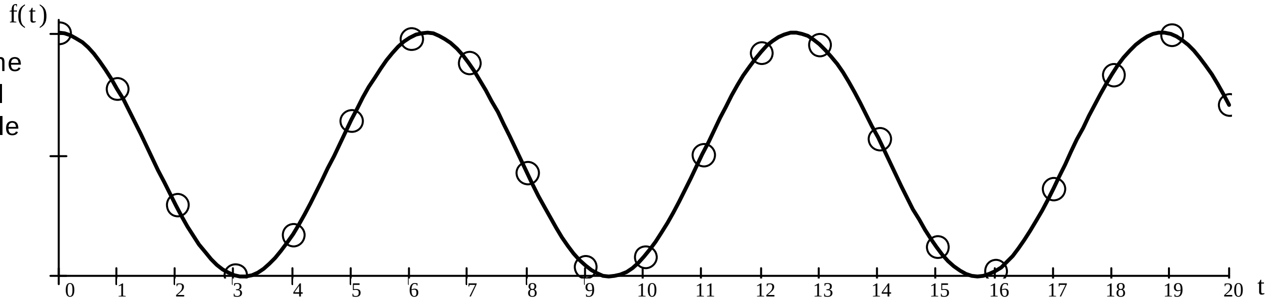
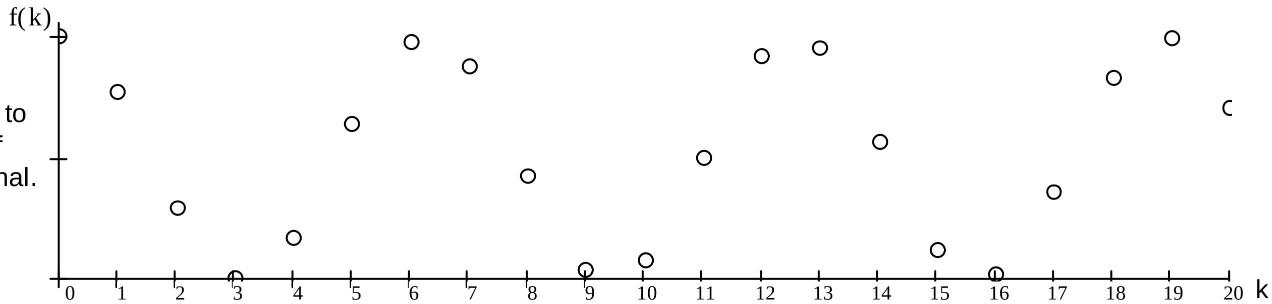
A Stolp
4/16/20

Continuous-time (analog) signal showing sample times



Discrete-time (digital) signal corresponding to the samples of the analog signal.

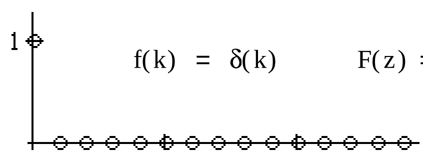


The z - transform

The z - transform will help us deal with discrete-time (digital) signals just like the Laplace transform helped us with continuous-time signals. So let's start making a table.

z - transform: $F(z) = \sum_{k=0}^{\infty} f(k)\cdot z^{-k}$

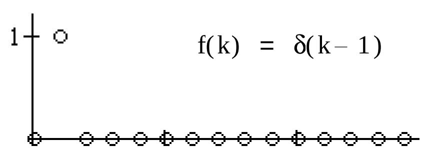
Impulse



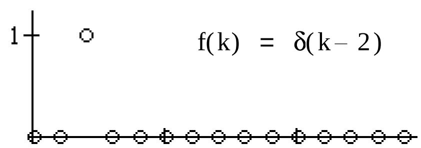
$f(k) = \delta(k)$ $\quad F(z) = \sum_{k=0}^{\infty} \delta(k)\cdot z^{-k} = 1 + 0 + 0 + 0 + \,.\,.\,.$ $\quad F(z) = 1$ no pole

