ECE 3510 Routh-Hurwitz Lecture

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Routh-Hurwitz Stability test

Denominator of transfer function or signal: $a_n \cdot s^n + a_{n-1} \cdot s^{n-1} + a_{n-2} \cdot s^{n-2} + a_{n-3} \cdot s^{n-3} \quad \ldots \quad a_1 \cdot s + a_0$

Usually of the **Closed-loop transfer function denominator** to test fo BIBO stability

Test denominator for poles in CRHP (RHP including imaginary axis)

1. For all poles to be in the LHP, all coefficients must be > 0

   For a second-order denominator, that is enough, skip the next step.

2. If all coefficients are > 0 & order > 2, then:
   Create Routh-Hurwitz array:

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	...	0
s^{n-1}	a_{n-1} & divide	a_{n-3}	a_{n-5}	a_{n-7}	...	0
s^{n-2}	b_1	b_2	b_3	b_4	...	0
s^{n-3}	c_1	c_2	c_3	c_4	...	0
.	.	.	.			
.	.	.	.			
s^0	z_1	0				

$$b_1 = \frac{a_{n-1} \cdot a_{n-2} - a_n \cdot a_{n-3}}{a_{n-1}} \qquad b_2 = \frac{a_{n-1} \cdot a_{n-4} - a_n \cdot a_{n-5}}{a_{n-1}} \qquad b_3 = \frac{a_{n-1} \cdot a_{n-6} - a_n \cdot a_{n-7}}{a_{n-1}}$$

$$c_1 = \frac{b_1 \cdot a_{n-3} - a_{n-1} \cdot b_2}{b_1} \qquad c_2 = \frac{b_1 \cdot a_{n-5} - a_{n-1} \cdot b_3}{b_1} \qquad c_3 = \frac{b_1 \cdot a_{n-7} - a_{n-1} \cdot b_4}{b_1}$$

Look at first column:

All positive = All roots left of imaginary axis

If any negative or 0, then there are poles on the Imaginary axis or in the RHP (Right-Half Plane)

Count sign reversals down the first column

Sign reversals = number of poles on the Imaginary axis or in the RHP (Right-Half Plane)

0 can be replaced by $-\varepsilon$ to see if there are any other sign reversals

The transfer functions of C(s) and P(s) are given below. In each case determine if the steady-state error will go to zero and whether disturbances will be completely rejected. Be sure to check for closed-loop stability when needed.

			0 steady-state error?	Reject Disturbances?
			$e_{ss}(\infty) = 0$?	$e_{ss}(\infty) = 0$ for disturbance?
a)	$C(s) = \dfrac{s+2}{s^2+5\cdot s+4}$	$P(s) = \dfrac{s+1}{s^2+5\cdot s+15}$	No no pole at zero	No no pole at zero
b)	$C(s) = \dfrac{s+5}{s^2+4\cdot s+3}$	$P(s) = \dfrac{s+1}{s^2+2\cdot s}$	Yes (Tentative answer) $P(s)$ has pole at zero	No $C(s)$ has no pole at zero

No stability test needed to answer those questions (part a)

Must test for stability: Closed loop transfer function $= \dfrac{C(s)\cdot P(s)}{1+C(s)\cdot P(s)} = \dfrac{N_C(s)\cdot N_P(s)}{D_C(s)\cdot D_P(s)+N_C(s)\cdot N_P(s)}$

Closed loop denominator $= D_C(s)\cdot D_P(s)+N_C(s)\cdot N_P(s)$

$$C(s)\cdot P(s) = \frac{s+5}{\left(s^2+4\cdot s+3\right)}\cdot\frac{s+1}{\left(s^2+2\cdot s\right)}$$

Closed loop denominator $= \left(s^2+4\cdot s+3\right)\cdot\left(s^2+2\cdot s\right)+(s+5)\cdot(s+1)$

$$D_H(s) = s^4+6\cdot s^3+12\cdot s^2+12\cdot s+5$$

Routh-Hurwitz Stability test

Test denominator for poles in CRHP (RHP including imaginary axis)

1. All coefficients must be > 0

   For a second-order denominator, that is enough

2. Create Routh-Hurwitz array:

$$D_H(s) = s^4+6\cdot s^3+12\cdot s^2+12\cdot s+5$$

(RH Ex.1)

s^4	1	12	5	0
s^3	6	12	0	
s^2	$\dfrac{6\cdot 12-1\cdot 12}{6}=10$	$\dfrac{6\cdot 5-1\cdot 0}{6}=5$		
s^1	$\dfrac{10\cdot 12-6\cdot 5}{10}=9$	$\dfrac{10\cdot 0-6\cdot 0}{10}=0$		
s^0	$\dfrac{9\cdot 5-10\cdot 0}{9}=5$			

Look at first column: All positive, so
All roots left of imaginary axis, so tentative answers above are correct

If any were negative or 0, then
$D_H(s)$ would have poles on the Imaginary axis or in the RHP (Right-Half Plane)

