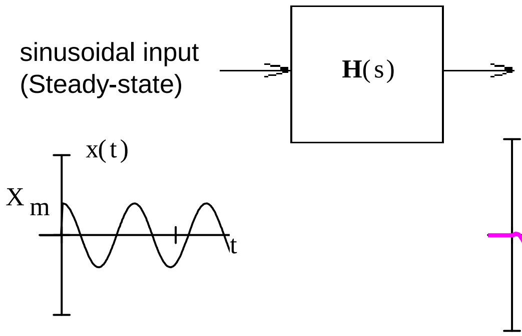
ECE 3510 # Sinusoidal Response Notes

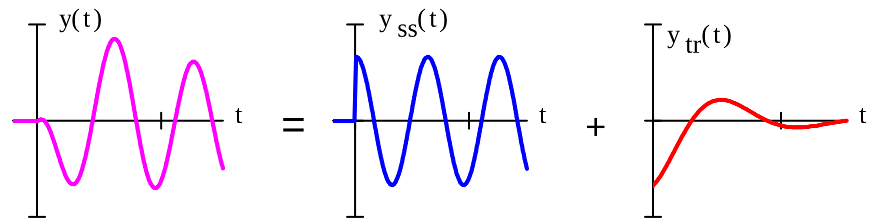
For BIBO Systems

A.Stolp
2/8/21,
1/28/24

The sinusoidal response of a system is the output when the input is a sinusoidal (which starts at time = 0).

System Sinusoidal Response





Complete step response = steady-state response + transient response

$\mathbf{Y}(s) = \mathbf{X}(s)\cdot\mathbf{H}(s) \quad = \quad X_m\cdot\mathbf{H}(j\omega)\cdot u(t) \quad + \quad \mathbf{Y}_{\mathbf{tr}}(s)$

$\mathbf{H}(j\omega)$ = phasor-type transfer function

Sinusoidal Input

$\cos(\omega t)\cdot u(t) \quad \Longleftrightarrow \quad \dfrac{s}{s^2+\omega^2}$ $\qquad \sin(\omega t)\cdot u(t) \quad \Longleftrightarrow \quad \dfrac{\omega}{s^2+\omega^2}$

General sinusoidal input: $\left[X_{mc}\cdot\cos(\omega t) + X_{ms}\cdot(\sin(\omega t)\cdot u(t))\right]\cdot u(t)$ $\qquad \mathbf{X}(s) = \dfrac{X_{mc}\cdot s + X_{ms}\cdot\omega}{s^2+\omega^2}$

OR $\quad X_m\cdot\cos(\omega t + \theta)\cdot u(t) = \left[X_{mc}\cdot\cos(\omega t) + X_{ms}\cdot(\sin(\omega t)\cdot u(t))\right]\cdot u(t)$

$X_{mc} = X_m\cdot\cos(\theta) \qquad X_{ms} = -X_m\cdot\sin(\theta)$ $\quad$ note that the sine carries the opposite sign as you might expect.

Steady-State Response & $\mathbf{H}(j\omega)$

$\mathbf{Y}(s) = \mathbf{X}(s)\cdot\mathbf{H}(s) = \dfrac{X_{mc}\cdot s + X_{ms}\cdot\omega}{s^2+\omega^2}\cdot\mathbf{H}(s)$ = Complete sinusoidal response

$= \dfrac{A\cdot s + B\cdot\omega}{s^2+\omega^2} + \dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)} + \dfrac{E}{(\blacksquare)} + \ldots$

partial fraction expansion: $\quad \mathbf{Y}(s) = \dfrac{X_{mc}\cdot s + X_{ms}\cdot\omega}{s^2+\omega^2}\cdot\mathbf{H}(s) = \dfrac{A\cdot s + B\cdot\omega}{s^2+\omega^2} + \left[\dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)} + \dfrac{E}{(\blacksquare)}\right]\cdot s$