Just like Laplace:
$f(t) = \delta(t)$ & $\mathbf{F}(s) = 1$

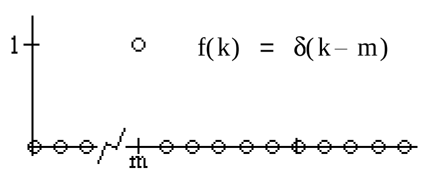
Delayed Impulses



$f(k) = \delta(k-1)$ $\quad F(z) = \sum_{k=0}^{\infty} \delta(k-1)\cdot z^{-k} = 0 + \frac{1}{z} + 0 + 0 + \,.\,.\,.$ $\quad F(z) = \frac{1}{z}$



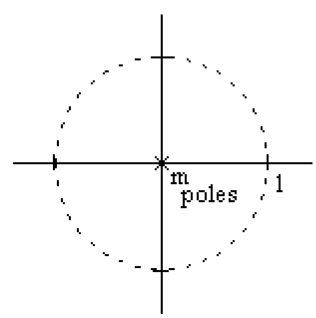
$f(k) = \delta(k-2)$ $\quad F(z) = \sum_{k=0}^{\infty} \delta(k-2)\cdot z^{-k} = 0 + 0 + \frac{1}{z^2} + 0 + \,.\,.\,.$ $\quad F(z) = \frac{1}{z^2}$



$f(k) = \delta(k-m)$ $\quad F(z) = \sum_{k=0}^{\infty} \delta(k-m)\cdot z^{-k}$

$= 0 + \,.\,.\,.\, 0 + \frac{1}{z^m} + 0 + 0 + \,.\,.\,. = \frac{1}{z^m}$

$F(z) = \frac{1}{z^m}$



Any finite-length signal can be made of delayed impulses, so all its poles are at the origin.

Power Series (Needed to solve the next few cases)

$$\text{SUM} = \sum_{k=0}^{n} \alpha^k = 1+\alpha+\alpha^2+\alpha^3+\alpha^4+\ldots\ \alpha^n$$

$$\text{SUM}\cdot(1-\alpha) = (1-\alpha)\left(1+\alpha+\alpha^2+\alpha^3+\alpha^4+\ldots\ \alpha^n\right)$$

$$= 1\left(1+\alpha+\alpha^2+\alpha^3+\alpha^4+\ldots\ \alpha^n\right) - \alpha\left(1+\alpha+\alpha^2+\alpha^3+\alpha^4+\ldots\ \alpha^n\right)$$

$$\begin{aligned} = \ & 1\left(1+\alpha+\alpha^2+\alpha^3+\alpha^4+\ldots\ \alpha^n\right) \\ & -\left(\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\ldots\ \alpha^n+\alpha^{n+1}\right) \\ \hline & 1 \qquad\qquad\qquad\qquad\qquad\qquad -\alpha^{n+1} \end{aligned}$$

$$\text{SUM}\cdot(1-\alpha) = 1-\alpha^{n+1}$$

$$\text{SUM} = \frac{1-\alpha^{n+1}}{(1-\alpha)} \qquad \text{If } n=\infty \quad \text{SUM} = \sum_{k=0}^{\infty}\alpha^{\infty} = \frac{1-\alpha^{\infty+1}}{(1-\alpha)} \overset{\text{if } (\alpha<1)}{=} \frac{1}{(1-\alpha)}$$

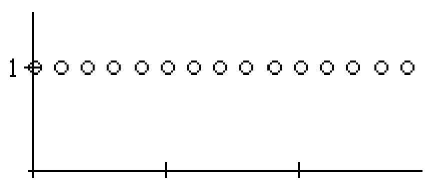
in region of convergence ($\alpha < 1$)

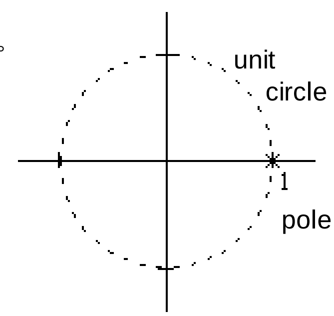
Unit Step

$$f(k) = u(k) \qquad F(z) = \sum_{k=0}^{\infty} u(k)\cdot z^{-k} = 1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\ldots\ \left(\frac{1}{z}\right)^{\infty}$$

$$= \frac{1}{\left(1-\frac{1}{z}\right)} = F(z) = \frac{z}{(z-1)}$$

in region of convergence





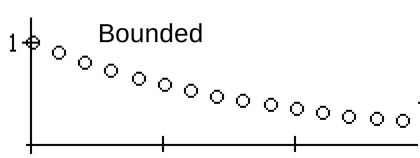
Like Laplace: $f(t) = u(t)$ & $\mathbf{F}(s) = \frac{1}{s}$

