ECE 3510 Partial Fraction Expansion Examples by the Mixed Method

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Ex. 1 Like Example 1 from page 12, but with more interesting numbers

$$F(s) = \frac{12\cdot s + 64}{(s+4)^2\cdot(s+6)} = \frac{A}{s+4} + \frac{B}{(s+4)^2} + \frac{C}{s+6}$$

Multiply both sides by: $(s+4)^2\cdot(s+6)$

$$12\cdot s + 64 = A\cdot(s+4)\cdot(s+6) + B\cdot(s+6) + C\cdot(s+4)^2$$

Set $s := -4$

$$12\cdot(-4) + 64 = 0 + B\cdot(-4+6) + 0$$

$$16 \qquad\qquad 2 \qquad\qquad B := 8$$

Set $s := -6$

$$12\cdot(-6) + 64 = 0 + 0 + C\cdot(-6+4)^2 \qquad C := -2$$

$$-8 \qquad\qquad (-2)^2$$

Back to equation above

$$12\cdot s + 64 = A\cdot(s+4)\cdot(s+6) + B\cdot(s+6) + C\cdot(s+4)^2$$

$$12\cdot s + 64 = A\cdot s^2 + A\cdot 10\cdot s + A\cdot 24 + 8\cdot s + 8\cdot 6 + C\cdot s^2 + C\cdot 8\cdot s + C\cdot 16$$

$$0\cdot s^2 = A\cdot s^2 + 0\cdot s^2 + C\cdot s^2 \qquad A := -C$$

no s^2 term on the left

$$A = 2$$

And the rule is: Get as many easy answers as possible before clearing fractions!

$$F(s) = \frac{12\cdot s + 64}{(s+4)^2\cdot(s+6)} = \frac{2}{s+4} + \frac{8}{(s+4)^2} + \frac{-2}{s+6}$$

$$f(t) = 2\cdot e^{-4\cdot t} + 8\cdot t\cdot e^{-4\cdot t} + -2\cdot e^{-6\cdot t}$$

Ex. 2 Like Example 2 from page 13

$$F(s) = \frac{2\cdot(s+1)}{s\cdot\left(s^2 + 4\cdot s + 13\right)}$$

Try to find factors of $\left(s^2 + 4\cdot s + 13\right)$ find roots $\frac{-4 + \sqrt{4^2 - 4\cdot 13}}{2} = -2 + 3j$ & $-2 - 3j$

= complex numbers.. STOP!

If there are complex poles, then expect sines and cosines in the time domain. If the poles have real components as well as imaginary components then the sines and cosines are multiplied by exponentials. The entries in the Laplace transform table for these are:

$$e^{a\cdot t}\cdot\cos(b\cdot t) \iff \frac{s-a}{(s-a)^2 + b^2}$$

and

$$e^{a\cdot t}\cdot\sin(b\cdot t) \iff \frac{b}{(s-a)^2 + b^2}$$

DON'T decompose like this:

$$F(s) = \frac{2\cdot(s+1)}{s\cdot\left(s^2 + 4\cdot s + 13\right)} = \frac{A}{s} + \frac{B'}{(s+2+3j)} + \frac{C'}{(s+2-3j)}$$

Because that will only lead to complex exponentials which then have to be changed to sines and cosine in a separate step.

It's much smarter to decompose to this form:

$$F(s) = \frac{2\cdot(s+1)}{s\cdot\left(s^2 + 2\cdot s + 2\right)} = \frac{A}{s} + \frac{B\cdot(s-a)}{(s-a)^2 + b^2} + \frac{C\cdot b}{(s-a)^2 + b^2}$$

Because these forms **are in the table!**

And the rule is: Only decompose to terms that are actually in the table!

Example 2, continued

Find a and b : $s^2 + 4\cdot s + 13 \;=\; (s - a)^2 + b^2 \;=\; s^2 - 2\cdot a\cdot s + a^2 + b^2$

$$4\cdot s \;=\; -2\cdot a\cdot s \qquad a := -2$$

$$13 \;=\; a^2 + b^2 = (-2)^2 + b^2$$

$$b^2 = 9 \qquad b := 3$$

Find a and b the easy way: Recall from above, roots of $\left(s^2 + 4\cdot s + 13\right) = \dfrac{-4 + \sqrt{4^2 - 4\cdot 13}}{2} = -2 + 3j$ & $-2 - 3j$

$$a := \mathrm{Re}(-2 + 3j) \qquad a = -2 \qquad b := \mathrm{Im}(-2 + 3j) \qquad b = 3$$

$$F(s) = \frac{2\cdot(s+1)}{s\cdot\left(s^2 + 4\cdot s + 13\right)} = \frac{2\cdot(s+1)}{s\cdot\left[(s-a)^2 + b^2\right]}$$

$$= \frac{2\cdot(s+1)}{s\cdot\left[(s+2)^2 + 3^2\right]} = \frac{A}{s} + \frac{B\cdot(s-a)}{(s-a)^2 + b^2} + \frac{C\cdot b}{(s-a)^2 + b^2}$$

