z - transforms

A. Stolp
4/15/09,
4/18/20,
4/18/23

f(k)		F(z)
$f(k) = \frac{1}{2\cdot\pi\cdot j}\cdot\oint F(z)\cdot z^{k-1}\,dz$ integral around a closed path in the complex plane		$\mathbf{F}(z) = \sum_{k=0}^{\infty} f(k)\cdot z^{-k}$
$\delta(k)$	impulse	1
$\delta(k-m)$	shifted impulse	$\frac{1}{z^m}$
$u(k)$	unit step	$\frac{z}{z-1}$

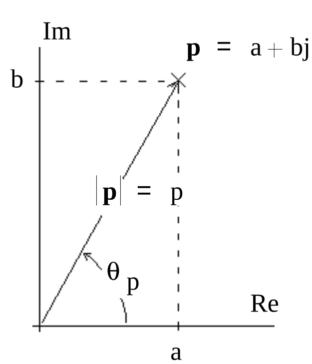
All the following are multiplied by $u(k)$

k	$\frac{z}{(z-1)^2}$
k^2	$\frac{z\cdot(z+1)}{(z-1)^3}$
k^3	$\frac{z\cdot\left(z^2+4\cdot z+1\right)}{(z-1)^4}$

Geometric Progression or Power Series

a^k	$\frac{z}{z-a}$
$k\cdot a^k$	$\frac{a\cdot z}{(z-a)^2}$
$k^2\cdot a^k$	$\frac{a\cdot z\cdot(z+a)}{(z-a)^3}$
$k^3\cdot a^k$	$\frac{a\cdot z\cdot\left(z^2+4\cdot a\cdot z+a^2\right)}{(z-a)^4}$

Sinusoids



$\cos\left(\Omega_o\cdot k\right)$	$\frac{z(z-a)}{z^2-2\cdot a\cdot z+1} = \frac{z\left(z-\cos\left(\Omega_o\right)\right)}{z^2-2\cdot\cos\left(\Omega_o\right)\cdot z+1}$
$\sin\left(\Omega_o\cdot k\right)$	$\frac{z\cdot b}{z^2-2\cdot a\cdot z+1} = \frac{z\cdot\sin\left(\Omega_o\right)}{z^2-2\cdot\cos\left(\Omega_o\right)\cdot z+1}$
$p^k\cdot\cos\left(\theta_p\cdot k\right)$	$\frac{z\cdot(z-a)}{z^2-2\cdot a\cdot z+\left(a^2+b^2\right)} = \frac{z\cdot\left(z-p\cdot\cos\left(\theta_p\right)\right)}{z^2-2\cdot p\cdot\cos\left(\theta_p\right)\cdot z+p^2}$
$p^k\cdot\sin\left(\theta_p\cdot k\right)$	$\frac{z\cdot b}{z^2-2\cdot a\cdot z+\left(a^2+b^2\right)} = \frac{z\cdot\left(p\cdot\sin\left(\theta_p\right)\right)}{z^2-2\cdot p\cdot\cos\left(\theta_p\right)\cdot z+p^2}$

z - transforms

Good for inverse z-transforms

A. Stolp
4/16/06
4/15/09

F(z)	f(k)
	All the following are multiplied by $u(k)$ unless specified otherwise
Poles at zero	
$\frac{A\cdot z}{z} = A$	$A\cdot\delta(k)$
$\frac{B\cdot z}{z^2} = \frac{B}{z}$	$B\cdot\delta(k-1)$
$\frac{C\cdot z}{z^3} = \frac{C}{z^2}$	$C\cdot\delta(k-2)$
$\frac{D\cdot z}{z^4} = \frac{D}{z^3}$	$D\cdot\delta(k-3)$
Poles on real axis (not at zero)	
$\frac{B\cdot z}{(z-p)}$	$B\cdot p^k$
$\frac{C\cdot z}{(z-p)^2}$	$C\cdot k\cdot p^{k-1}$
$\frac{D\cdot z}{(z-p)^3}$	$D\cdot\frac{1}{2}\cdot k\cdot(k-1)\cdot p^{k-2}$
$\frac{E\cdot z}{(z-p)^4}$	$E\cdot\frac{1}{6}\cdot k\cdot(k-1)\cdot(k-2)\cdot p^{k-3}$
Complex poles	
$\frac{B\cdot z}{(z-p)} + \frac{\overline{B}\cdot z}{(z-\overline{p})}$	$2\cdot\lvert B\rvert\cdot(\lvert p\rvert)^k\cdot\cos(\theta_p\cdot k+\theta_B)$
$\frac{B\cdot z}{(z-p)^2} + \frac{\overline{B}\cdot z}{(z-\overline{p})^2}$	$2\cdot\lvert B\rvert\cdot k\cdot(\lvert p\rvert)^{k-1}\cdot\cos[\theta_p\cdot(k-1)+\theta_B]$
$\frac{B\cdot z}{(z-p)^3} + \frac{\overline{B}\cdot z}{(z-\overline{p})^3}$	$\lvert B\rvert\cdot k\cdot(k-1)\cdot(\lvert p\rvert)^{k-2}\cdot\cos[\theta_p\cdot(k-2)+\theta_B]$
$\frac{B\cdot z}{(z-p)^4} + \frac{\overline{B}\cdot z}{(z-\overline{p})^4}$	$\frac{1}{3}\cdot\lvert B\rvert\cdot k\cdot(k-1)\cdot(k-2)\cdot(\lvert p\rvert)^{k-3}\cdot\cos[\theta_p\cdot(k-3)+\theta_B]$

