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Partial Fraction Expansion

Ex.1 $F(z) = \dfrac{1}{(z-1)\cdot(z+1)}$ Example 1 from Bodson, page 197

Divide by z first, because all the table entries have a z in the numerator, you can remultiply by z at the end.

$$\frac{F(z)}{z} = \frac{1}{z\cdot(z-1)\cdot(z+1)} = \frac{A}{z} + \frac{B}{(z-1)} + \frac{C}{z+1}$$

Multiply both sides by: $z\cdot(z-1)\cdot(z+1)$

$$1 = A\cdot(z-1)\cdot(z+1) + B\cdot z\cdot(z+1) + C\cdot z\cdot(z-1)$$

Set $z := 0$

$$1 = \underbrace{A\cdot(0-1)\cdot(0+1)}_{-1} + 0 + 0 \qquad A := \frac{1}{-1} \qquad A = -1$$

Set $z := 1$

$$1 = 0 + \underbrace{B\cdot 1\cdot(1+1)}_{2} + 0 \qquad B := \frac{1}{2} \qquad B = 0.5$$

Set $z := -1$

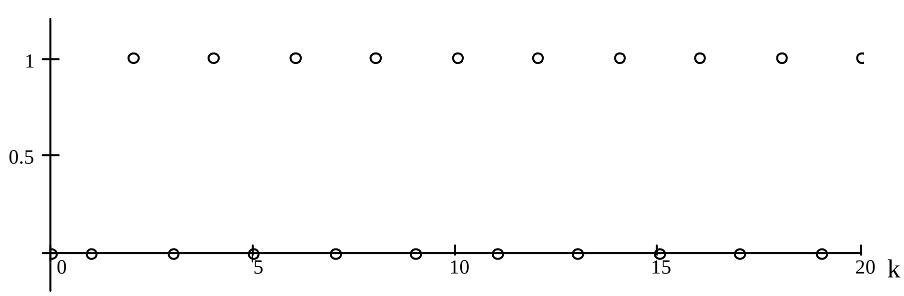
$$1 = 0 + 0 + \underbrace{C\cdot(-1)\cdot(-1-1)}_{2} \qquad C := \frac{1}{2} \qquad C = 0.5$$

$$\frac{F(z)}{z} = \frac{1}{z\cdot(z-1)\cdot(z+1)} = \frac{-1}{z} + \frac{1}{2}\cdot\frac{1}{(z-1)} + \frac{1}{2}\cdot\frac{1}{z+1}$$

Now multiply back through by z to get partial fractions that can actually be found in the table.

$$F(z) = \frac{1}{(z-1)\cdot(z+1)} = \frac{-1\cdot z}{z} + \frac{1}{2}\cdot\frac{z}{(z-1)} + \frac{1}{2}\cdot\frac{z}{z+1}$$

$$f(k) := \left[-1\cdot\delta(k) + \frac{1}{2} + \frac{1}{2}\cdot(-1)^k\right]\cdot u(k)$$



By long division, as shown in section 6.3.2 in Bodson text.

$$(z-1)\cdot(z+1) = \left(z^2-1\right)$$

$$\left(z^2-1\right)\,\overline{)\;1\;}\quad = z^{-2} + z^{-4} + z^{-6} + \ldots$$

$1 - z^{-2}$

z^{-2}

$z^{-2} - z^{-4}$

z^{-4}

$z^{-4} - z^{-6}$

z^{-6} etc

never ends

Ex.2 $F(z) = \dfrac{z}{(z-0.9)^2\cdot(z+0.8)}$

$$\frac{F(z)}{z} = \frac{1}{(z-0.9)^2\cdot(z+0.8)} = \frac{A}{z-0.9} + \frac{0.9\cdot B}{(z-0.9)^2} + \frac{C}{z+0.8}$$

Multiply both sides by: $(z-0.9)^2\cdot(z+0.8)$

$$1 = A\cdot(z-0.9)\cdot(z+0.8) + 0.9\cdot B\cdot(z+0.8) + C\cdot(z-0.9)^2$$

Set $z := 0.9$

$$1 = 0 + 0.9\cdot B\cdot \underbrace{(0.9+0.8)}_{1.7} + 0 \qquad B := \frac{1}{0.9\cdot 1.7} \qquad B = 0.654$$

