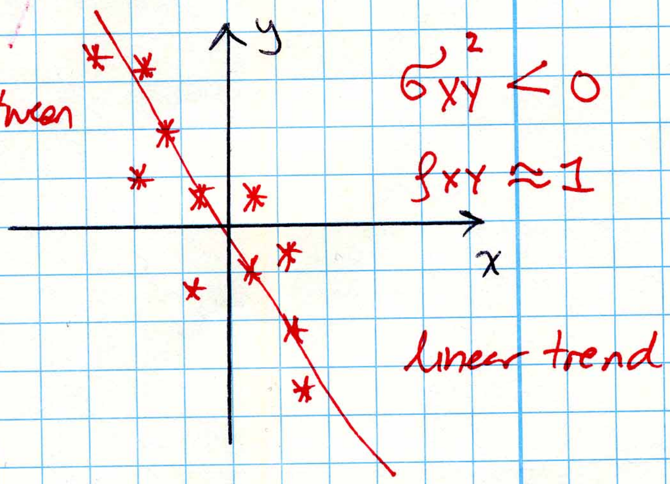
Summary: X, Y independent $\Longrightarrow \sigma_{XY}^2 = 0$

$\sigma_{XY}^2 = 0 \;\not\Longrightarrow\;$ X, Y independent

not necessarily.

Example: Here is what the scatter plot might look like if X and Y are correlated:





Example: (Advanced topic : multivariate normal density)

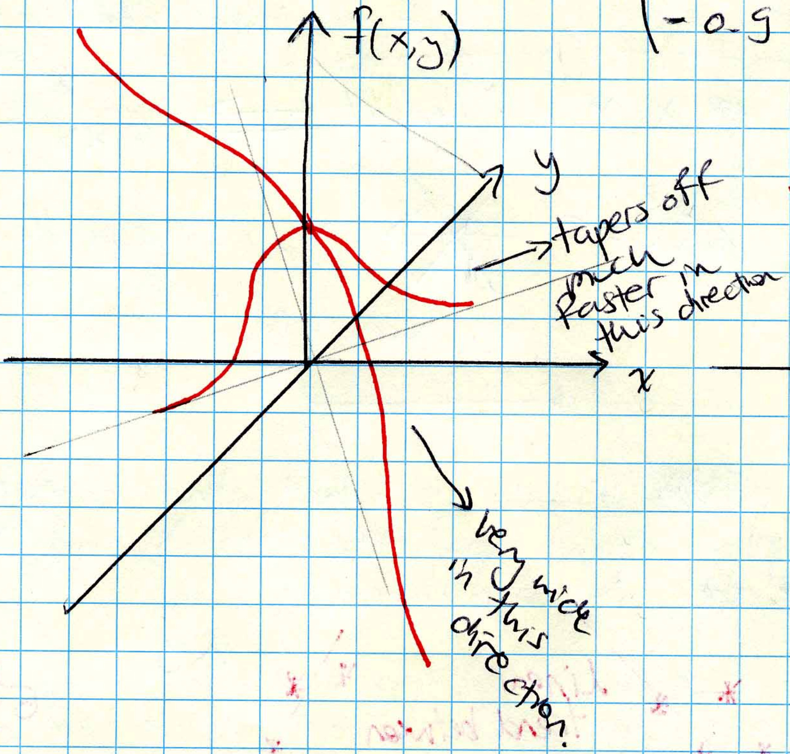
$$f(x,y) = \frac{1}{2\pi|\Sigma|} e^{-\underbrace{\begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix}}_{\text{vector}}^{T} \Sigma^{-1} \begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix}}$$

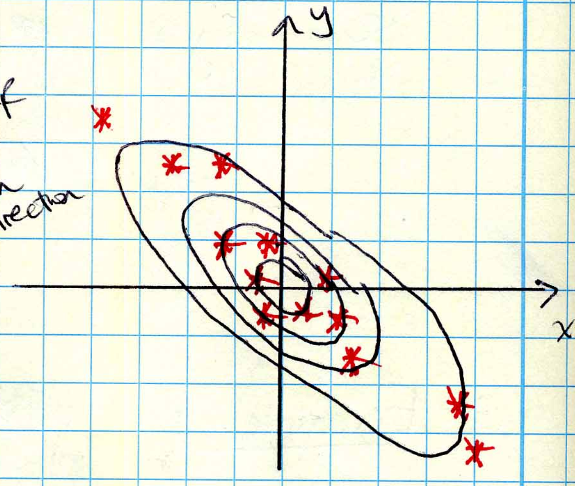
transpose

Σ : 2x2 covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy}^2 \\ \sigma_{xy}^2 & \sigma_y^2 \end{pmatrix}$$

