ECE 3510 Lecture notes Inverse z-transforms

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Partial Fraction Expansion

Ex.1 $F(z) = \dfrac{1}{(z-1)\cdot(z+1)}$ Example 1 from Bodson, page 168

$$\frac{F(z)}{z} = \frac{1}{z\cdot(z-1)\cdot(z+1)} = \frac{A}{z} + \frac{B}{(z-1)} + \frac{C}{z+1}$$

Multiply both sides by: $z\cdot(z-1)\cdot(z+1)$

$$1 = A\cdot(z-1)\cdot(z+1) + B\cdot z\cdot(z+1) + C\cdot z\cdot(z-1)$$

Set $z := 0$

$$1 = \underset{-1}{A\cdot(0-1)\cdot(0+1)} + 0 + 0 \qquad A := \frac{1}{-1} \qquad A = -1$$

Set $z := 1$

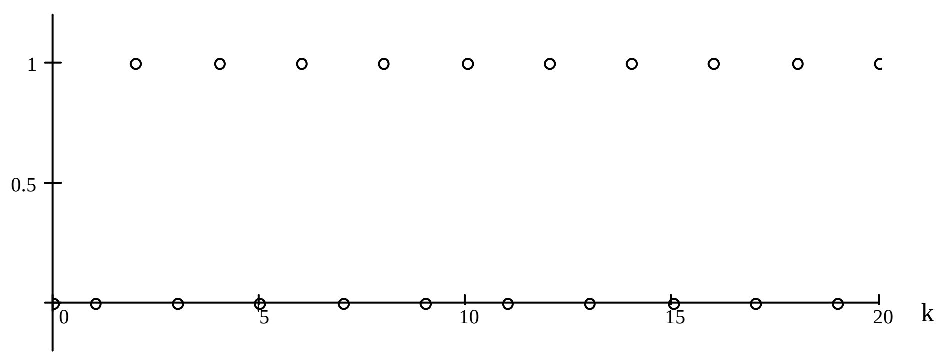
$$1 = 0 + \underset{2}{B\cdot 1\cdot(1+1)} + 0 \qquad B := \frac{1}{2} \qquad B = 0.5$$

Set $z := -1$

$$1 = 0 + 0 + \underset{2}{C\cdot(-1)\cdot(-1-1)} \qquad C := \frac{1}{2} \qquad C = 0.5$$

$$F(z) = \frac{1}{(z-1)\cdot(z+1)} = \frac{-1\cdot z}{z} + \frac{1}{2}\cdot\frac{z}{(z-1)} + \frac{1}{2}\cdot\frac{z}{z+1}$$

$$f(k) := -1\cdot\delta(k) + \frac{1}{2} + \frac{1}{2}\cdot(-1)^k \qquad k := 0, 1 .. 20$$



Ex.2 $F(z) = \dfrac{z}{(z-0.9)^2\cdot(z+0.8)}$

$$\frac{F(z)}{z} = \frac{1}{(z-0.9)^2\cdot(z+0.8)} = \frac{A}{z-0.9} + \frac{0.9\cdot B}{(z-0.9)^2} + \frac{C}{z+0.8}$$

Multiply both sides by: $(z-0.9)^2\cdot(z+0.8)$

$$1 = A\cdot(z-0.9)\cdot(z+0.8) + 0.9\cdot B\cdot(z+0.8) + C\cdot(z-0.9)^2$$

Set $z := 0.9$

$$1 = 0 + 0.9\cdot B\cdot \underbrace{(0.9+0.8)}_{1.7} + 0$$

$B := \dfrac{1}{0.9\cdot 1.7}$

$B = 0.654$

Set $z := -0.8$

$$1 = 0 + 0 + C\cdot \underbrace{(-0.8-0.9)^2}_{(-1.7)^2}$$

$C := \dfrac{1}{(-1.7)^2}$

$C = 0.346$

Back to equation above

$$1 = A\cdot(z+0.9)\cdot(z+0.8) + 0.9\cdot B\cdot(z+0.8) + C\cdot(z+0.9)^2$$

$$1 = A\cdot z^2 + 1.7\cdot A\cdot z + .72\cdot A + 0.9\cdot B\cdot z + 0.72\cdot B + C\cdot z^2 + 1.8\cdot C\cdot z + .81\cdot C$$