Set $z := -0.8$

$$1 = 0 + 0 + C\cdot \underbrace{(-0.8-0.9)^2}_{(-1.7)^2} \qquad C := \frac{1}{(-1.7)^2} \qquad C = 0.346$$

Back to equation above

$$1 = A\cdot(z+0.9)\cdot(z+0.8) + 0.9\cdot B\cdot(z+0.8) + C\cdot(z+0.9)^2$$

$$1 = A\cdot z^2 + 1.7\cdot A\cdot z + .72\cdot A + 0.9\cdot B\cdot z + 0.72\cdot B + C\cdot z^2 + 1.8\cdot C\cdot z + .81\cdot C$$

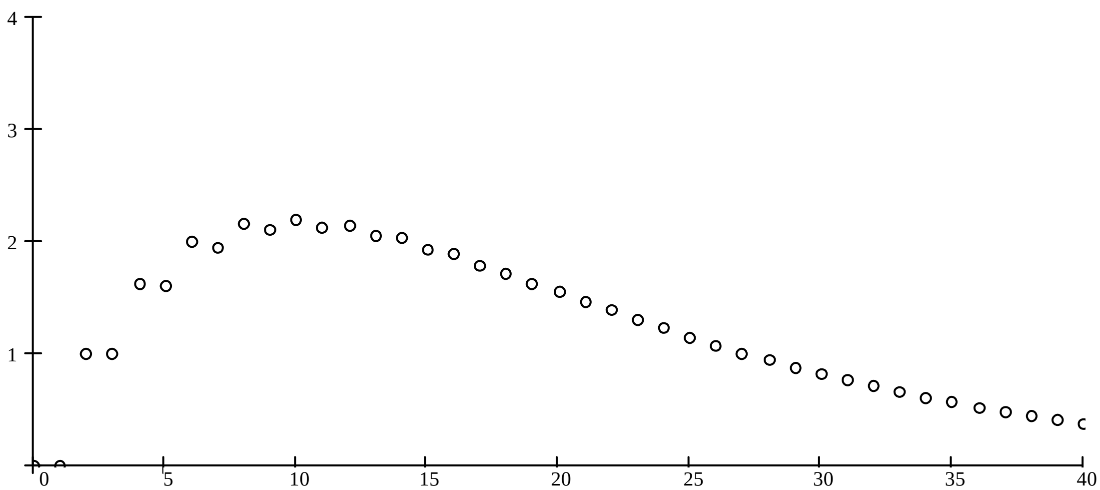
$$0\cdot z^2 = A\cdot z^2 + 0 + C\cdot z^2 \qquad A := -C \qquad A = -0.346$$

no z^2 term on the left

$$F(z) = \frac{z}{(z-0.9)^2\cdot(z+0.8)} = \frac{-0.346\cdot z}{z-0.9} + \frac{0.654\cdot 0.9\cdot z}{(z-0.9)^2} + \frac{0.346\cdot z}{z+0.8}$$

$$f(k) = -0.346\cdot(0.9)^k + 0.654\cdot k\cdot(0.9)^k + 0.346\cdot(-0.8)^k$$

$$f(k) := \left[-0.346\cdot(0.9)^k + 0.654\cdot k\cdot(0.9)^k + 0.346\cdot(-0.8)^k\right]\cdot u(k) \qquad k := 0, 1 .. 40$$



Ex.3 $F(z) = \dfrac{z}{z^2 - 2\cdot z + 2}$

The complex coefficient way (not recommended)

$$\frac{F(z)}{z} = \frac{1}{(z-(1+j))\cdot(z-(1-j))} = \frac{A}{(z-(1+j))} + \frac{B}{(z-(1-j))}$$

$$\left.\frac{1}{(z-(1-j))}\right|_{z\,=\,(1+j)} = A = \frac{1}{((1+j)-(1-j))} = -0.5j$$

$$\left.\frac{1}{(z-(1+j))}\right|_{z\,=\,1-j} = B = \frac{1}{((1-j)-(1+j))} = 0.5j$$

