ECE 3510 Partial Fraction Expansion Examples by the Mixed Method

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Ex. 1 Example 1 from page 12

$$F(s) = \frac{1}{(s+1)^2\cdot(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$$

Multiply both sides by: $(s+1)^2\cdot(s+2)$

$$1 = A\cdot(s+1)\cdot(s+2) + B\cdot(s+2) + C\cdot(s+1)^2$$

Set $s := -1$

$$1 = 0 + \underbrace{B\cdot(-1+2)}_{1} + 0 \qquad B := 1$$

Set $s := -2$

$$1 = 0 + 0 + \underbrace{C\cdot(-2+1)^2}_{1} \qquad C := 1$$

Back to equation above

$$1 = A\cdot(s+1) + B\cdot(s+2) + C\cdot(s+1)^2$$

$$1 = A\cdot s^2 + A\cdot 3\cdot s + A\cdot 2 + 1\cdot s + 1\cdot 2 + C\cdot s^2 + C\cdot 2\cdot s + C\cdot 1$$

$$0\cdot s^2 = A\cdot s^2 + 0 + C\cdot s^2 \qquad A := -C$$

no s^2 term on the left

$$A = -1$$

And the rule is: Get as many easy answers as possible before clearing fractions!

$$F(s) = \frac{1}{(s+1)^2\cdot(s+2)} = \frac{-1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2}$$

$$f(t) = -1\cdot e^{-1\cdot t} + 1\cdot t\cdot e^{-1\cdot t} + 1\cdot t\cdot e^{-2\cdot t}$$

Ex. 2 Example 2 from page 13

$$F(s) = \frac{2\cdot(s+1)}{s\cdot\left(s^2+2\cdot s+2\right)}$$

Try to find factors of $\left(s^2+2\cdot s+2\right)$: $\dfrac{-2+\sqrt{2^2-4\cdot 2}}{2} = -1 + j$ & $-1-j$

= complex numbers.. STOP!

If there are complex poles, then expect sines and cosines in the time domain. If the poles have real components as well as imaginary components then the sines and cosines are multiplied by exponentials. The entries in the Laplace transform table for these are:

$$e^{at}\cdot\cos(b\cdot t) \iff \frac{s-a}{(s-a)^2+b^2}$$

and

$$e^{at}\cdot\sin(b\cdot t) \iff \frac{b}{(s-a)^2+b^2}$$

DON'T decompose like this:

$$F(s) = \frac{2\cdot(s+1)}{s\cdot\left(s^2+2\cdot s+2\right)} = \frac{A}{s} + \frac{B}{(s+1+j)} + \frac{C}{(s+1-j)}$$

Because that will only lead to complex exponentials which then have to be changed to sines and cosine in a separate step.

It's much smarter to decompose to this form:

$$F(s) = \frac{2\cdot(s+1)}{s\cdot\left(s^2+2\cdot s+2\right)} = \frac{A}{s} + \frac{B\cdot(s-a)}{(s-a)^2+b^2} + \frac{C\cdot b}{(s-a)^2+b^2}$$

Because these forms **are in the table!**

And the rule is: Only decompose to terms that are actually in the table!

Example 2 from page 13, continued

Find a and b : $s^2 + 2\cdot s + 2 = (s-a)^2 + b^2 = s^2 - 2\cdot a\cdot s + a^2 + b^2$

$$2\cdot s = -2\cdot a\cdot s \qquad a := -1$$

$$2 = a^2 + b^2$$

$$b^2 = 1 \qquad b := 1$$

Find a and b the easy way: Recall from above: $s^2 + 2\cdot s + 2 = \dfrac{-2+\sqrt{2^2 - 4\cdot 2}}{2} = -1 + j$ & $-1 - j$

$a := \mathrm{Re}(-1+j)$ $\quad a := -1$ $\quad b := \mathrm{Im}(-1+j)$ $\quad b := 1$

$$F(s) = \frac{2\cdot(s+1)}{s\cdot\left(s^2+2\cdot s+2\right)} = \frac{2\cdot(s+1)}{s\cdot\left[(s-a)^2+b^2\right]}$$

$$= \frac{2\cdot(s+1)}{s\cdot\left[(s+1)^2+1^2\right]} = \frac{A}{s} + \frac{B\cdot(s-a)}{(s-a)^2+b^2} + \frac{C\cdot b}{(s-a)^2+b^2}$$