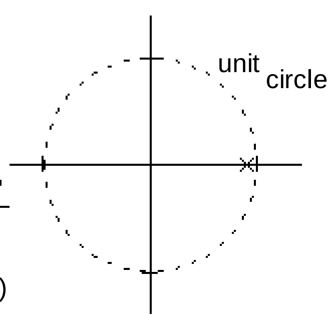
Geometric Progression

$$f(k) = a^k\cdot u(k) \qquad F(z) = \sum_{k=0}^{\infty} a^k\cdot z^{-k} = 1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\left(\frac{a}{z}\right)^3+\ldots\ \left(\frac{a}{z}\right)^{\infty} = \frac{1}{\left(1-\frac{a}{z}\right)} = F(z) = \frac{z}{(z-a)}$$

If ($a < 1$)

Bounded

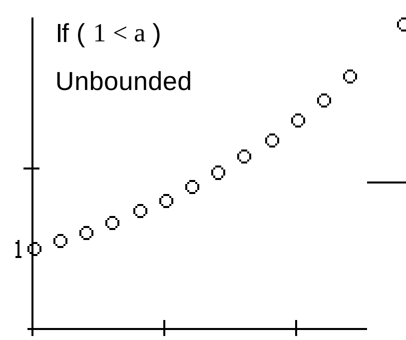


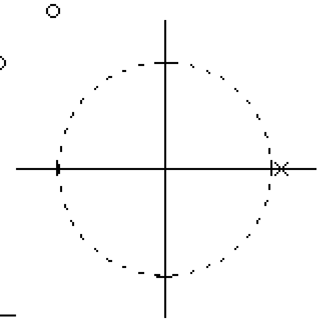


Unit step is a special case: ($a = 1$)

If ($1 < a$)

Unbounded



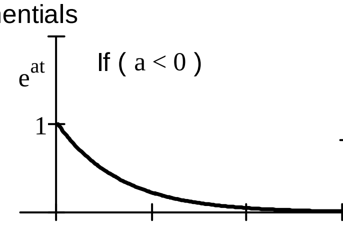


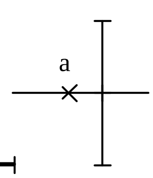
Like Laplace exponentials

$f(t) = e^{a\cdot t}\cdot u(t)$

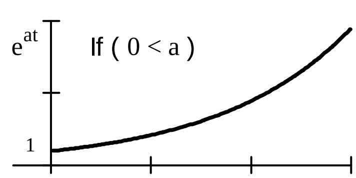
$\mathbf{F}(s) = \frac{1}{s-a}$

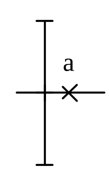
If ($a < 0$)





If ($0 < a$)





$f(k) = \left(C_1 \cdot a_1^{\,k} + C_2 \cdot a_2^{\,k}\right) \cdot u(k)$ $\qquad F(z) = C_1 \cdot \dfrac{z}{z - a_1} + C_2 \cdot \dfrac{z}{z - a_2}$ Linearity

Sinusoidals

If $C_2 = \overline{C_1}$ (complex conjugate) and $a_2 = \overline{a_1}$ and we'll now call $C_1 = C$ and $a_1 = p$

Then $f(k) = \left[C \cdot p^k + \overline{C} \cdot \left(\overline{p}\right)^k\right] \cdot u(t)$ and $F(z) = C \cdot \dfrac{z}{z - p} + \overline{C} \cdot \dfrac{z}{z - \overline{p}}$

$$f(k) = \left[C \cdot p^k + \overline{C} \cdot \left(\overline{p}\right)^k\right] \cdot u(k)$$

$$= \left[|C| \cdot e^{j \cdot \theta_C} \cdot (|p|)^k \cdot e^{j \cdot \theta_p \cdot k} + |C| \cdot e^{-j \cdot \theta_C} \cdot (|p|)^k \cdot e^{-j \cdot \theta_p \cdot k} \right] \cdot u(k)$$

