Second-Order Systems

Objectives

The objective of this lab is to study the characteristics of step responses and of sinusoidal responses for second-order systems. Critically damped and underdamped systems are considered. Concepts of rise time, settling time, percent overshoot, and frequency of oscillations are introduced for step responses. For sinusoidal inputs, the steady-state responses are studied first, and peaking in the frequency domain is observed for underdamped systems. Transient responses are also investigated.

Introduction

In the lab on first-order systems, the response of a brush DC motor with the voltage v (V or volts) considered as an input, and the angular velocity ω (rad/s or s^{-1}) considered as an output, was found to be approximately described by a first-order model

$$P(s) = \frac{\omega(s)}{v(s)} = \frac{k}{s+a}. \tag{8.8}$$

If θ, *the angular position*, is considered to be the output, a slightly different model results. Since the angular position is the integral of the velocity, a second-order transfer function is obtained

$$P(s) = \frac{\theta(s)}{v(s)} = \frac{k}{s(s+a)}. \tag{8.9}$$

Note that the second-order system has a pole at $s = -a$ and a pole at $s = 0$.

We consider the motor under a feedback control law of the form

$$v(t) = k_P(r(t) - \theta(t)), \tag{8.10}$$

where k_P is a parameter to be selected, called the *proportional gain.* The objective is for $\theta(t)$ to track $r(t)$, the reference input, which is now considered to be the input to the system. The control law is called a *proportional* control law because the control input is proportional to the error between the reference input and the output. In the Laplace domain, equation (8.10) becomes

$$v(s) = k_P(r(s) - \theta(s)). \tag{8.11}$$

The transfer function from the input r to the output θ can be verified to be

$$P_{CL}(s) \triangleq \frac{\theta(s)}{r(s)} = \frac{kk_P}{s^2 + as + kk_P}. \tag{8.12}$$

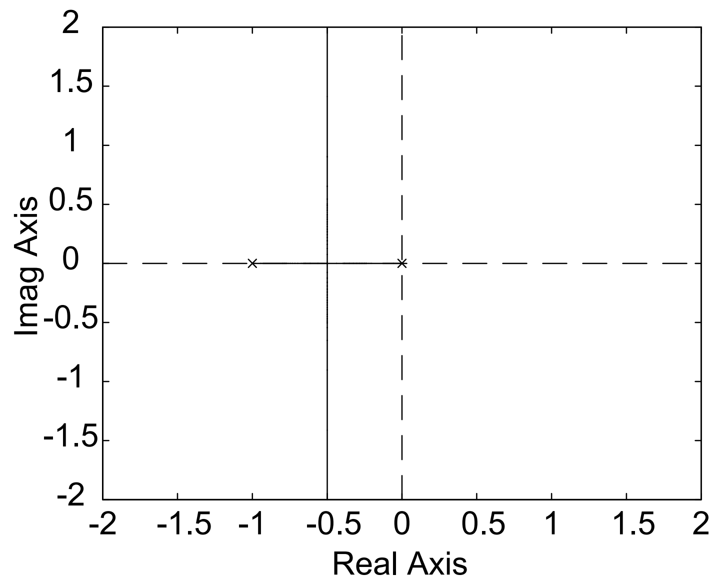

Figure 8.23: Root-locus for second-order system

For different values of the parameter k_P, various second-order systems are obtained. Fig. 8.23 shows the locus of the poles of the transfer function (8.12) as k_P varies from 0 to ∞, assuming $k = 1$ and $a = 1$. For small k_P, the two poles are close to the original values at $s = 0$ and $s = -1$. As the gain increases, the two poles move closer together, eventually merging and splitting from the real axis as a complex pair. For larger values of k_P, the real parts of the poles remain constant, and the imaginary parts continue to grow. A system for which the two poles are real is called *overdamped*. If the two poles are complex, it is called *underdamped*. The limit case where the poles are real and equal is called *critically damped*. This situation occurs for $a^2 = 4kk_P$.

The step response of a system is the response to an input $r(t) = r_0$. Fig. 8.24 shows the step responses for two values of k_P. The magnitude of the input is $r_0 = 1$. The dashed line is the response that corresponds to $k_P = 0.25$, the value such that the two poles are real and equal (critical damping). The solid line corresponds to $k_P = 1.25$, the value such that