If X, Y are correlated with $\rho_{xy} \approx -1$ Σ: might look like $\Sigma = \begin{pmatrix} 1 & -0.9 \\ -0.9 & 1 \end{pmatrix}$





Contours of constant density and scatter plot

Example:

$$f(x,y) = \begin{cases} \frac{6}{5}\left[1-(x-y)^2\right] & , \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 \end{array} \\ 0 & , \text{elsewhere} \end{cases}$$

compute σ^2_{XY} , ρ_{XY}

Soln: Lets first compute the marginal densities

$$g(x) = \int_0^1 \frac{6}{5}\left(1 - x^2 - y^2 + 2xy\right)dy = \frac{6}{5}\left(y\Big|_0^1 - x^2 y\Big|_0^1 - \frac{y^3}{3}\Big|_0^1 + xy^2\Big|_0^1 \right)$$

$$= \frac{6}{5}\left(\frac{2}{3} + x - x^2\right) \quad \text{For } 0 \le x \le 1$$

then $\mu_X = \int_0^1 x\, g(x)\,dx = \frac{6}{5}\int_0^1 \frac{2x}{3} + x^2 - x^3\,dx$

$$= \frac{6}{5}\left(\frac{x^2}{3}\Big|_0^1 + \frac{x^3}{3}\Big|_0^1 - \frac{x^4}{4}\Big|_0^1 \right) = \frac{6}{5}\left(\frac{1}{3} + \frac{1}{3} - \frac{1}{4}\right)$$

$$= 1/2.$$

We can compt $h(y)$ and μ_Y in the same way to find $\mu_Y = 1/2$ also.

Next compute $E[XY]$

$$E[XY] = \int_0^1\int_0^1 xy\,\frac{6}{5}\left(1 - x^2 - y^2 + 2xy\right)dx\,dy$$

$$= \frac{6}{5}\int_0^1\int_0^1 xy - x^3y - xy^3 + 2x^2y^2\,dx\,dy$$

$$= \frac{6}{5}\int_0^1 \left[\frac{x^2y}{2}\Big|_0^1 - \frac{x^4y}{4}\Big|_0^1 - \frac{x^2y^3}{2}\Big|_0^1 + \frac{2x^3y^2}{3}\Big|_0^1 \right] dy$$

$$= \frac{6}{5}\int_0^1 \frac{y}{2} - \frac{y}{4} - \frac{y^3}{2} + \frac{2y^2}{3}\,dy$$

$$= \frac{6}{5} \int_0^1 \frac{y}{4} + \frac{2y^2}{3} - \frac{y^3}{2} \, dy$$

$$= \frac{6}{5} \left(\frac{y^2}{8}\Big|_0^1 + \frac{2y^3}{9}\Big|_0^1 - \frac{y^4}{8}\Big|_0^1 \right) = \frac{6}{5}\left(\cancel{\frac{1}{8}} + \frac{2}{9} - \cancel{\frac{1}{8}} \right)$$

$$= 4/15$$

Then $\sigma_{XY}^2 = E[XY] - \mu_x \mu_y = \frac{4}{15} - \frac{1}{2} \times \frac{1}{2} = \frac{1}{60}$

To compute ρ_{XY} we also need σ_X, σ_Y

$$\sigma_X^2 = E[X^2] - \mu_x^2$$

$$E[X^2] = \int_0^1 \int_0^1 x^2 \frac{6}{5}\left(1 - x^2 - y^2 + 2xy\right) dx\, dy$$

$$\approx 0.3267$$

so $\sigma_X^2 = 0.3267 - (0.5)^2 = 0.0767$

similarly $\sigma_Y^2 = 0.0767$

Then $\rho_{XY} = \dfrac{\sigma_{XY}^2}{\sigma_X \sigma_Y} = \dfrac{1/60}{\sqrt{0.0767}\sqrt{0.0767}} = 0.2177$