$$= |C| \cdot (|p|)^k \cdot \left(e^{j \cdot \theta_C} \cdot e^{j \cdot \theta_p} + e^{-j \cdot \theta_C} \cdot e^{-j \cdot \theta_p \cdot k}\right) \cdot u(k)$$

$$= |C| \cdot (|p|)^k \cdot \left[e^{j \cdot \left(\theta_C + \theta_p \cdot k\right)} + e^{-j \cdot \left(\theta_C + \theta_p \cdot k\right)}\right] \cdot u(k)$$

$$= 2 \cdot |C| \cdot (|p|)^k \cdot \left[\frac{e^{j \cdot \left(\theta_p \cdot k + \theta_C\right)} + e^{-j \cdot \left(\theta_p \cdot k + \theta_C\right)}}{2}\right] \cdot u(k)$$

$$= 2 \cdot |C| \cdot (|p|)^k \cdot \cos\left(\theta_p \cdot k + \theta_C\right) \cdot u(k)$$

$$= \frac{C \cdot z \cdot \left(z - \overline{p}\right) + \overline{C} \cdot z \cdot (z - p)}{(z - p) \cdot \left(z - \overline{p}\right)}$$

2 complex-conjugate poles at p and $\overline{p}$

Recall Euler's eq.: $\cos(\theta \cdot t) = \dfrac{e^{j \cdot \theta \cdot t} + e^{-j \cdot \theta \cdot t}}{2}$

If C is real ($\theta_C = 0$)

$f(k) = 2 \cdot (|p|)^k \cdot \cos\left(\theta_p \cdot k\right) \cdot u(k)$

$$F(z) = \frac{C \cdot z \cdot \left[\left(z - \overline{p}\right) + (z - p)\right]}{(z - p) \cdot \left(z - \overline{p}\right)} = \frac{C \cdot z \cdot \left(2 \cdot z - \overline{p} - p\right)}{z^2 + z\left(\overline{p} + p\right) + p \cdot \overline{p}}$$

$$\overline{p} + p = |p| \cdot \cos\left(\theta_p\right) - j \cdot |p| \cdot \sin\left(\theta_p\right) + |p| \cdot \cos\left(\theta_p\right) + j \cdot |p| \cdot \sin\left(\theta_p\right)$$

$$= |p| \cdot \cos\left(\theta_p\right) + |p| \cdot \cos\left(\theta_p\right) \qquad = 2 \cdot |p| \cdot \cos\left(\theta_p\right)$$

$$F(z) = \frac{2 \cdot C \cdot z \cdot \left(z - |p| \cdot \cos\left(\theta_p\right)\right)}{z^2 + z \cdot \left(2 \cdot |p| \cdot \cos\left(\theta_p\right)\right) + p \cdot \overline{p}}$$

This leads directly to:

If $f(k) = (|p|)^k \cdot \cos\left(\theta_p \cdot k\right) \cdot u(k)$ $\qquad F(z) = \dfrac{z \cdot \left(z - |p| \cdot \cos\left(\theta_p\right)\right)}{z^2 - 2 \cdot |p| \cdot \cos\left(\theta_p\right) \cdot z + (|p|)^2}$

And If $|p| = 1$ (poles are right in the unit circle)

$f(k) = \cos\left(\theta_p \cdot k\right) \cdot u(k)$ $\qquad F(z) = \dfrac{z \cdot \left(z - \cos\left(\theta_p\right)\right)}{z^2 - 2 \cdot \cos\left(\theta_p\right) \cdot z + 1}$

Then sometimes θ_p is replaced by Ω_o

If C is -j |C|, imaginary (θ_C = -90º) ($\overline{C} = j \cdot |C|$)