$p := (1+j) \qquad |p| = \sqrt{2} \qquad \underline{/p} = \theta_p = \dfrac{\pi}{4}$

$c = -\dfrac{1}{2}\cdot j \qquad |c| = \dfrac{1}{2} \qquad \underline{/c} = \theta_c = -\dfrac{\pi}{2}$

Use this Table entry $\dfrac{C\cdot z}{(z-p)} + \dfrac{\overline{C}\cdot z}{\left(z-\overline{p}\right)}$ <--> $2\cdot|C|\cdot(|p|)^k\cdot\cos\left(\theta_p\cdot k + \theta_C\right)$ Note: table shows B, where I've changed to C for clarity here

$$f(k) = 2\cdot|c|\cdot(|p|)^k\cdot\cos\left(\theta_p\cdot k + \theta_c\right) = 2\cdot\frac{1}{2}\cdot\left(\sqrt{2}\right)^k\cdot\cos\left(\frac{\pi}{4}\cdot k - \frac{\pi}{2}\right)\cdot u(k) = \left(\sqrt{2}\right)^k\cdot\sin\left(\frac{\pi}{4}\cdot k\right)\cdot u(k)$$

The easy way

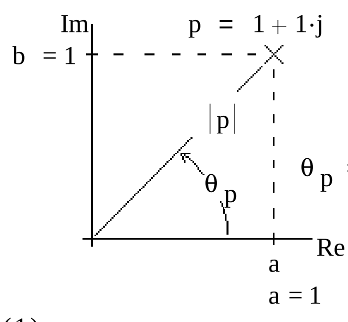
$$(|p|)^k\cdot\cos\left(\theta_p\cdot k\right) \quad \text{<-->} \quad \frac{z\cdot(z-a)}{z^2 - 2\cdot a\cdot z + \left(a^2+b^2\right)} = \frac{z\cdot\left(z - |p|\cdot\cos\left(\theta_p\right)\right)}{z^2 - 2\cdot|p|\cdot\cos\left(\theta_p\right)\cdot z + (|p|)^2}$$

$$(|p|)^k\cdot\sin\left(\theta_p\cdot k\right) \quad \text{<-->} \quad \frac{z\cdot b}{z^2 - 2\cdot a\cdot z + \left(a^2+b^2\right)} = \frac{z\cdot\left(|p|\cdot\sin\left(\theta_p\right)\right)}{z^2 - 2\cdot|p|\cdot\cos\left(\theta_p\right)\cdot z + (|p|)^2}$$

Fit to our denominator: $z^2 - 2\cdot z + 2$ $\qquad |p| = \sqrt{2}$

$a := 1 \qquad b := \sqrt{2 - a^2} \qquad b = 1$

|p| and θ_p are found here, but only needed in the very last step, to find f(k).



$$\theta_p = \operatorname{acos}\left(\frac{1}{\sqrt{2}}\right) = 0.785 = \frac{\pi}{4}$$

Continue partial fraction expansion

$$\frac{F(z)}{z} = \frac{1}{z^2 - 2\cdot z + 2} = \frac{A(z-1)}{z^2 - 2\cdot z + 2} + \frac{B(1)}{z^2 - 2\cdot z + 2}$$

Let: $z = 1 \qquad B := 1$

$1 = A(z-1) + B$

$0\cdot z = A\cdot z \qquad A := 0$

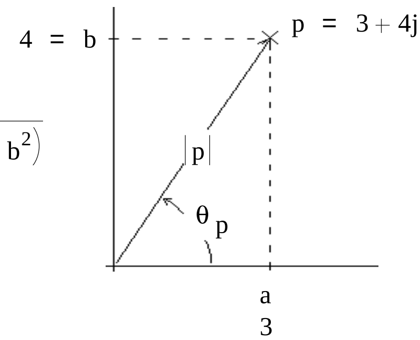
|p| and θ_p are needed here, to find f(k).

$$f(k) = A\cdot\left[(|p|)^k\cdot\cos\left(\theta_p\cdot k\right)\right]\cdot u(k) + B\cdot\left[(|p|)^k\cdot\sin\left(\theta_p\cdot k\right)\right]\cdot u(k)$$

$$f(k) = \left(\sqrt{2}\right)^k\cdot\sin\left(\frac{\pi}{4}\cdot k\right)\cdot u(k)$$