steady-state response + transient response

$\mathbf{Y}_{\mathbf{ss}}(s) \quad + \quad \mathbf{Y}_{\mathbf{tr}}(s)$

multiply both sides by: $\left(s^2+\omega^2\right)$ $\qquad \left(X_{mc}\cdot s + X_{ms}\cdot\omega\right)\cdot\mathbf{H}(s) = A\cdot s + B\cdot\omega + \left[\dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)} + \dfrac{E}{(\blacksquare)}\right]\cdot\left(s^2+\omega^2\right)$

set $s := j\omega$ $\qquad \left(X_{mc}\cdot j\omega + X_{ms}\cdot\omega\right)\cdot\mathbf{H}(j\omega) = A\cdot j\omega + B\cdot\omega + \left[\dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)} + \dfrac{E}{(\blacksquare)}\right]\cdot 0$

divide both sides by $j\omega$ $\qquad \left(X_{mc} - X_{ms}\cdot j\right)\cdot\mathbf{H}(j\omega)$

$\mathbf{X}(\omega)\cdot\mathbf{H}(j\omega) = A - B\cdot j = \mathbf{Y}_{\mathbf{ss}}(\omega)$ = steady-state response in phasor form

(real is cosine, imaginary is **-**sine)

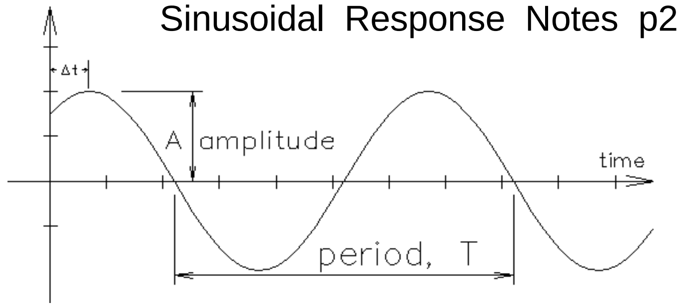
$\mathbf{X}(\omega)$ = the input expressed in phasor form NOT $\mathbf{X}(s)$ with $s := \omega$ or $s := j\omega$ (that would be ∞)

$\mathbf{H}(j\omega)$ = the steady-state sinusoidal transfer function

= phasor-type transfer function

The **transient part** would be found by finishing the partial-fraction expansion.

Steady-State Response by Phasors



Time-domain sinusoidals

T = Period

f = frequency, cycles / second $\quad f = \frac{1}{T} = \frac{\omega}{2\cdot\pi}$

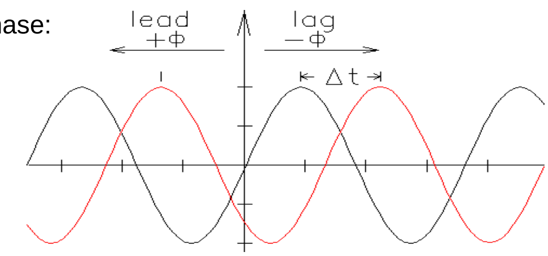
ω = radian frequency, radians/sec $\quad \omega = 2\cdot\pi\cdot f$

A = amplitude

Phase: $\phi = -\frac{\Delta t}{T}\cdot 360\cdot deg$ or: $\phi = -\frac{\Delta t}{T}\cdot 2\cdot\pi\cdot rad$

$y(t) = A\cdot\cos(\omega\cdot t + \phi)$

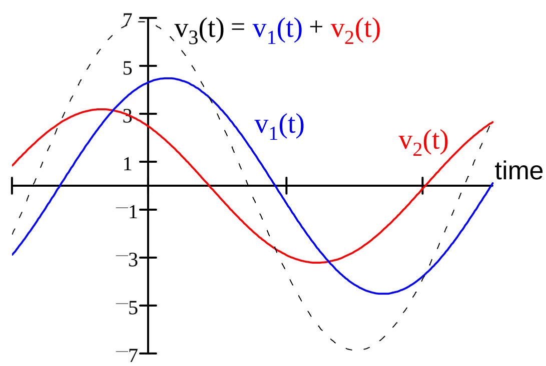
Phase:



Expression of signals as phasors

		Phasor
voltage:	$v(t) = V_p\cdot\cos(\omega\cdot t + \phi)$	$\mathbf{V}(\omega) = V_p\cdot e^{j\cdot\phi}$
current:	$i(t) = I_p\cdot\cos(\omega\cdot t + \phi)$	$\mathbf{I}(\omega) = I_p\cdot e^{j\cdot\phi}$

Ex1 What if a signal is the sum of two sinusoids of the same frequency.