Multiply both sides by: $s\cdot\left[(s+1)^2+1^2\right]$

$$2\cdot(s+1) = A\cdot\left[(s+1)^2+1^2\right] + B\cdot(s+1)\cdot s + C\cdot 1\cdot s$$

Set $s := 0$

$$2\cdot(1) = A\cdot\left[(1)^2+1^2\right] + 0 + 0 \qquad A := 1$$

Set $s := -1$

$$0 = A\cdot\left[(0)^2+1^2\right] + 0 + C\cdot(-1) \qquad C := 1$$

Back to equation above

$$2\cdot(s+1) = A\cdot\left[(s+1)^2+1^2\right] + B\cdot(s+1)\cdot s + C\cdot 1\cdot s$$

$$2\cdot s+2 = A\cdot s^2 + A\cdot 2\cdot s + A\cdot 2 + B\cdot s^2 + B\cdot s + C\cdot s$$

$$0\cdot s^2 = A\cdot s^2 + B\cdot s^2 \qquad B := -A$$

no s^2 term on the left $\qquad B = -1$

$$F(s) = \frac{2\cdot(s+1)}{s\cdot\left(s^2+2\cdot s+2\right)} = \frac{A}{s} + \frac{B\cdot(s-a)}{(s-a)^2+b^2} + \frac{C\cdot b}{(s-a)^2+b^2}$$

$$= \frac{1}{s} + \frac{-1\cdot(s+1)}{(s+1)^2+1^2} + \frac{1\cdot 1}{(s+1)^2+1^2}$$

$$f(t) = 1 + -1\cdot e^{-1\cdot t}\cdot\cos(1\cdot t) + 1\cdot e^{-1\cdot t}\cdot\sin(1\cdot t)$$

$$= 1 - e^{-t}\cdot\cos(t) + e^{-t}\cdot\sin(t)$$

Ex. 3
$$F(s) := \frac{3\cdot s}{(s^2+1)\cdot(s^2+4)}$$

DON' T decompose like this:

$$F(s) := \frac{3\cdot s}{(s^2+1)\cdot(s^2+4)} = \frac{A\cdot s+B}{(s^2+1)} + \frac{C\cdot s+D}{(s^2+4)}$$

It' s much smarter to decompose to this form:

$$F(s) := \frac{3\cdot s}{(s^2+1)\cdot(s^2+4)} = \frac{A\cdot s}{(s^2+a_1{}^2)} + \frac{B\cdot a_1}{(s^2+a_1{}^2)} + \frac{C\cdot s}{(s^2+a_2{}^2)} + \frac{D\cdot a_2}{(s^2+a_2{}^2)}$$

$$a_1 := \sqrt{1} \qquad a_1 = 1 \qquad a_2 := \sqrt{4} \qquad a_2 = 2$$

$$= \frac{A\cdot s}{(s^2+1^2)} + \frac{B\cdot 1}{(s^2+1^2)} + \frac{C\cdot s}{(s^2+2^2)} + \frac{D\cdot 2}{(s^2+2^2)}$$

Multiply both sides by: $(s^2+1)\cdot(s^2+4)$

$$3\cdot s = A\cdot s\cdot(s^2+2^2) + B\cdot 1\cdot(s^2+2^2) + C\cdot s\cdot(s^2+1^2) + D\cdot 2\cdot(s^2+1^2)$$

Set $s := \sqrt{-1}$
$$3\cdot j = A\cdot j\cdot\underbrace{(-1+2^2)}_{3} + B\cdot 1\cdot\underbrace{(-1+2^2)}_{3} + 0 + 0$$
$$A := 1 \qquad B := 0$$

Set $s := \sqrt{-4}$
$$3\cdot 2\cdot j = 0 + 0 + C\cdot 2\cdot j\cdot\underbrace{(-4+1^2)}_{-3} + D\cdot 2\cdot\underbrace{(-4+1^2)}_{-3}$$
$$C := -1 \qquad D := 0$$

$$F(s) := \frac{3\cdot s}{(s^2+1)\cdot(s^2+4)} = \frac{1\cdot s}{(s^2+1^2)} + \frac{0\cdot 1}{(s^2+1^2)} + \frac{-1\cdot s}{(s^2+2^2)} + \frac{0\cdot 2}{(s^2+2^2)}$$

$$f(t) = (1\cdot\cos(1\cdot t) + 0\cdot\sin(1\cdot t) + (-1)\cdot\cos(2\cdot t) + 0\cdot\sin(2\cdot t))\cdot u(t)$$
$$= (\cos(t) - \cos(2\cdot t))\cdot u(t)$$