Ex.4

$$F(z) = \frac{2\cdot z\cdot(3\cdot z+17)}{(z-1)\cdot\left(z^2-6\cdot z+25\right)}$$



$$\frac{F(z)}{z} = \frac{2\cdot(3\cdot z+17)}{(z-1)\cdot\left(z^2-6\cdot z+25\right)} = \frac{A}{z-1} + \frac{B\cdot(z-a)}{z^2-2\cdot a\cdot z+\left(a^2+b^2\right)} + \frac{C\cdot b}{z^2-2\cdot a\cdot z+\left(a^2+b^2\right)}$$

$$= \frac{A}{z-1} + \frac{B(z-a)}{z^2-6\cdot z+25} + \frac{C\cdot b}{z^2-6\cdot z+25}$$

$$z^2-6\cdot z+25$$

$$z^2-2\cdot a\cdot z+\left(a^2+b^2\right)$$

$$a := 3 \qquad b := \sqrt{25-a^2} \qquad b = 4$$

|p| and θ_p are found here, but only needed in the very last step, to find f(k).

$$|p| = \sqrt{25} = 5$$

$$\theta_p = \operatorname{asin}\left(\frac{4}{5}\right) = 0.927 = \operatorname{acos}\left(\frac{3}{5}\right) = 0.927 = \operatorname{atan}\left(\frac{4}{3}\right) = 0.927$$

(in radians)

several ways to find θ_p (in radians)

$$(z-a) \qquad a = 3 \qquad b = 4$$

$$\frac{F(z)}{z} = \frac{2\cdot(3\cdot z+17)}{(z-1)\cdot\left(z^2-6\cdot z+25\right)} = \frac{A}{z-1} + \frac{B(z-3)}{z^2-6\cdot z+25} + \frac{C\cdot 4}{z^2-6\cdot z+25}$$

$$\left.\frac{2\cdot(3\cdot z+17)}{\left(z^2-6\cdot z+25\right)}\right|_{z\,=\,1} = A = \frac{2\cdot(3\cdot 1+17)}{\left(1^2-6\cdot 1+25\right)} = 2$$

$$2\cdot(3\cdot z+17) = A\cdot\left(z^2-6\cdot z+25\right) + B\cdot(z-3)\cdot(z-1) + C\cdot 4\cdot(z-1)$$

$$6\cdot z+34 = 2\cdot\left(z^2-6\cdot z+25\right) + B\cdot\left(z^2-4\cdot z+3\right) + C\cdot 4\cdot(z-1)$$

$$6\cdot z+34 = 2\cdot z^2-12\cdot z+50 + B\cdot z^2-4\cdot B\cdot z+3\cdot B + C\cdot 4\cdot z-C\cdot 4$$

$$B := -2$$

$$6\cdot z = -12\cdot z + 4\cdot 2\cdot z + C\cdot 4\cdot z$$

$$C = \frac{6+12-8}{4} = 2.5$$

OR

$$\frac{34-50+6}{-4} = 2.5$$

$$F(z) = \frac{2\cdot z\cdot(3\cdot z+17)}{(z-1)\cdot\left(z^2-6\cdot z+25\right)} = \frac{2\cdot z}{z-1} + \frac{-2\cdot z(z-3)}{z^2-2\cdot z+2} + \frac{2.5\cdot 4}{z^2-2\cdot z+2}$$

|p| and θ_p are needed here, to find f(k).

$$u(k) \qquad (|p|)^k\cdot\cos\left(\theta_p\cdot k\right) \qquad (|p|)^k\cdot\sin\left(\theta_p\cdot k\right)$$

$$f(k) = 2\cdot u(k) - 2\cdot 5^k\cdot\cos(0.927\cdot k) + 2.5\cdot 5^k\cdot\sin(0.927\cdot k)$$

$$f(k) = \left(2\cdot u(k) - 2\cdot 5^k\cdot\cos(0.927\cdot k) + 2.5\cdot 5^k\cdot\sin(0.927\cdot k)\right)u(k)$$