Multiply both sides by: $s\cdot\left[(s+2)^2 + 3^2\right]$

$$2\cdot(s+1) = A\cdot\left[(s+2)^2 + 3^2\right] + B\cdot(s+2)\cdot s + C\cdot 3\cdot s$$

Set $s := 0$

$$2\cdot(1) = A\cdot\left[(2)^2 + 3^2\right] + 0 + 0 \qquad A := \frac{2}{13}$$

$$(2)^2 + 3^2 = 13$$

Set $s := -2$

$$2\cdot(-1) = A\cdot\left[(0)^2 + 3^2\right] + 0 + C\cdot 3\cdot(-2)$$

$$-2 = 9\cdot A + -6\cdot C \qquad C := \frac{2 + 9\cdot A}{6} = \frac{22}{39}$$

Back to equation above

$$2\cdot(s+1) = A\cdot\left(s^2 + 4\cdot s + 13\right) + B\cdot(s+2)\cdot s + C\cdot 3\cdot s$$

$$2\cdot s + 2 = A\cdot s^2 + A\cdot 4\cdot s + A\cdot 13 + B\cdot s^2 + 2\cdot B\cdot s + 3\cdot C\cdot s$$

$$0\cdot s^2 = A\cdot s^2 + B\cdot s^2 \qquad B := -A = -\frac{2}{13}$$

no s^2 term on the left

$$F(s) = \frac{2\cdot(s+1)}{s\cdot\left(s^2 + 4\cdot s + 13\right)} = \frac{A}{s} + \frac{B\cdot(s-a)}{(s-a)^2 + b^2} + \frac{C\cdot b}{(s-a)^2 + b^2}$$

$$= \frac{2}{13}\cdot\frac{1}{s} + -\frac{2}{13}\cdot\frac{(s+2)}{(s+2)^2 + 3^2} + \frac{22}{39}\cdot\frac{3}{(s+2)^2 + 3^2}$$

$$f(t) = \frac{2}{13} + -\frac{2}{13}\cdot e^{-2\cdot t}\cdot\cos(3\cdot t) + \frac{3}{13}\cdot e^{-2\cdot t}\cdot\sin(3\cdot t)$$

$$= \left(\frac{2}{13} - \frac{2}{13}\cdot e^{-2\cdot t}\cdot\cos(3\cdot t) + \frac{22}{39}\cdot e^{-2\cdot t}\cdot\sin(3\cdot t)\right)\cdot u(t)$$

Ex. 3 $F(s) := \dfrac{9 \cdot s^2 - 9 \cdot s + 36}{(s^2+1)\cdot(s^2+4)}$

DON'T decompose like this:

$$F(s) := \frac{9 \cdot s^2 - 9 \cdot s + 36}{(s^2+1)\cdot(s^2+4)} = \frac{A'}{(s+j)} + \frac{B'}{(s-j)} + \frac{C'}{(s+2j)} + \frac{D'}{(s-2j)}$$

or even this:

$$= \frac{A'' \cdot s + B''}{(s^2+1)} + \frac{C'' \cdot s + D''}{(s^2+4)}$$

Look at the table first, to see what you should aim for.

It's much smarter to decompose to this form:

$$F(s) := \frac{9 \cdot s^2 - 9 \cdot s + 36}{(s^2+1)\cdot(s^2+4)} = \frac{A \cdot s}{(s^2 + a_1{}^2)} + \frac{B \cdot a_1}{(s^2 + a_1{}^2)} + \frac{C \cdot s}{(s^2 + a_2{}^2)} + \frac{D \cdot a_2}{(s^2 + a_2{}^2)}$$

$a_1 := \sqrt{1}$ $a_1 = 1$ $a_2 := \sqrt{4}$ $a_2 = 2$

$$= \frac{A \cdot s}{(s^2 + 1^2)} + \frac{B \cdot 1}{(s^2 + 1^2)} + \frac{C \cdot s}{(s^2 + 2^2)} + \frac{D \cdot 2}{(s^2 + 2^2)}$$

or, if you prefer:

$$= \frac{A \cdot s + B \cdot a_1}{(s^2+1)} + \frac{C \cdot s + D \cdot a_2}{(s^2+4)}$$

which is the same thing

Multiply both sides by: $(s^2+1)\cdot(s^2+4)$

$$9 \cdot s^2 - 9 \cdot s + 36 = A \cdot s \cdot (s^2 + 2^2) + B \cdot 1 \cdot (s^2 + 2^2) + C \cdot s \cdot (s^2 + 1^2) + D \cdot 2 \cdot (s^2 + 1^2)$$

Set $s := \sqrt{-1}$

$$9 \cdot s^2 - 9 \cdot s + 36 = A \cdot j \cdot \underbrace{(-1 + 2^2)}_{3} + B \cdot 1 \cdot \underbrace{(-1 + 2^2)}_{3} + 0 + 0$$

$$9 \cdot j^2 - 9 \cdot j + 36 = A \cdot j \cdot 3 + B \cdot 3$$

$-9 \cdot j + 27$ $A := -3$ $B := 9$

A and B must be real because of the way that we have decomposed the transform. The time functions cannot have unreal coefficients