$$0\cdot z^2 = A\cdot z^2 + 0 + C\cdot z^2$$

no z^2 term on the left

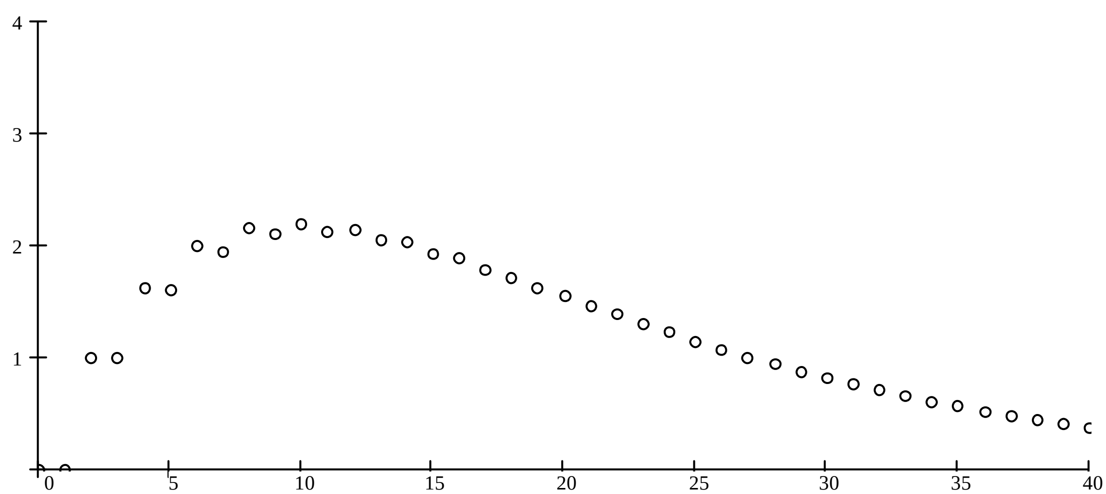
$A := -C$

$A = -0.346$

$$F(z) = \frac{z}{(z-0.9)^2\cdot(z+0.8)} = \frac{-0.346\cdot z}{z-0.9} + \frac{0.654\cdot 0.9\cdot z}{(z-0.9)^2} + \frac{0.346\cdot z}{z+0.8}$$

$$f(k) = -0.346\cdot(0.9)^k + 0.654\cdot k\cdot(0.9)^k + 0.346\cdot(-0.8)^k$$

$f(k) := -0.346\cdot(0.9)^k + 0.654\cdot k\cdot(0.9)^k + 0.346\cdot(-0.8)^k$ $\quad k := 0, 1 .. 40$



Ex.3 $F(z) = \dfrac{z}{z^2 - 2\cdot z + 2} = \dfrac{z}{(z-(1+j))\cdot(z-(1-j))}$

$$\frac{F(z)}{z} = \frac{1}{(z-(1+j))\cdot(z-(1-j))} = \frac{A}{(z-(1+j))} + \frac{B}{(z-(1-j))}$$

$$\left.\frac{1}{(z-(1-j))}\right|_{z = (1+j)} = A = \frac{1}{((1+j)-(1-j))} = -0.5j$$

$$\left.\frac{1}{(z-(1+j))}\right|_{z = 1-j} = B = \frac{1}{((1-j)-(1+j))} = 0.5j$$

$p := (1+j)$ $\quad |p| = \sqrt{2}$ $\quad \underline{/p} = \theta_p = \dfrac{\pi}{4}$

$c = -\dfrac{1}{2}\cdot j$ $\quad |c| = \dfrac{1}{2}$ $\quad \underline{/c} = \theta_c = -\dfrac{\pi}{2}$

$$f(k) = 2\cdot|c|\cdot(|p|)^k\cdot\cos(\theta_p\cdot k + \theta_c) = 2\cdot\frac{1}{2}\cdot\left(\sqrt{2}\right)^k\cdot\cos\left(\frac{\pi}{4}\cdot k - \frac{\pi}{2}\right) = \left(\sqrt{2}\right)^k\cdot\sin\left(\frac{\pi}{4}\cdot k\right)$$

A little different way

Recall: $\quad (|p|)^k\cdot\cos(\theta_p\cdot k) \quad$ <--> $\quad \dfrac{z\cdot(z - |p|\cdot\cos(\theta_p))}{z^2 - 2\cdot|p|\cdot\cos(\theta_p)\cdot z + (|p|)^2}$