$f(k) = 2 \cdot (|p|)^k \cdot \sin\left(\theta_p \cdot k\right) \cdot u(k)$

$$F(z) = \frac{-j \cdot |C| \cdot z \cdot \left(z - \overline{p}\right) + j \cdot |C| \cdot z \cdot (z - p)}{(z - p) \cdot \left(z - \overline{p}\right)}$$

$$= \frac{|C| \cdot z \cdot \left(-j \cdot z + j \cdot z + j \cdot \overline{p} - j \cdot p\right)}{z^2 - 2 \cdot |p| \cdot \cos\left(\theta_p\right) \cdot z + (|p|)^2}$$

$$j \cdot \overline{p} - j \cdot p = \left(j \cdot |p| \cdot \cos\left(\theta_p\right) - j \cdot j \cdot |p| \cdot \sin\left(\theta_p\right)\right) - j \cdot |p| \cdot \cos\left(\theta_p\right) - j \cdot j \cdot |p| \cdot \sin\left(\theta_p\right)$$

$$= j \cdot |p| \cdot \cos\left(\theta_p\right) + |p| \cdot \sin\left(\theta_p\right) - j \cdot |p| \cdot \cos\left(\theta_p\right) + |p| \cdot \sin\left(\theta_p\right)$$

$$F(z) = \frac{z \cdot \left(2 \cdot |p| \cdot \sin\left(\theta_p\right)\right)}{z^2 - 2 \cdot |p| \cdot \cos\left(\theta_p\right) \cdot z + (|p|)^2}$$

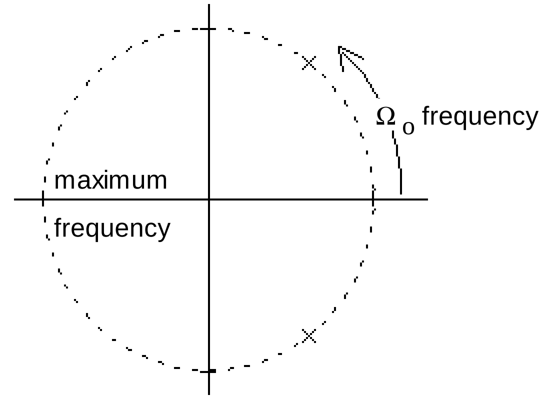
which gets us to the sine terms in the table

<u>Sinusoidals</u>

$$f(k) = \cos(\Omega_o \cdot k)\cdot u(k) \qquad F(z) = \frac{z\left(z - \cos(\Omega_o)\right)}{z^2 - 2\cdot\cos(\Omega_o)\cdot z + 1}$$

AND

$$f(k) = \sin(\Omega_o \cdot k)\cdot u(k) \qquad F(z) = \frac{z\cdot\sin(\Omega_o)}{z^2 - 2\cdot\cos(\Omega_o)\cdot z + 1}$$