Ex. 4 Example 3 from page 15

$$F(s) = \frac{1}{\left(s^2+2\cdot s+2\right)^2}$$

Try to find factors of $\left(s^2+2\cdot s+2\right)$: $\dfrac{-2+\sqrt{2^2-4\cdot 2}}{2} = -1+j$ & $-1-j$

= complex numbers.. STOP!

$a := \mathrm{Re}(-1+j)$ $\quad a := -1$

$b := \mathrm{Re}(-1+j)$ $\quad b := 1$

Decompose to:

$$e^{a t}\cdot\cos(b\cdot t) \quad <=> \quad \frac{s-a}{(s-a)^2+b^2}$$

$$e^{a t}\cdot\sin(b\cdot t) \quad <=> \quad \frac{b}{(s-a)^2+b^2}$$

and

$$t\cdot e^{a t}\cdot\cos(b\cdot t) \quad <=> \quad \frac{(s-a)^2-b^2}{\left[(s-a)^2+b^2\right]^2}$$

$$t\cdot e^{a t}\cdot\sin(b\cdot t) \quad <=> \quad \frac{2\cdot b\cdot(s-a)}{\left[(s-a)^2+b^2\right]^2}$$

Decompose to this form:

$$F(s) = \frac{1}{\left[(s-a)^2+b^2\right]^2} = \frac{A\cdot(s-a)}{(s-a)^2+b^2} + \frac{B\cdot b}{(s-a)^2+b^2} + \frac{C\cdot\left[(s-a)^2-b^2\right]}{\left[(s-a)^2+b^2\right]^2} + \frac{D\cdot(2\cdot b\cdot(s-a))}{\left[(s-a)^2+b^2\right]^2}$$

Multiply both sides by: $\left[(s+1)^2+1^2\right]^2$

$$1 = A\cdot(s+1)\cdot\left[(s+1)^2+1^2\right] + B\cdot 1\cdot\left[(s+1)^2+1^2\right] + C\cdot\left[(s+1)^2-1^2\right] + D\cdot(2\cdot 1\cdot(s+1))$$

Set $s := -1$

$$1 = 0 + B\cdot 1 + \underset{-1}{C\cdot(0-1)} + 0$$

$1 = B - C$

$-C = 1 - B$

Back to equation above

$$1 = A\cdot s^3 + A\cdot 2\cdot s^2 + A\cdot 2\cdot s + A\cdot s^2 + A\cdot 2\cdot s + A\cdot 2 + B\cdot s^2 + B\cdot 2\cdot s + B\cdot 2 + C\cdot s^2 + C\cdot 2\cdot s + D\cdot 2\cdot s + D\cdot 2$$

$$0\cdot s^3 = A\cdot s^3 \qquad A := 0$$

$$1 = 0 + B\cdot s^2 + B\cdot 2\cdot s + B\cdot 2 + C\cdot s^2 + C\cdot 2\cdot s + D\cdot 2\cdot s + D\cdot 2$$

$$0\cdot s^2 = B\cdot s^2 + C\cdot s^2$$

$B := -C = 1 - B$

$B := \dfrac{1}{2}$

$C := -B = -\dfrac{1}{2}$

$$1 = B\cdot 2 + D\cdot 2$$

$\dfrac{1}{2}\cdot 2 = 1$ $\qquad D := 0$

$$F(s) = \frac{1}{\left(s^2+2\cdot s+2\right)^2} = 0 + \frac{\frac{1}{2}\cdot 1}{(s-1)^2+1^2} - \frac{\frac{1}{2}\cdot\left[(s+1)^2-1^2\right]}{\left[(s+1)^2+1^2\right]^2} + 0$$

$$f(t) = \frac{1}{2}\cdot e^{-t}\cdot\sin(t) - \frac{1}{2}\cdot t\cdot e^{-t}\cdot\cos(t)$$