Example counting: A president, vice-president, and treasurer are to be chosen in a student club which has 40 members.

* How many choices of officers are possible?

\[ P_3 = \frac{40!}{(40-3)!} = 40 \times 39 \times 38 = 59,280 \]

We use permutation because who serves in which office matters.

* How many choices if A will only serve if he is president?

2 cases: Case 1 - A serves as president

39 \ P_2 \text{ choices for other two offices}

Case 2 - A doesn't serve at all

39 \ P_3 \text{ choices}

Total # choices = 39 \ P_2 + 39 \ P_3 = 39 \times 38 + 39 \times 38 \times 37 = 1482 + 54,834 = 56,316

* How many choices if A is not eligible to serve as president but can serve in other offices?

Case 1: A doesn't serve \ 39 \ P_3 \text{ choices}

Case 2: A serves as vice-president \ 39 \ P_2 \text{ choices}

Case 3: A serves as treasurer \ 39 \ P_2 \text{ choices}

\text{Sum these three}
How many choices if A and B will serve together or not at all.

Case 1: A & B don't serve \(38 \cdot P_3\)

Case 2: A & B serve -

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<tr>
<th>Sub cases</th>
<th>President</th>
<th>Vice-President</th>
<th>Treasurer</th>
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<td>A</td>
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Could have found with \(3! = 6\) because this is how many ways we can order A, B and X.

For each of these there are \(38 \cdot P_1 = 38\) remaining choices for the \(X\) position.

Therefore \(6 \cdot 38\)

So total choices = \(38 \cdot P_3 + 6 \cdot 38\)

\[= 38 \cdot 37 \cdot 36 + 6 \cdot 38\]

\[= 50,616 + 228\]

\[= 50,844\]
Example counting. A deck of 13 cards contains one of each card 1-10 and Jack, Queen, and King. If we draw a hand of 4 cards without replacement from this deck, what is the probability of getting a King and a Queen? Assume all outcomes are equally likely.

* First, let's find $\#_{\text{all possible outcomes}}$ (order doesn't matter)

$$\binom{13}{4} = \frac{13!}{9!4!} = \frac{13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = \frac{715}{715}$$

* Now, find $\#_{\text{outcomes with a K and Q}}$ (again, order doesn't matter)

The K and Q take up 2 spots, and there are 2 spots left. Also, there are 11 cards left to choose from. So

$$\binom{11}{2} = \frac{11!}{9!2!} = \frac{11 \times 10}{2 \times 1} = 55$$

$$P(\text{K, Q in hand of 4}) = \frac{55}{715} = 0.077$$
Example counting: Four married couples are going to sit in 8 seats in a row at a concert.
* How many ways can they sit if there are no restrictions?
\[ 8! = 40320 \]

* If each couple must sit next to each other

Think of each couple as a unit. There are 4 units.
The 4 units can be organized in \( 4! = 24 \) ways.

Now each couple could sit in 2 different ways: husband left, wife right or vice versa.
Then for any unit organization from above, we have 2^4 possibilities.
So answer is \( 24 \times 2^4 = 384 \)
Example counting: Given the digits 0, 1, 2, 3, 4, 5, 6, how many 4-digit numbers can be written? Note you can't use 0 in the thousands position for a 4-digit number.

Step 1: $6 \times x \times x \times x = 6$ possibilities for thousands position
\{1, 2, 3, 4, 5, 6\}

Step 2: $6 \times 6 \times 5 \times 4 = 720$ possibilities for hundreds position
5 for tens
4 for units

Alternative solution:

# possibilities if 0 can be used in the thousands position is $^7P_4$

Of these how many have 0 in the thousands position? Well, if 0 is used in the thousands position, the rest is the same as selecting 3 digits out of the 6 so $^6P_3$

Therefore answer is $^7P_4 - ^6P_3 = 7 \times 6 \times 5 \times 4 - 6 \times 5 \times 4$

$= 840 - 120 = 720$
Example: Lottery: 60 participants 3 prizes: car, laptop, and $5. What is the probability Joe wins the car, Brian wins the laptop and Kate wins $5?

Soln. Who wins what (order) matters, so total # outcomes $60 \choose 3$. The event includes only 1 outcome; therefore the answer is $1/60 \choose 3 = \frac{1}{60 \times 59 \times 58}$

* What is the probability Joe wins the car?

Soln. # outcomes with Joe winning the car is $59 \choose 2$

This is because if Joe wins the car the other 2 prizes will go to two of the 59 remaining people.

Here $P(\text{Joe wins car}) = \frac{59 \choose 2}{60 \choose 3}$

$$= \frac{59 \times 58}{60 \times 59 \times 58} = \frac{1}{60}$$ which is probably what you'd have guessed intuitively

* Now lets say all three prizes are the same, i.e. a laptop. So order doesn't matter. What is the probability that Joe, Brian & Kate win?

Soln. # outcomes with Joe, Brian & Kate winning disregarding order is $1$

# total outcomes disregarding order is $60 \choose 3$

so the prob is $1/60 \choose 3 = \frac{3 \times 2 \times 1}{60 \times 59 \times 58}$
Example: Dice game (Yahtzee). 5 dice are rolled. To win you must get three of a kind and two of a kind (not 5 of the same kind though). Find the probability of winning.

Sln. (1) Total # outcomes \(6^5\) 5 dice, each six faces

(2) Total # winning outcomes

Example winning outcome: \[\begin{array}{c}
4 \\
2 \\
4 \\
2 \\
2 \\
\end{array}\]

red white blue yellow white

Imagine the dice have colors, so we can keep track of the combinations.

\[P(\text{win}) = \frac{300}{6^5} = 0.0386 \approx 1/26\]

2a) In how many ways can I choose 3 dice out of 5?

2b) In how many ways can I choose the remaining 2 dice?

2c) For the triplet, how many possible values?

2d) " " double, " " "

\[2a \times 2b \times 2c \times 2d = \# \text{ winning outcomes}\]

\[
\left\{
\begin{array}{l}
2a) \quad 5 \binom{3}{3} = 10 \\
2b) \quad 2 \binom{2}{2} = 1
\end{array}
\right.
\]

2c) The triplet in 2a could have been 1s, 2s, 3s, 4s or 6s. So 6 ways

2d) 5 ways to avoid having 5 dice of the same kind