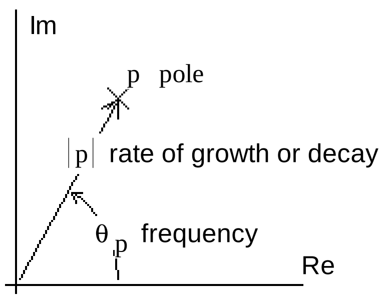


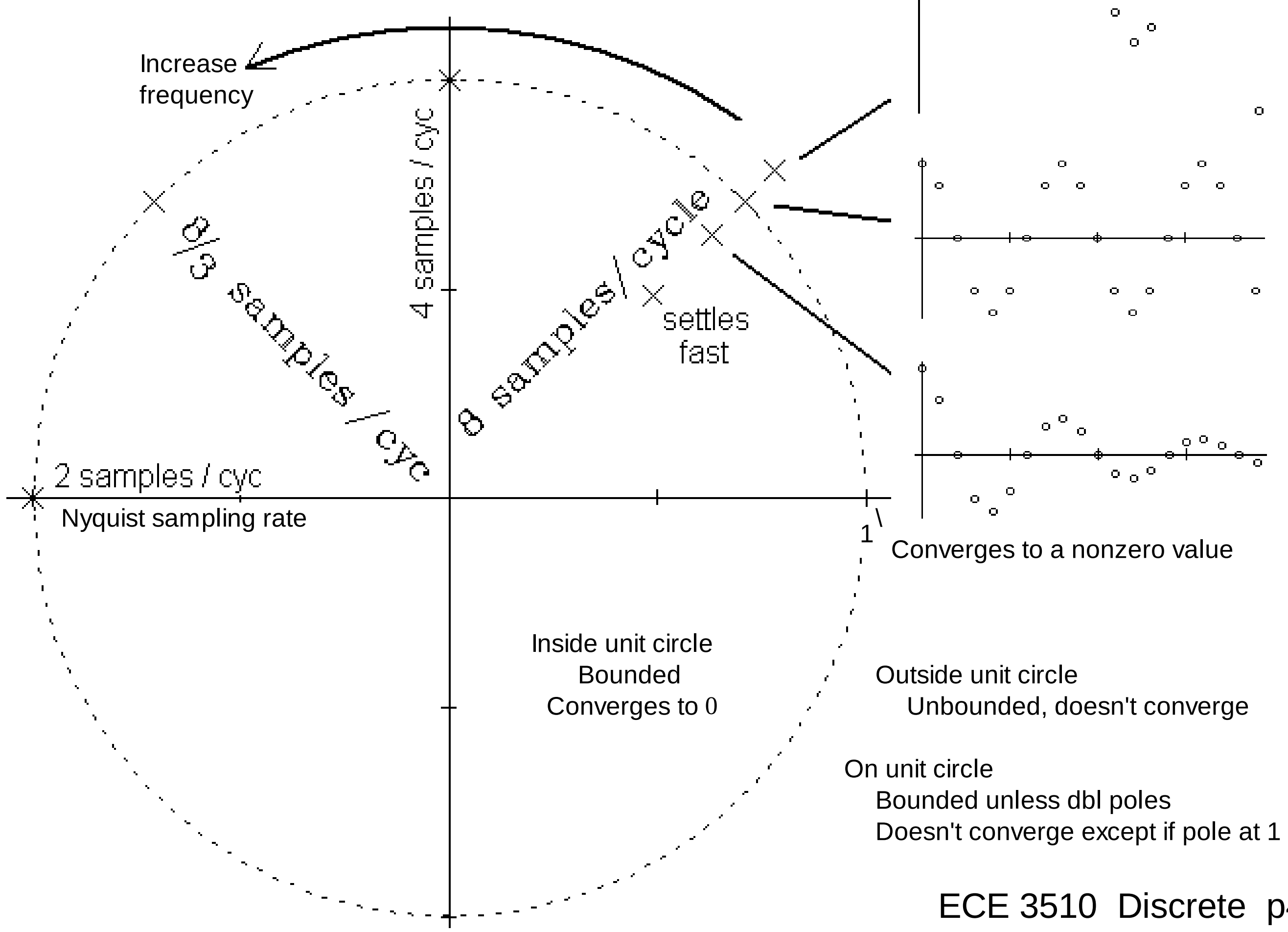
<u>Sinusoidals with growth or decay</u>

$$f(k) = \left|p^k\right|\cdot\cos(\theta_p \cdot k)\cdot u(k) \qquad F(z) = \frac{z\cdot\left(z - |p|\cdot\cos(\theta_p)\right)}{z^2 - 2\cdot|p|\cdot\cos(\theta_p)\cdot z + (|p|)^2}$$