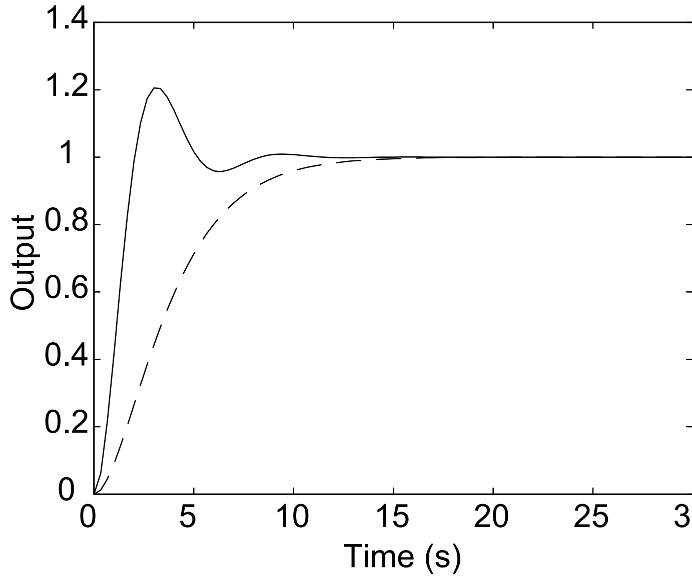

Figure 8.24: Step responses of second-order systems

the poles are complex and with imaginary parts equal to twice the real parts (underdamped). When the poles are complex, with imaginary parts greater than or equal to the real parts, the response exhibits overshoot and oscillations. The angular frequency of the oscillations is equal to the imaginary part of the poles (in rad/s). In Fig. 8.24, the period of the oscillations is approximately 6 seconds, and the imaginary part of the poles is 1 rad/s.

One defines the *rise time* as the time it takes for the output to reach the steady-state value. Because the value is only reached asymptotically, it is common to define the *10-90% rise time*, as the time it takes for the output to move from 10% to 90% of the steady-state value. The *settling time* is the time it takes for the output to reach and stay within a certain percentage of the final value. Several values of the percentage are used, with 2% being common. We will refer to this settling time as the *98% settling time*. One also defines the *percent overshoot* as the percentage of overshoot over the steady-state value.

In this lab, we also consider the responses to sinusoidal inputs

$$r(t) = r_0 \sin(\omega_0 t + \phi_0) \tag{8.13}$$

In the steady-state, the response of the system is given by

$$\theta_{ss}(t) = M(\omega_0)\; r_0 \sin(\omega_0 t + \phi_0 + \alpha(\omega_0)), \tag{8.14}$$

where the magnitude $M(\omega_0)$ and the phase $\alpha(\omega_0)$ satisfy

$$M(\omega_0) = |P_{CL}(j\omega_0)|, \qquad \alpha(\omega_0) = \arg(P_{CL}(j\omega_0)). \tag{8.15}$$

and P_{CL} is the transfer function from r to θ given in (8.12).

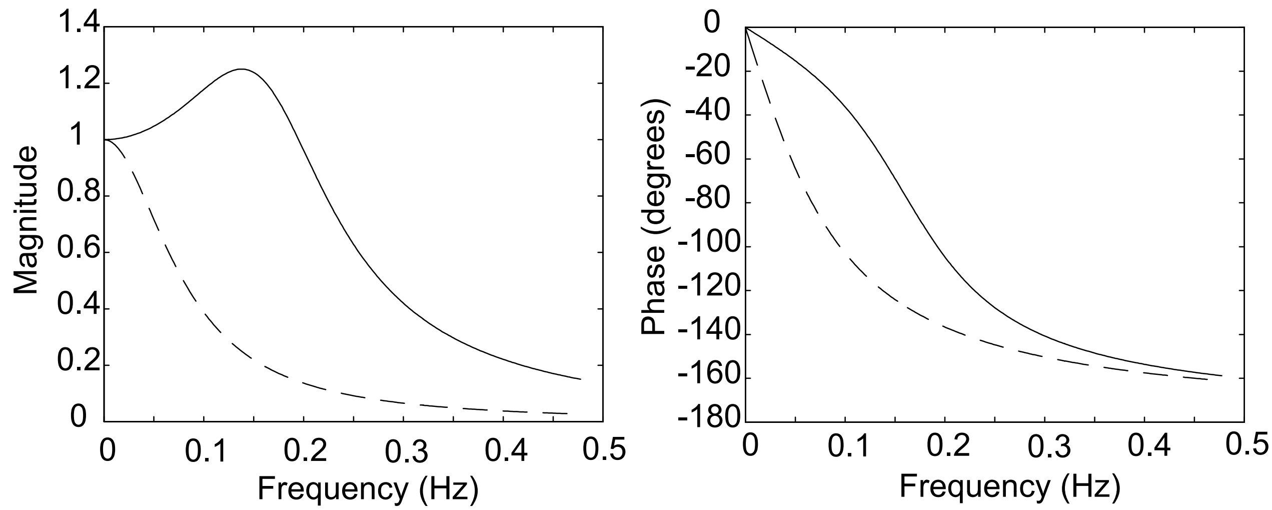

Figure 8.25: Frequency responses of second-order systems

Fig. 8.25 shows the frequency responses of the two second-order systems for $k_P = 0.25$ and $k_P = 1.25$, assuming $a = k = 1$. The dashed lines are for $k_P = 0.25$ (critical damping) and the solid lines are for $k_P = 1.25$ (underdamped). On the left are the magnitudes of the frequency responses (that is, M in equation (8.15)), and on the right are the phases in degrees (that is, α in equation (8.15)). The x-axis has been labelled in Hz, rather than rad/s, with 0.16 Hz being approximately 1 rad/s. Note that, for the underdamped case, the magnitude of the frequency response peaks to a value about 25% higher than the DC value. The steady-state response for that frequency will, therefore, be higher than for any other frequency. This phenomenon is usually called *resonance*. For a lightly damped system, the frequency of peaking is close to the value of the imaginary part of the pole, which is 1 rad/s in this example.

