conditional probability

notation: \( P(A \mid B) \) is the probability of event \( A \) given that event \( B \) occurred.

dice roll : \( S = \{1, 2, 3, 4, 5, 6\} \)

\( A = \{1, 2, 3\} \quad P(A) = 1/2 \)

\( B = \text{dice roll is even} = \{2, 4, 6\} \)

\( P(A \mid B) = ? \)

Since we know for sure that \( B \) occurred, i.e., the outcome must be either 2, 4 or 6, we are working with a reduced sample space. To find \( P(A \mid B) \), we have to count how many of the new reduced possible set of outcomes is in \( A \).

In this example, this corresponds to the outcome 2.

Hence \( P(A \mid B) = 1/3 \)

Notice \( P(A \mid B) < P(A) \) in this case. In other words, knowing that the outcome was even (event \( B \) occurred)
the probability of \( A \) occurring.

\[ \begin{align*}
\text{Sample space becomes } & B \\
\text{Probabilities are scaled accordingly.} \\
\text{Note: } & P(A \mid B) \neq P(B \mid A)
\end{align*} \]
Theorem: \( P(A \mid B) + P(A' \mid B) = 1 \)

Using defn. of conditional probability, we have:

\[
\frac{P(A \cap B)}{P(B)} + \frac{P(A' \cap B)}{P(B)} = \frac{P(A \cap B) + P(A' \cap B)}{P(B)}
\]

but since \( A \) and \( A' \) form a partition of \( S \), we have:

\[
P(B) = P(A \cap B) + P(A' \cap B)
\]

Then substituting into (**) above we have \( \frac{P(B)}{P(B)} = 1 \)

There is no similar simple relationship between \( P(A \mid B) \) and \( P(A \mid B') \).

\[
P(A \mid B') = \frac{P(A \cap B')}{P(B')} = \frac{P(A) - P(A \cap B)}{1 - P(B)}
\]

Example: The probability of a flight departing on time is \( P(D) = 0.83 \). The probability of a flight arriving on time is \( P(A) = 0.82 \). We are also told \( P(D \cap A) = 0.78 \) (The probability that a flight both departs and arrives on time).

* What is the probability of a flight arriving on time if we know it departed on time?

\[
P(A \mid D) = \frac{P(A \cap D)}{P(D)} = \frac{0.78}{0.83} = 0.94
\]

* What is the probability that a flight departed on time if we know it arrived on time?

\[
P(D \mid A) = \frac{P(A \cap D)}{P(D)} = \frac{0.78}{0.82} = 0.95
\]
INDEPENDENT EVENTS

**Defn.** Two events $A$ and $B$ are independent if and only if

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B).$$

Otherwise $A$ and $B$ are dependent.

**Intuition** If knowing that event $B$ occurred doesn't change the probability that $A$ will occur, then $B$ must carry no information about $A$. In other words, $A$ & $B$ are independent.

**Obvious example** If we flip two coins $A$ & $B$ and I tell you that $B$ came up heads, what is the probability that $A$ was heads?

$$P(A=\text{heads} | B=\text{heads}) = \frac{P(A=\text{heads} \cap B=\text{heads})}{P(B=\text{heads})}$$

**Sample space**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>Prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heads</td>
<td>heads</td>
<td>heads</td>
<td>1/4</td>
</tr>
<tr>
<td>Heads</td>
<td>tails</td>
<td>tails</td>
<td>1/4</td>
</tr>
<tr>
<td>Tails</td>
<td>heads</td>
<td>heads</td>
<td>1/4</td>
</tr>
<tr>
<td>Tails</td>
<td>tails</td>
<td>tails</td>
<td>1/4</td>
</tr>
</tbody>
</table>

$$P(A=\text{heads} | B=\text{heads}) = \frac{1/4}{1/2} = \frac{1}{2}$$

But also $P(A=\text{heads}) = 1/2$

Therefore, these two events are independent.
Dice roll. \( S = \{1, 2, 3, 4, 5, 6\} \)