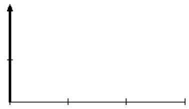
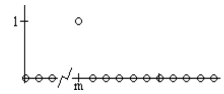
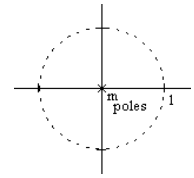
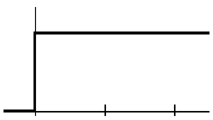
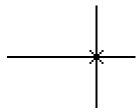
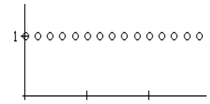
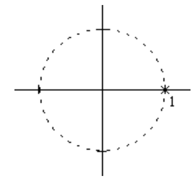
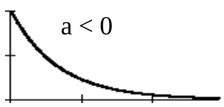
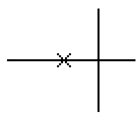
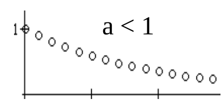
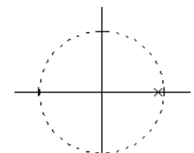
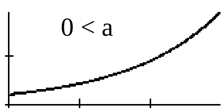
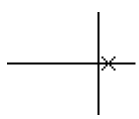
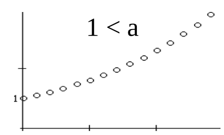
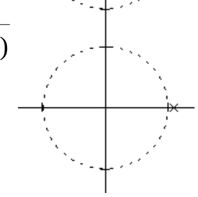
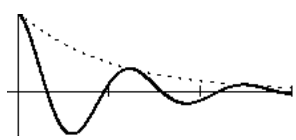
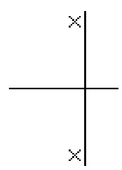
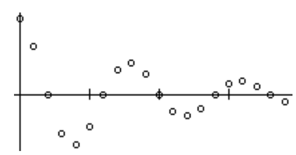
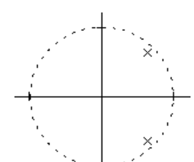
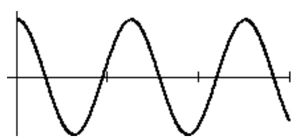
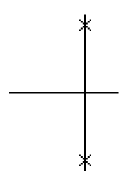
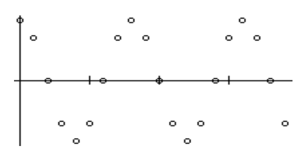
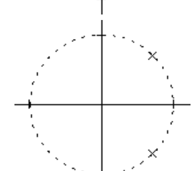
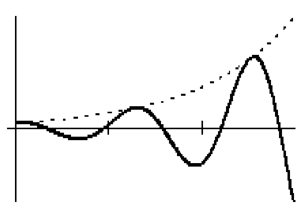
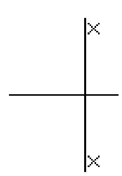
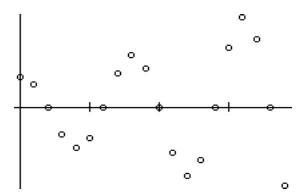
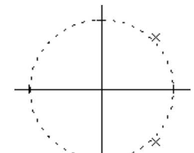
AND

$$f(k) = (|p|)^k\cdot\sin(\theta_p \cdot k)\cdot u(k) \qquad F(z) = \frac{z\cdot\left(|p|\cdot\sin(\theta_p)\right)}{z^2 - 2\cdot|p|\cdot\cos(\theta_p)\cdot z + (|p|)^2}$$





Laplace Transform (Unilateral) | z - transforms

	f(t)		F(s)	pole	f(k)		F(z)	pole
impulse	$\delta(t)$		1	none	$\delta(k)$		1	none
delayed impulse					$\delta(k-m)$		$\frac{1}{z^m}$	
unit step	$u(t)$		$\frac{1}{s}$		$u(k)$		$\frac{z}{z-1}$	
Exponential or Geometric Progression	$e^{a\cdot t}\cdot u(t)$	$a < 0$ 	$\frac{1}{s-a}$		$a^k\cdot u(k)$	$a < 1$ 	$\frac{z}{(z-a)}$	
		$0 < a$ 				$1 < a$ 		
Sinusoidals Only cosine shown here	$e^{a\cdot t}\cdot\cos(b\cdot t)\cdot u(t)$		$\frac{s-a}{(s-a)^2+b^2}$		$\lvert p^k\rvert\cdot\cos(\theta_p\cdot k)\cdot u(k)$		$\frac{z\cdot(z-\lvert p\rvert\cdot\cos(\theta_p))}{z^2-2\cdot\lvert p\rvert\cdot\cos(\theta_p)\cdot z+(\lvert p\rvert)^2}$	
	$a < 0$				$\lvert p\rvert < 1$			
	$a = 0$				$\lvert p\rvert = 1$			
	$0 < a$				$\lvert p\rvert > 1$			