Alternatively, check the actual roots

Using your calculator, find the roots of:

$$0 = s^4+6\cdot s^3+12\cdot s^2+12\cdot s+5$$

Roots: $\begin{bmatrix} -1 \\ -3.359 \\ -0.82-0.903\cdot j \\ -0.82+0.903\cdot j \end{bmatrix}$ roots all negative, stable

So tentative answers above are correct

More Routh-Hurwitz method examples

RH Ex.2 Given a cloosed-loop denominator: $D(s) = s^4 + 10 \cdot s^3 + 3 \cdot s^2 + 5 \cdot s + 2$ Are all the poles in the OLHP?

s^4	1	3	2	0
s^3	10	5	0	
s^2	$\frac{10 \cdot 3 - 1 \cdot 5}{10} = 2.5$	$\frac{10 \cdot 2 - 1 \cdot 0}{10} = 2$		
s^1	$\frac{2.5 \cdot 5 - 10 \cdot 2}{2.5} = -3$	0		
s^0	$\frac{-3 \cdot 2 - 1 \cdot 0}{-3} = 2$			

Two sign reversals = two problem poles, in the RHP NO

Actual roots: $\begin{bmatrix} 0.062 + 0.732 \cdot j \\ 0.062 - 0.732 \cdot j \\ -0.381 \\ -9.743 \end{bmatrix}$ Two roots positive

RH Ex.3

$C(s) = \frac{3 \cdot s^2 + 8}{s^3 + 2 \cdot s^2 + 4 \cdot s}$ $P(s) = \frac{2}{s^2 + 3}$ (Notice that the Plant is not inherently stable)

$C(s) \cdot P(s) = \frac{3 \cdot s^2 + 8}{\left(s^3 + 2 \cdot s^2 + 4 \cdot s\right)} \cdot \frac{2}{\left(s^2 + 3\right)}$ Closed loop denominator $= \left(s^3 + 2 \cdot s^2 + 4 \cdot s\right) \cdot \left(s^2 + 3\right) + \left(3 \cdot s^2 + 8\right) \cdot 2$

$= s^5 + 2 \cdot s^4 + 7 \cdot s^3 + 12 \cdot s^2 + 12 \cdot s + 16$

Routh-Hurwitz array:

s^5	1	7	12	0
s^4	2	12	16	
s^3	$\frac{2 \cdot 7 - 1 \cdot 12}{2} = 1$	$\frac{2 \cdot 12 - 1 \cdot 16}{2} = 4$	0	
s^2	$\frac{1 \cdot 12 - 2 \cdot 4}{1} = 4$	$\frac{1 \cdot 16 - 2 \cdot 0}{1} = 16$		
s^1	$\frac{4 \cdot 4 - 1 \cdot 16}{4} = 0$ (-ε)	0		
s^0	$\frac{-\varepsilon \cdot 16}{-\varepsilon}$			

Problem, some root(s) in CRHP
Consider this -ε & you get 2 sign changes, 2 unstable poles

Doesn't make sense to progress to the next row if all you want to know is stability, but if you count above as -ε, this answer would come out +, indicating two problem poles

Actual roots: $\begin{bmatrix} 2 \cdot j \\ -2 \cdot j \\ -1.651 \\ -0.175 - 1.547 \cdot j \\ -0.175 + 1.547 \cdot j \end{bmatrix}$ First 2 roots are on imaginary axis, unstable

Closed-loop transfer-function denominator	Transfer function stable?	
a) $D(s) = s^5 + 3 \cdot s^4 - 18 \cdot s^3 + 3 \cdot s^2 + s + 2$	No	The third coefficient is negative, there must be root(s), & thus poles, in the closed right half plane.
b) $D(s) = s^6 + 3 \cdot s^4 + 18 \cdot s^3 + 3 \cdot s^2 + s + 2$	No	The s^5 coefficient is zero, there must be root(s) in the closed right half plane.
c) $D(s) = s^6 + 3 \cdot s^5 + 18 \cdot s^4 + 3 \cdot s^3 + s^2 + 2 \cdot s$	No	The last coefficient is zero, there must be root(s) in the closed right half plane.
d) $D(s) = \left(s^2 + 2 \cdot s + 5\right) \cdot \left(s^2 + 4 \cdot s + 4\right)$ (Example 1 in text)	Yes	Neither factor has unstable poles so together they also have none. Don't multiply and complicate matters
e) $D(s) = \left(s^2 - 2 \cdot s + 5\right) \cdot \left(s^2 + 4 \cdot s + 4\right)$ (Example 2 in text)	No	First factor has at least one unstable pole, so together they also have at least one. Don't multiply and complicate matters
f) $D(s) = s^5 + 4 \cdot s^4 + 2 \cdot s^3 + 6 \cdot s^2 + 2 \cdot s + 1$	Can't tell without the full array	

RH Ex.4

s^4	1	2	2	0
s^3	4	6	1	0
s^2	$\frac{4 \cdot 2 - 1 \cdot 6}{4} = 0.5$	$\frac{4 \cdot 2 - 1 \cdot 1}{4} = 1.75$		
s^1	$\frac{0.5 \cdot 6 - 4 \cdot 1.75}{0.5} = -8$	$\frac{0.5 \cdot 1 - 4 \cdot 0}{0.5} = 1$		
s^0	$\frac{-8 \cdot 1.75 - 0.5 \cdot 1}{-8} = 1.813$			