- \( A: \) dice roll is even
- \( B: \) dice roll is greater than 2

\[
\begin{align*}
A &= \{2, 4, 6\} \quad & P(A) = 1/2 \\
B &= \{3, 4, 5, 6\} \quad & P(B) = 4/6
\end{align*}
\]

\[
P(A|B) = \frac{P(A \cap B)}{P(B)}
\]

\[
\begin{align*}
A \cap B &= \{4, 6\} \\
P(A|B) &= \frac{2/6}{4/6} = \frac{2}{4} = 1/2
\end{align*}
\]

so \( P(A|B) = P(A) \)

*Also \( P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{2/6}{1/2} = 4/6 = P(B) \)*

In fact, if \( P(A|B) = P(A) \) then \( P(B|A) = P(B) \)

\[
P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)
\]

\[
\Rightarrow \left\{ P(A \cap B) = P(A)P(B) \right\}
\]

\[
\Rightarrow P(A \cap B) = P(B) \frac{P(A)}{P(B)}
\]

But this is \( P(B|A) \)

\[
\left\{ \begin{array}{l}
A: \text{College graduate} \\
B: \text{Smoker} \\
C: \text{Heart disease}
\end{array} \right.
\]

\[
P(A) = 0.7 \\
P(B) = 0.1 \\
P(C) = 0.05
\]

\[
P(A \cap C) = 0.035 \\
P(B \cap C) = 0.03
\]

\[
P(C|A) = \frac{P(C \cap A)}{P(A)} = \frac{0.035}{0.7} = 0.05 \neq P(C) \text{ independent}
\]

\[
P(C|B) = \frac{P(C \cap B)}{P(B)} = \frac{0.03}{0.1} = 0.3 \neq P(C) \text{ dependent}
\]
MULTIPLICATIVE RULES

**Theorem.** Directly from the definition of conditional probability we also have \( P(AnB) = P(A|B)P(B) \).

**Theorem.** If \( A \) & \( B \) are independent events \( \iff P(AnB) = P(A)P(B) \).

**Example:** An electrical engineering lab has 20 probes of which 3 are bad. A student selects 2 probes randomly, what is the probability that both are bad?

- Events: \( A \): First probe is bad
- \( B \): Second probe is bad
- Question is asking \( P(AnB) \)
- **Multiplicative rule:** \( P(AnB) = P(A)P(B|A) \)

\[
P(A) = \frac{3}{20}
\]

Now if we know \( A \) occurred then there are 19 probes remaining of which 2 are bad, so

\[
P(B|A) = \frac{2}{19}
\]

Therefore \( P(AnB) = \frac{3}{20} \cdot \frac{2}{19} = \frac{3}{190} \)

**Example** A rigged coin is twice as likely to come up heads than tails. If the coin is flipped 3 times what is the probability of getting 3 heads?

**Solu.** \( P(H) = \frac{2}{3} \) \( P(T) = \frac{1}{3} \)

Since the three flips are independent

\[
P(HHH) = P(H)P(H)P(H) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}
\]

**What is the probability of event \( A \) that 2 heads occur?**

\[
P(A) = P(HHT) + P(HTH) + P(THH)
\]

\[
= \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}
\]
Generalization of the multiplicative rules

If, in an experiment, the events \( A_1, A_2, \ldots, A_k \) can occur, then
\[
P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1) P(A_2|A_1) P(A_3|A_1\cap A_2) \ldots P(A_k|A_1\cap \ldots \cap A_{k-1})
\]

If the events \( A_1, A_2, \ldots, A_k \) are independent, then
\[
P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1) P(A_2) \ldots P(A_k) = \prod_{n=1}^{k} P(A_n)
\]

Example: In the Yahtzee dice game 5 dice are rolled and a winning roll has three of a kind and two of a kind (but not five of a kind). Previously, we computed the probability of winning as 0.0386.