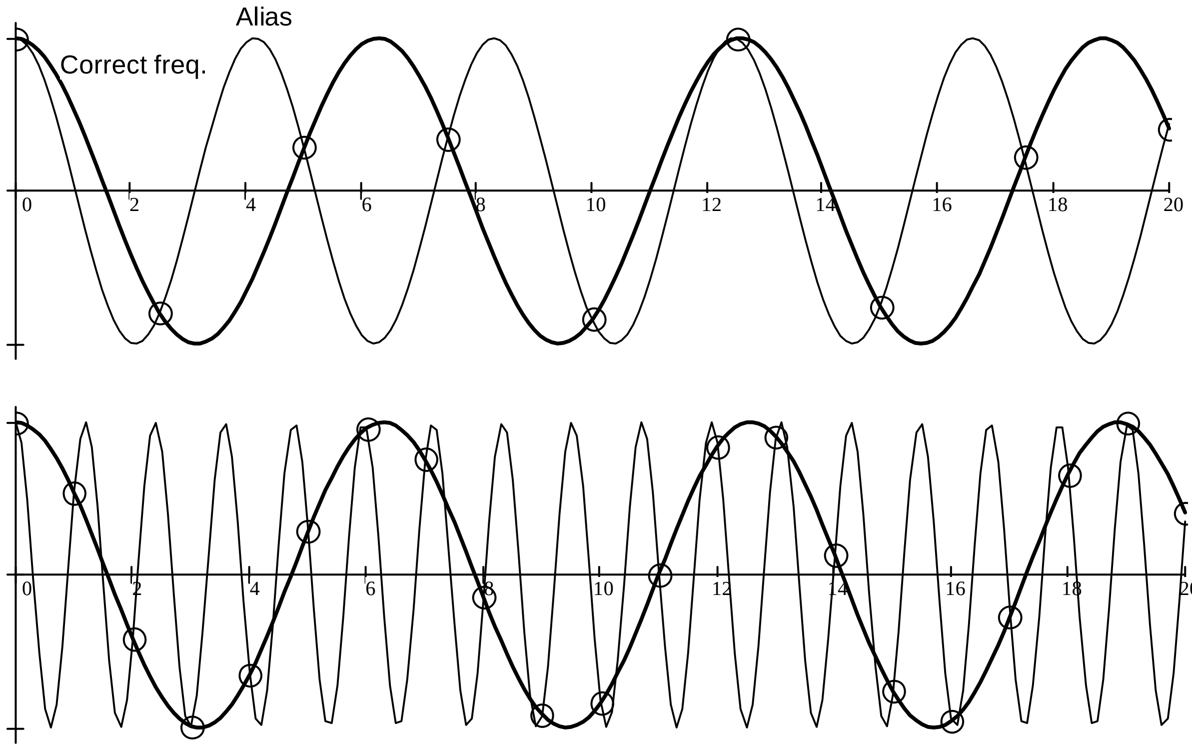
Time constant $\tau = -\frac{1}{\ln(\lvert p\rvert)}$ Settling time $T_s = 4\cdot\tau$

Damping factor $\zeta = \frac{-\ln(\lvert p\rvert)}{\sqrt{\ln(\lvert p\rvert)^2-\theta_p^{\,2}}}$

Aliasing

Close to the maximum frequency (2 samples per cycle)

Far from the maximum frequency.



z - transform Properties

All the following are multiplied by u(k) unless specified otherwise

Operation	f(k)	F(z)
Linearity	$c \cdot f(k) + d \cdot g(k)$	$c \cdot F(z) + d \cdot G(z)$
Right shift (Delay) $m \ge 0$	$f(k-m) \cdot u(k-m)$	$\frac{1}{z^m} \cdot F(z) = z^{-m} \cdot F(z)$
	$f(k-1)$	$z^{-1} \cdot F(z) + f(-1)$
	$f(k-2)$	$z^{-2} \cdot F(z) + z^{-1} \cdot f(-1) + f(-2)$
	$f(k-m)$	$\frac{1}{z^m} \cdot F(z) + \frac{1}{z^m} \cdot \sum_{k=1}^{m} f(-k) \cdot z^k$
Left shift $m \ge 0$	$f(k+m)$	$z^m \cdot F(z) - z^m \cdot \sum_{k=0}^{m-1} f(k) \cdot z^{-k}$
not common	$f(k+1)$	$z \cdot F(z) - z \cdot f(0)$
	$f(k+2)$	$z^2 \cdot F(z) - z^2 \cdot f(0) - z \cdot f(1)$
Initial value	$f(0)$	$\lim_{z \to \infty} F(z)$
Final value	$f(\infty)$	$\lim_{z \to 1} (z-1) \cdot F(z)$ (all poles of $(z-1)F(z)$ inside unit circle)