—— $v_1(t) = 3.2\cdot V\cdot\cos(\omega\cdot t + 40\cdot deg)$

—— $v_2(t) = 4.5\cdot V\cdot\sin(\omega\cdot t + 75\cdot deg)$

- - - - $v_3(t) = v_1(t) + v_2(t)$

I'm going to drop the (w) notation from the phasor notation, it gets cumbersome, but remember that phasors are in the frequency domain.

Express the time-domain voltages as phasors:

$\mathbf{V_1} := 3.2\cdot V\cdot e^{j\cdot 40\cdot deg}$ = 3.2V /40º

Convert to rectangular: $3.2\cdot V\cdot\cos(40\cdot deg) = 2.451\cdot V$ $\quad 3.2\cdot V\cdot\sin(40\cdot deg) = 2.057\cdot V$ $\quad \mathbf{V_1} = 2.451 + 2.057j\ \cdot V$

Phasors are based on cosines, so express $v_2(t)$ as a cosine. Remember: $\sin(\omega t) = \cos(\omega\cdot t - 90\cdot deg)$

So: $v_2(t) = 4.5\cdot V\cdot\cos(\omega\cdot t + 75\cdot deg - 90\cdot deg) = 4.5\cdot V\cdot\cos(\omega\cdot t - 15\cdot deg)$

$\mathbf{V_2}$ = 4.5V /-15º or: $\mathbf{V_2} := 4.5\cdot V\cdot e^{-j\cdot 15\cdot deg}$

$4.5\cdot V\cdot\cos(-15\cdot deg) = 4.347\cdot V$ $\quad 4.5\cdot V\cdot\sin(-15\cdot deg) = -1.165\cdot V$

$\mathbf{V_2} = 4.347 - 1.165j\ \cdot V$ \
$\quad$ } add
$\mathbf{V_1} = 2.451 + 2.057j\ \cdot V$ /

Add real parts: $4.347 + 2.451 = 6.798$ $\quad \mathbf{V_3} := \mathbf{V_1} + \mathbf{V_2}$

Add imaginary parts: $-1.165 + 2.057 = 0.892$ $\quad \mathbf{V_3} = 6.798 + 0.892j\ \cdot V$ sum

Change $\mathbf{V_3}$ back to polar coordinates:

$\sqrt{6.798^2 + 0.892^2} = 6.856$ $\quad \operatorname{atan}\left(\frac{0.892}{6.798}\right) = 7.475\cdot deg$

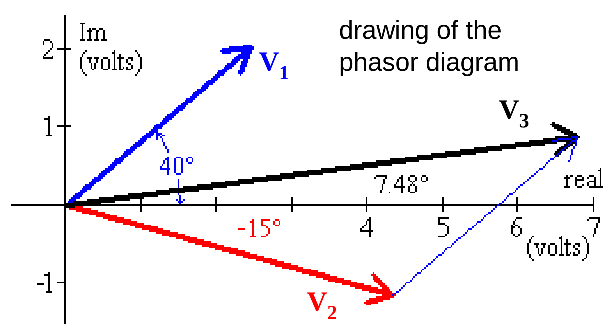
OR, in Mathcad notation (which you'll see a lot more):

$|\mathbf{V_3}| = 6.856\cdot V$ $\quad \arg(\mathbf{V_3}) = 7.48\cdot deg$

$\mathbf{V_3}(\omega) = 6.856V\ \underline{/\ 7.48º}$ or: $\mathbf{V_3}(\omega) = 6.856\cdot V\cdot e^{-j\cdot 7.48\cdot deg}$

$\mathbf{V_3}$ may also be converted back to the time domain:

$v_3(t) = v_1(t) + v_2(t) = 6.856\cdot\cos(\omega\cdot t + 7.48\cdot deg)\cdot V$



drawing of the phasor diagram

Impedances

Resistor

$v_R = i_R \cdot R$

$\mathbf{V_R}(s) = R \cdot \mathbf{I}(s)$

$\mathbf{Z_R} = R$

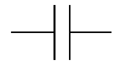
Inductor

$v_L(t) = L \cdot \frac{d}{dt} i_L(t)$

$\mathbf{V_L}(s) = s \cdot L \cdot \mathbf{I_L}(s)$

$\mathbf{Z_L} = L \cdot s$

Capacitor



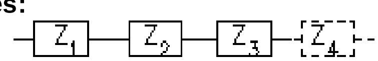
$i_C(t) = C \cdot \frac{d}{dt} v_C(t)$

$\mathbf{I_C}(s) = C \cdot s \cdot \mathbf{V_C}(s)$

$\mathbf{V_C}(\omega) = \frac{1}{C \cdot s} \cdot \mathbf{I}(s)$

$\mathbf{Z_C} = \frac{1}{C \cdot s}$

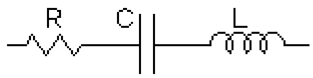
series:



$\mathbf{Z_{eq}} = \mathbf{Z_1} + \mathbf{Z_2} + \mathbf{Z_3} + \ldots$

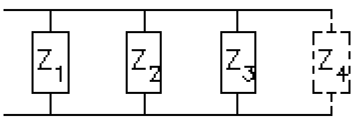
Voltage divider: $\mathbf{V_{Zn}} = \mathbf{V_{total}} \cdot \frac{\mathbf{Z_n}}{\mathbf{Z_1} + \mathbf{Z_2} + \mathbf{Z_3} + \ldots}$

Example:



$\mathbf{Z_{eq}} = R + \frac{1}{C \cdot s} + L \cdot s$

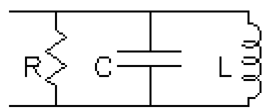
parallel:



$\mathbf{Z_{eq}} = \frac{1}{\frac{1}{\mathbf{Z_1}} + \frac{1}{\mathbf{Z_2}} + \frac{1}{\mathbf{Z_3}} + \ldots}$

Current divider: $\mathbf{I_{Zn}} = \mathbf{I_{total}} \cdot \frac{\frac{1}{\mathbf{Z_n}}}{\frac{1}{\mathbf{Z_1}} + \frac{1}{\mathbf{Z_2}} + \frac{1}{\mathbf{Z_3}} + \ldots}$

Example:



$\mathbf{Z_{eq}} = \frac{1}{\frac{1}{R} + \frac{1}{\frac{1}{C \cdot s}} + \frac{1}{L \cdot s}} = \frac{1}{\frac{1}{R} + C \cdot s + \frac{1}{L \cdot s}}$

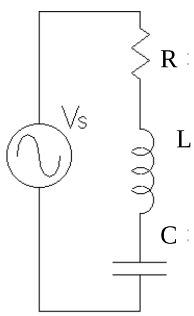
For Steady-State Response, replace s with jω & 1/s with -jω .

Ex2 Find $\mathbf{V_R}$, $\mathbf{V_L}$, and $\mathbf{V_C}$ in polar phasor form. $f := 2 \cdot kHz$ Steady-state

$\omega := 2 \cdot \pi \cdot f$ $\omega = 12566 \cdot \frac{rad}{sec}$

$\mathbf{V_S}(\omega) = 6 \cdot V \cdot e^{j0}$

$f := 2 \cdot kHz$



$R := 500 \cdot \Omega$

$L := 80 \cdot mH$

$C := 0.4 \cdot \mu F$

$\mathbf{Z_L} := j \cdot \omega \cdot L$ $\mathbf{Z_L} = 1.005j \cdot k\Omega$

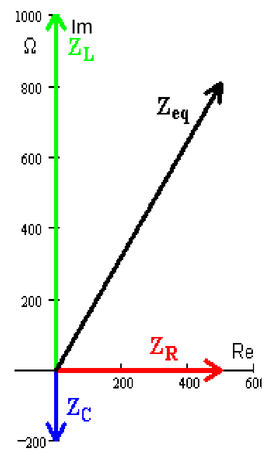
$= 1005\Omega \ \underline{/ \ 90^o}$

$\mathbf{Z_C} := \frac{1}{j \cdot \omega \cdot C}$ $\mathbf{Z_C} = -0.199j \cdot k\Omega$