Set $s := \sqrt{-4}$

$$9 \cdot (2 \cdot j)^2 - 9 \cdot (2 \cdot j) + 36 = 0 + 0 + C \cdot 2 \cdot j \cdot \underbrace{(-4 + 1^2)}_{-3} + D \cdot 2 \cdot \underbrace{(-4 + 1^2)}_{-3}$$

$-36 - 18 \cdot j + 36$ $-6 \cdot j \cdot C$ $-6 \cdot D$

$C := 3$ $D := 0$

$$F(s) := \frac{9 \cdot s^2 - 9 \cdot s + 36}{(s^2+1)\cdot(s^2+4)} = \frac{-3 \cdot s}{(s^2 + 1^2)} + \frac{9 \cdot 1}{(s^2 + 1^2)} + \frac{3 \cdot s}{(s^2 + 2^2)} + \frac{0 \cdot 2}{(s^2 + 2^2)}$$

$$f(t) = ((-3) \cdot \cos(1 \cdot t) + 9 \cdot \sin(1 \cdot t) + 3 \cdot \cos(2 \cdot t) + 0 \cdot \sin(2 \cdot t)) \cdot u(t)$$

$$= ((-3) \cdot \cos(1 \cdot t) + 9 \cdot \sin(1 \cdot t) + 3 \cdot \cos(2 \cdot t)) \cdot u(t)$$

Ex. 4 Example 3 from page 15

$F(s) = \dfrac{1}{\left(s^2+2\cdot s+2\right)^2}$ Try to find factors of $\left(s^2+2\cdot s+2\right)$: $\dfrac{-2+\sqrt{2^2-4\cdot 2}}{2} = -1+j$ & $-1-j$

= complex numbers.. STOP!

$a := \mathrm{Re}(-1+j)$ $a = -1$

$b := \mathrm{Im}(-1+j)$ $b = 1$

Decompose to:

$e^{a\cdot t}\cdot\cos(b\cdot t)$ <=> $\dfrac{s-a}{(s-a)^2+b^2}$

$e^{a\cdot t}\cdot\sin(b\cdot t)$ <=> $\dfrac{b}{(s-a)^2+b^2}$

and

$t\cdot e^{a\cdot t}\cdot\cos(b\cdot t)$ <=> $\dfrac{(s-a)^2-b^2}{\left[(s-a)^2+b^2\right]^2}$

$t\cdot e^{a\cdot t}\cdot\sin(b\cdot t)$ <=> $\dfrac{2\cdot b\cdot(s-a)}{\left[(s-a)^2+b^2\right]^2}$

Decompose to this form:

$$F(s) = \frac{1}{\left[(s-a)^2+b^2\right]^2} = \frac{A\cdot(s-a)}{(s-a)^2+b^2} + \frac{B\cdot b}{(s-a)^2+b^2} + \frac{C\cdot\left[(s-a)^2-b^2\right]}{\left[(s-a)^2+b^2\right]^2} + \frac{D\cdot(2\cdot b\cdot(s-a))}{\left[(s-a)^2+b^2\right]^2}$$

Multiply both sides by: $\left[(s+1)^2+1^2\right]^2$

$$1 = A\cdot(s+1)\cdot\left[(s+1)^2+1^2\right] + B\cdot 1\cdot\left[(s+1)^2+1^2\right] + C\cdot\left[(s+1)^2-1^2\right] + D\cdot(2\cdot 1\cdot(s+1))$$

Set $s := -1$

$$1 = 0 + B\cdot 1 + C\cdot(0-1) + 0$$

-1

$1 = B - C$

$-C = 1 - B$

Back to equation above

$$1 = A\cdot s^3 + A\cdot 2\cdot s^2 + A\cdot 2\cdot s + A\cdot s^2 + A\cdot 2\cdot s + A\cdot 2 + B\cdot s^2 + B\cdot 2\cdot s + B\cdot 2 + C\cdot s^2 + C\cdot 2\cdot s + D\cdot 2\cdot s + D\cdot 2$$

$0\cdot s^3 = A\cdot s^3$ $A := 0$

$$1 = 0 + B\cdot s^2 + B\cdot 2\cdot s + B\cdot 2 + C\cdot s^2 + C\cdot 2\cdot s + D\cdot 2\cdot s + D\cdot 2$$

$0\cdot s^2 = B\cdot s^2 + C\cdot s^2$ $B := -C = 1 - B$

$B := \dfrac{1}{2}$

$C := -B = -\dfrac{1}{2}$

$1 = B\cdot 2 + D\cdot 2$

$\dfrac{1}{2}\cdot 2 = 1$ $D := 0$

$$F(s) = \frac{1}{\left(s^2+2\cdot s+2\right)^2} = 0 + \frac{\frac{1}{2}\cdot 1}{(s-1)^2+1^2} - \frac{\frac{1}{2}\cdot\left[(s+1)^2-1^2\right]}{\left[(s+1)^2+1^2\right]^2} + 0$$

$$f(t) = \frac{1}{2}\cdot e^{-t}\cdot\sin(t) - \frac{1}{2}\cdot t\cdot e^{-t}\cdot\cos(t)$$