**What is the probability of not getting a single winning roll in 5 tries?**

**Solution:** Each roll of the 5 dice together is one try. Each try is independent of the other tries. In other words, the result of try one has no influence on try 2.

Therefore:
\[
P(\text{lose, lose, lose, lose, lose}) = P(\text{lose}) \cdot P(\text{lose}) \cdot \ldots \cdot P(\text{lose})
\]
\[
= P(\text{lose})^5 = (1 - 0.0386)^5 = 0.82
\]

**What is the probability of getting 3 winners in 44 tries?**

Let's consider the sequence win, win, win, followed by 44 - 3 = 41 losses with order
\[
P(3 \text{ win, 41 loose}) = P(\text{win})^3 P(\text{lose})^{41} = 0.0386^3 (1 - 0.0386) = 1.14 \times 10^{-5}
\]

But there are \( \binom{44}{3} \) ways to choose three winning rolls in 44 tries without order
\[
P(3 \text{ win, 41 loose without order}) = \binom{44}{3} \times 1.14 \times 10^{-5} \approx 0.15
\]
Example:

A bag contains 10 red stones and 10 blue stones. What is the probability of selecting 2 red stones without replacement? What is the probability of selecting 2 red stones with replacement?

- 1st selection
  - 10 out of 20 red
  - 10 out of 20 blue

- Sample space (possible outcomes) changes

- Probability of red on 2nd selection is 9/19

- Probability of red on 3rd selection is 9/18

- Probability of blue on 2nd selection is 9/19

Notice: 

\[ P(RRR) = \frac{10 \times 9 \times 8}{10 \times 9 \times 8} = \frac{9}{19} \]

\[ P(RBB) = \frac{10 \times 9 \times 8}{10 \times 9 \times 8} = \frac{8}{19} \]

\[ P(BRR) = \frac{10 \times 9 \times 8}{10 \times 9 \times 8} = \frac{9}{19} \]

\[ P(BRB) = \frac{10 \times 9 \times 8}{10 \times 9 \times 8} = \frac{8}{19} \]

There are 12 dependent outcomes.

Given: Red, Red, Blue.

Over two selections, there are 3 red.

Due to multiplicative rule,

\[ P(\text{RR}) = \frac{9}{19} \]

\[ P(\text{RB}) = \frac{8}{19} \]

\[ P(\text{BR}) = \frac{8}{19} \]

\[ P(\text{BB}) = \frac{9}{19} \]

Also, \( P(RBB) = P(BRB) \)

# ways we can arrange 2 red and 1 blue

\[ \frac{3 \times 9 \times 8}{9 \times 8} = \frac{27}{36} = 0.225 \]
Ex. A bag contains 10 red stones and 90 blue stones. What is the probability of selecting red, blue, red in that order when sampling with replacement? What is the probability of selecting two reds and one blue?

1st selection, 10 out of 100 red. So probability of red on first selection is 10/100.

2nd selection, 90 out of 100 red (sample space hasn’t changed because we are sampling with replacement). So probability of blue on second selection is 90/100.

3rd selection, still 10 out of 100 red. So prob of red is 10/100.

Independent events

Again using independence of the selections

\[ P(\text{RBR}) = \frac{10}{100} \times \frac{90}{100} \times \frac{10}{100} = \frac{9}{1000} \]

Similarly

\[ P(\text{RCB}) = \frac{10}{100} \times \frac{10}{100} \times \frac{90}{100} = \frac{9}{1000} \]

and

\[ P(\text{BRB}) = \frac{90}{100} \times \frac{10}{100} \times \frac{10}{100} = \frac{9}{1000} \]

\[ P(\text{two out of 3 red}) = \frac{3 \times 9}{10000} = 0.027 \]

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Ex. At the press of a button, a random number generator is used in a computer game to display a red stone with probability 0.1 or a blue stone with probability 0.9. What is the probability of getting two red stones and one blue stone when the button is pressed 3 times?

Solu. This equivalent to the previous example because the scenario described is sampling with replacement.