$= -199\Omega \ \underline{/ \ 90^o} = 199\Omega \ \underline{/ -90^o}$

$\mathbf{Z_{eq}} := R + j \cdot \omega \cdot L + \frac{1}{j \cdot \omega \cdot C}$ $\mathbf{Z_{eq}} = 500 + 806.366j \cdot \Omega$

$\sqrt{500^2 + 806^2} = 948.491$ $\text{atan}\left(\frac{806}{500}\right) = 58.187 \cdot deg$ $\mathbf{Z_{eq}} = 948.5\Omega \ \underline{/ \ 58.2^o}$



b) find the current: $\mathbf{I} := \frac{6 \cdot V \cdot e^{j \cdot 0}}{\mathbf{Z_{eq}}}$ magnitude: $\frac{6 \cdot V}{948.5 \cdot \Omega} = 6.326 \cdot mA$ angle: $0 \cdot deg - 58.2 \cdot deg = -58.2 \cdot deg$

$\mathbf{I} = 6.326mA \ \underline{/ -58.2^o}$

c) Draw a phasor diagram of $\mathbf{V_S}$, $\mathbf{V_R}$, $\mathbf{V_L}$, and $\mathbf{V_C}$

$\mathbf{V_R} := \mathbf{I} \cdot R$

magnitude $6.326 \cdot mA \cdot 500 \cdot \Omega = 3.163 \cdot V$

angle $-58.2 \cdot deg + 0 \cdot deg = -58.2 \cdot deg$

$\mathbf{V_R} = 3.163V \ /{-58.2^o}$

$\mathbf{V_L} := \mathbf{I} \cdot \mathbf{Z_L}$ $6.326 \cdot mA \cdot 1005 \cdot \Omega = 6.358 \cdot V$

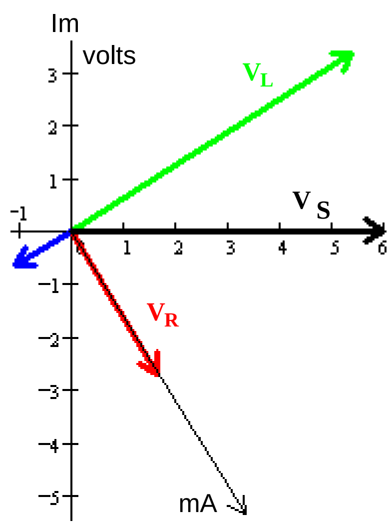
$-58.2 \cdot deg + 90 \cdot deg = 31.8 \cdot deg$

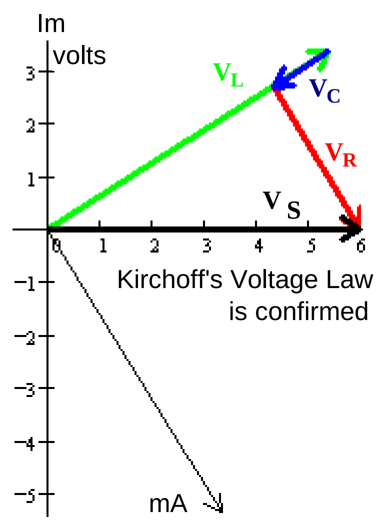
$\mathbf{V_L} = 6.358V \ / \ 31.8^o$

$\mathbf{V_C} := \mathbf{I} \cdot \mathbf{Z_C}$ $6.326 \cdot mA \cdot (199) \cdot \Omega = 1.259 \cdot V$