Pre-lab

Verify analytically equation (8.12). Then, for the values of k and a corresponding to the DC motor (for simplicity, you may let $k = 1000$ and $a = 100$), calculate the values of k_P such that the closed-loop poles are real and equal. Also compute k_P such that the imaginary parts are equal to twice the real parts. In Matlab, calculate and plot the step responses of the system for the two values of k_P. The Matlab function *step* may be used. You should check the help files for the functions *step* and *tf* to get the correct syntax.

Determine from the plots the values of the 10-90% rise time and of the 98% settling time. For the underdamped case, determine the percent overshoot and the frequency of oscillation. To obtain the frequency of oscillation, you may estimate the half period as the time between the first peak (overshoot) and the second peak (undershoot). Compare the frequency of oscillation to the imaginary part of the pole.

In Matlab, calculate and plot the frequency response of the DC motor under proportional control, for the critical and for the underdamped cases. In other words, plot the equivalent of Fig. 8.25 for the specific values of the motor. The Matlab function *freqs* may be used for that purpose. With *freqs*, you will need to enter the frequencies in rad/s, but be sure to label your plots in Hz, as in Fig. 8.25. From the plots, determine the value of the frequency where peaking occurs, and give the amount of peaking as a percentage of excess over the value of the DC gain. Also determine the phase lag of the response at the frequency of peaking.

Experiments

Equipment needed

You will need a brush DC motor, a dual power amplifier, and a cable rack.

Preliminary testing

Carry out the same testing procedure as in the lab on first-order systems. Transfer the files LAB2.C and LAB3.C to your account. Compile and create executable programs for these files.

Step responses

The program LAB2 lets you first enter a value of the proportional gain. Then, you apply the value of the reference input r through the keyboard, and may change it in real-time. The reference input is for the angular position of the rotor, and is given in degrees. Using the program, obtain the step responses for the critical and underdamped cases and for a reference input equal to 90°. From the plots, determine the rise times and settling times, and for the underdamped case, the percent overshoot and the frequency of the oscillations. Follow the same procedure as in the pre-lab. You should find that the overshoot and the frequency of oscillation are somewhat higher than expected. The origin of the discrepancy is in the additional dynamics of the motor related to the inductance of the motor, an effect that was observed in the step responses of the lab on first-order systems, but was neglected in the model.

Sinusoidal responses

The program LAB3 lets you enter values for the proportional gain, for the magnitude of the reference input r_0, and for the phase ϕ_0. Then, the value of the frequency ω_0 is entered and may be changed in real-time. Note that the magnitude of the reference input is expressed in degrees (giving the reference angular position of the motor in degrees), and that the frequency of the reference input is given in Hz. All the experiments concern the underdamped case, so that the value of k_P that should be used is the value calculated for that case.

Test the program by applying a sinusoidal input with $r_0 = 10$ degrees, $\phi_0 = 0$, and a frequency of 1 Hz. Having checked that all works according to expectations, run an experiment where you step the frequency every second according to the sequence 5, 10, 15, 20, 25, and 30 (Hz). Plot the response and estimate the frequency of peaking, as well as the percentage of peaking (you may run separate experiments if you want to estimate the frequency of peaking more precisely). Compare the results to those predicted by the analysis. You should find that the percentage and the frequency of peaking are somewhat higher than expected. As for the step responses, this effect is due to the neglected dynamics related to the inductance of the motor. From the plot of the response with steps in frequency, measure the magnitude of the response for each frequency, and draw a crude plot of the magnitude of the frequency response as a function of frequency (in Hz) using the results.

Run a separate experiment at the frequency of peaking, and plot both the input and output signals on the same graphs. Estimate the phase lag of the frequency response α and compare to the value of the pre-lab. Next, run an experiment with a frequency of 50 Hz and a phase $\phi_0 = 0$ and observe that the transient response reaches values that are considerably higher than the steady-state value. However, the transient response varies significantly with the phase of the input signal. To illustrate this point, repeat the experiment with a different phase, adjusted to minimize or at least significantly reduce the transient response. Calculate the amount of overshoot, or peaking in the time domain, as a percentage of excess over the steady-state response (for $\phi_0 = 0$ and for the phase that minimizes the transient response).

Report at a glance

Be sure to include:

- Pre-lab calculations.
- Plots of the step responses, with the values of the rise time, settling time, percent overshoot, and frequency of oscillation.
- Plot of the response with steps in frequency. Value of the percentage of peaking in the frequency domain and of the frequency of peaking. Crude magnitude plot.
- Plot of the response at the frequency of peaking, with the value of α.
- Plot of the transient responses at 50Hz for two values of the phase. Percentage value of the peaking in the time domain.
- Comparison of the results of the pre-lab and experiments.
- Comments.