$(|p|)^k\cdot\sin(\theta_p\cdot k) \quad$ <--> $\quad \dfrac{z\cdot(|p|\cdot\sin(\theta_p))}{z^2 - 2\cdot|p|\cdot\cos(\theta_p)\cdot z + (|p|)^2}$

$$z^2 - 2\cdot z + 2 = z^2 - 2\cdot|p|\cdot\cos(\theta_p)\cdot z + (|p|)^2$$

$$|p| = \sqrt{2}$$

$$|p|\cdot\cos(\theta_p) = \mathrm{Re}(p) = 1$$

$$|p|\cdot\sin(\theta_p) = \mathrm{Im}(p) = \sqrt{(|p|)^2 - \mathrm{Re}(p)^2} = 1$$

$$\frac{F(z)}{z} = \frac{1}{z^2 - 2\cdot z + 2} = \frac{A(z-1)}{z^2 - 2\cdot z + 2} + \frac{B(1)}{z^2 - 2\cdot z + 2}$$

Let: $z = 1$ $\quad B := 1$

$1 = A\cdot z - A + B$

$A := 0$

$$f(k) = \left(\sqrt{2}\right)^k\cdot\sin\left(\frac{\pi}{4}\cdot k\right)$$

Ex.4

$$F(z) = \frac{2\cdot z\cdot(3\cdot z+17)}{(z-1)\cdot\left(z^2-6\cdot z+25\right)}$$

$$\frac{F(z)}{z} = \frac{2\cdot(3\cdot z+17)}{(z-1)\cdot\left(z^2-6\cdot z+25\right)}$$

$$|p| = \sqrt{25} = 5$$

$$6 = 2\cdot|p|\cdot\cos\left(\theta_p\right)$$

$$|p|\cdot\cos\left(\theta_p\right) = 3 = \mathrm{Re}(p)$$

$$|p|\cdot\sin\left(\theta_p\right) = \mathrm{Im}(p) = \sqrt{(|p|)^2-\mathrm{Re}(p)^2} = \sqrt{25-3^2} = 4$$

$$z - |p|\cdot\cos\left(\theta_p\right) = (z-3) \qquad |p|\cdot\sin\left(\theta_p\right) = 4$$

$$\theta_p = \mathrm{asin}\left(\frac{4}{5}\right) = 0.927$$

$$\frac{F(z)}{z} = \frac{2\cdot(3\cdot z+17)}{(z-1)\cdot\left(z^2-6\cdot z+25\right)} = \frac{A}{z-1} + \frac{C(z-3)}{z^2-2\cdot z+2} + \frac{D\cdot 4}{z^2-2\cdot z+2}$$

$$\left.\frac{2\cdot(3\cdot z+17)}{\left(z^2-6\cdot z+25\right)}\right|_{z\,=\,1} = A = \frac{2\cdot(3\cdot 1+17)}{\left(1^2-6\cdot 1+25\right)} = 2$$

$$2\cdot(3\cdot z+17) = A\cdot\left(z^2-6\cdot z+25\right) + C\cdot(z-3)\cdot(z-1) + D\cdot 4\cdot(z-1)$$

$$6\cdot z+34 = 2\cdot\left(z^2-6\cdot z+25\right) + C\cdot\left(z^2-4\cdot z+3\right) + D\cdot 4\cdot(z-1)$$

$$6\cdot z+34 = 2\cdot z^2-12\cdot z+50 + C\cdot z^2-4\cdot C\cdot z+3\cdot C + D\cdot 4\cdot z-D\cdot 4$$

$$C := -2$$

$$6\cdot z = -12\cdot z + 4\cdot 2\cdot z + D\cdot 4\cdot z$$

$$D = \frac{6+12-8}{4} = 2.5$$

OR $$\frac{34-50+6}{-4} = 2.5$$

$$F(z) = \frac{2\cdot z\cdot(3\cdot z+17)}{(z-1)\cdot\left(z^2-6\cdot z+25\right)} = \frac{2\cdot z}{z-1} + \frac{-2\cdot z(z-3)}{z^2-2\cdot z+2} + \frac{2.5\cdot 4}{z^2-2\cdot z+2}$$

$$f(k) = 2\cdot u(k) - 2\cdot 5^k\cdot\cos(0.927\cdot k) + 2.5\cdot 5^k\cdot\sin(0.927\cdot k)$$