where $B = \lvert B\rvert\cdot e^{j\cdot\theta_B}$ and $p = \lvert p\rvert\cdot e^{j\cdot\theta_p}$

if $B = C + D\cdot j$ and $p = q + r\cdot j$

then $\lvert B\rvert = \sqrt{C^2+D^2}$ and $\lvert p\rvert = \sqrt{q^2+r^2}$

$\theta_B = \operatorname{atan}\left(\frac{D}{C}\right)$ $\qquad$ $\theta_p = \operatorname{atan}\left(\frac{r}{q}\right)$

ECE 3510 z - transform Properties

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4/15/09

Operation	f(k)	F(z)	
	All the following are multiplied by u(k) unless specified otherwise		
Addition	$f(k)+g(k)$	$\mathbf{F}(z)+\mathbf{G}(z)$	
Scalar multiplication	$c\cdot f(k)$	$c\cdot\mathbf{F}(z)$	
Linearity	$c\cdot f(k)+d\cdot g(k)$	$c\cdot\mathbf{F}(z)+d\cdot\mathbf{G}(z)$	
Right shift $m \geq 0$	$f(k-m)\cdot u(k-m)$	$\frac{1}{z^m}\cdot\mathbf{F}(z) = z^{-m}\cdot\mathbf{F}(z)$	
	$f(k-m)$	$\frac{1}{z^m}\cdot\mathbf{F}(z)+\frac{1}{z^m}\cdot\sum_{k=1}^{m} f(-k)\cdot z^k$	
	$f(k-1)$	$z^{-1}\cdot\mathbf{F}(z)+f(-1)$	
	$f(k-2)$	$z^{-2}\cdot F(z)+z^{-1}\cdot f(-1)+f(-2)$	
	$f(k-3)$	$z^{-3}\cdot\mathbf{F}(z)+z^{-2}\cdot f(-1)+z^{-1}\cdot f(-2)+f(-3)$	
Left shift $m \geq 0$	$f(k+m)$	$z^m\cdot\mathbf{F}(z)-z^m\cdot\sum_{k=0}^{m-1} f(k)\cdot z^{-k}$	
	$f(k+1)$	$z\cdot\mathbf{F}(z)-z\cdot f(0)$	
	$f(k+2)$	$z^2\cdot\mathbf{F}(z)-z^2\cdot f(0)-z\cdot f(1)$	
	$f(k+3)$	$z^3\cdot\mathbf{F}(z)-z^3\cdot f(0)-z^2\cdot f(1)-z\cdot f(2)$	
Multiplication by p^k	$p^k\cdot f(k)$	$\mathbf{F}\left(\frac{z}{p}\right)$	Frequency scaling
Multiplication by k	$k\cdot f(k)$	$-z\cdot\frac{d}{dz}F(z)$	Frequency differentiation
Time convolution	$f(k) * g(k)$	$\mathbf{F}(z)\cdot\mathbf{G}(z)$	
Initial value	$f(0)$	$\lim_{z\to\infty}\mathbf{F}(z)$	
Final value	$f(\infty)$	$\lim_{z\to 1}(z-1)\cdot\mathbf{F}(z)$ (all poles of $(z-1)\mathbf{F}(z)$ inside unit circle)	

ECE 3510 Lines of Constant Damping in the z-plane

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4/4/07