\ Problem, some root(s) in CRHP

\ No need to progress to the next row if all you want to know is stability, but this extra steps can tell you there are two problem poles

Actual roots: $\begin{bmatrix} -3.855 \\ -0.187 - 0.4 \cdot j \\ -0.187 + 0.4 \cdot j \\ 0.114 - 1.147 \cdot j \\ 0.114 + 1.147 \cdot j \end{bmatrix}$ Last two roots are in the RHP

No Not stable

RH Ex.5 Use the Routh-Hurwitz method to determine the value range of K that will produce a stable system.

$D(s) = s^4 + 2 \cdot s^3 + 1 \cdot s^2 + s + K$ Characteristic equation of a feedback sytem.

s^4	1	1	K	0
s^3	2	1	0	
s^2	$\frac{2 \cdot 1 - 1 \cdot 1}{2} = 0.5$	$\frac{2 \cdot K - 1 \cdot 0}{2} = K$		
s^1	$\frac{0.5 \cdot 1 - 2 \cdot K}{0.5} = 1 - 4 \cdot K$	0		
s^0	$\frac{(1 - 4 \cdot K) \cdot K - 0.5 \cdot 0}{1 - 4 \cdot K} = K$			

$K > 0$ $\qquad 0 = 1 - 4 \cdot K$ $\qquad K < \frac{1}{4} = 0.25$

$0 < K < 0.25$

RH Ex.6 Use the Routh-Hurwitz method to determine the value range of K that will produce a stable system.

$D(s) = s^4 + 4\cdot K\cdot s^3 + 12\cdot s^2 + 2\cdot K\cdot s + K$

s^4	1	12	K	0
s^3	$4\cdot K$	$2\cdot K$	0	
s^2	$\frac{4\cdot K\cdot 12 - 1\cdot 2\cdot K}{4\cdot K} = 11.5$	$\frac{4\cdot K\cdot K - 1\cdot 0}{4\cdot K} = K$		
s^1	$\frac{11.5\cdot 2\cdot K - 4\cdot K\cdot K}{11.5} = 2\cdot K - \frac{4}{11.5}\cdot K^2$	0		
s^0	$\frac{\left(2\cdot K - \frac{4}{11.5}\cdot K^2\right)\cdot K}{2\cdot K - \frac{4}{11.5}\cdot K^2} = K$			

$K > 0$ This could have been seen from the original expression

$0 < 2 - \frac{4}{11.5}\cdot K$ $\quad K < 2\cdot\frac{11.5}{4} = 5.75$

$0 < K < 5.75$

RH Ex.7 Use the Routh-Hurwitz method to determine if all the poles are to the left of - 5.

$D(s) = s^3 + 44\cdot s^2 + 320\cdot s + 648$ Characteristic equation of a feedback sytem.

Replace all occurances of s with (s - 5)

$(s-5)^3 + 44\cdot(s-5)^2 + 320\cdot(s-5) + 648$

$\left(s^3 - 15\cdot s^2 + 75\cdot s - 125\right) + 44\cdot\left(s^2 - 10\cdot s + 25\right) + 320\cdot(s-5) + 648$

$s^3 - 15\cdot s^2 + 75\cdot s - 125 + 44\cdot s^2 - 44\cdot 10\cdot s + 44\cdot 25 + 320\cdot s - 320\cdot 5 + 648 \quad = \quad s^3 + 29\cdot s^2 - 45\cdot s + 23$

No, this has a negative coefficient

RH Ex.7b Are all the poles are to the left of - 4?

Replace all occurances of s with (s - 4)

$(s-4)^3 + 44\cdot(s-4)^2 + 320\cdot(s-4) + 648$

$\left(s^3 - 12\cdot s^2 + 48\cdot s - 64\right) + 44\cdot\left(s^2 - 8\cdot s + 16\right) + 320\cdot(s-4) + 648$

$s^3 - 12\cdot s^2 + 48\cdot s - 64 + 44\cdot s^2 - 44\cdot 8\cdot s + 44\cdot 16 + 320\cdot s - 320\cdot 4 + 648 \quad = \quad s^3 + 32\cdot s^2 + 16\cdot s + 8$

s^3	1	16	0
s^2	32	8	0
s^1	$\frac{32\cdot 16 - 1\cdot 8}{32} = 15.75$	$\frac{32\cdot 0 - 1\cdot 0}{32} = 0$	
s^0	$\frac{15.75\cdot 8 - 32\cdot 0}{15.75} = 8$		

Look at first column: All positive, so all roots are indeed left of -4.

Actual roots of:

$0 = s^3 + 44\cdot s^2 + 320\cdot s + 648$ $\quad \begin{pmatrix} -35.5 \\ -4.25 - 0.438\cdot j \\ -4.25 + 0.438\cdot j \end{pmatrix}$ Sure enough, all roots are left of -4, and not all left of -5