$-58.2 \cdot deg + (-90) \cdot deg = -148.2 \cdot deg$

$\mathbf{V_C} = 1.259V \ /{-148.2^o}$





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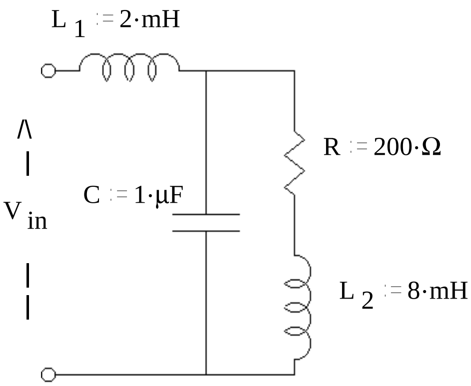


Ex3 a) Find the steady-state V_C and $v_C(t)$ given $v_{in}(t)$ is a 12-V cosine wave at: $f := 2.5\cdot kHz$

with a 20º leading phase angle. $\mathbf{V_{in}} := 12\cdot V\cdot e^{j\cdot 20\cdot deg}$

Transfer function for $\mathbf{V_C}$ as the output: $\mathbf{H}(s) = \dfrac{\mathbf{V_C}(s)}{\mathbf{V_{in}}(s)}$

$$\mathbf{H}(s) = \frac{\dfrac{1}{\dfrac{1}{R+L_2\cdot s}+C\cdot s}}{L_1\cdot s+\dfrac{1}{\dfrac{1}{R+L_2\cdot s}+C\cdot s}} \qquad \frac{\left(\dfrac{1}{R+L_2\cdot s}+C\cdot s\right)}{\left(\dfrac{1}{R+L_2\cdot s}+C\cdot s\right)}$$

$$= \frac{1}{L_1\cdot s\cdot\left(\dfrac{1}{R+L_2\cdot s}+C\cdot s\right)+1} = \frac{1}{L_1\cdot(j\cdot\omega)\cdot\left[\dfrac{1}{R+L_2\cdot(j\cdot\omega)}+C\cdot(j\cdot\omega)\right]+1}$$

$f = 2.5\cdot kHz$

$\omega := 2\cdot\pi\cdot f \qquad \omega = 15708\cdot\dfrac{rad}{sec}$

$$\mathbf{H}(j\omega) = \frac{1}{0.002\cdot(j\cdot 15708)\cdot\left[\dfrac{1}{200+0.008\cdot(j\cdot 15708)}+10^{-6}\cdot(j\cdot 15708)\right]+1}$$

$$= \frac{1}{31.416\cdot j\cdot\left[3.585\cdot 10^{-3}-2.252\cdot 10^{-3}\cdot j\ +(j\cdot 0.015708)\right]+1} = \frac{1}{1-0.423+0.113\cdot j} = \frac{1}{0.588\cdot e^{j\cdot 11.039\cdot deg}}$$

$$= 1.7\cdot e^{-j\cdot 11.039\cdot deg}$$

$$\mathbf{V_C} = \mathbf{V_{in}}(\omega)\cdot\mathbf{H}(j\omega) = 12\cdot V\cdot e^{j\cdot 20\cdot deg}\cdot 1.7\cdot e^{-j\cdot 11.039\cdot deg} = 12\cdot V\cdot 1.7\ \underline{/\ 20 - 11.039^o} = 20.4V\ \underline{/\ 8.96^o}$$

$$v_c(t) = 20.4\cdot V\cdot\cos\left(15708\cdot\frac{rad}{sec}\cdot t+8.96\cdot deg\right)$$

a) Find the steady-state I_{L2} and $i_{L2}(t)$.

Transfer function for $\mathbf{I_{L2}}$ as the output: $\mathbf{H}(s) = \dfrac{\mathbf{I_{L2}}(s)}{\mathbf{V_{in}}(s)} = \dfrac{\left(\dfrac{\mathbf{V_C}(s)}{R+L_2\cdot s}\right)}{\mathbf{V_{in}}(s)} = \dfrac{\mathbf{V_C}(s)}{\mathbf{V_{in}}(s)}\cdot\dfrac{1}{R+L_2\cdot s}$

$$\mathbf{H}(s) = \frac{1}{L_1\cdot s\cdot\left(\dfrac{1}{R+L_2\cdot s}+C\cdot s\right)+1}\cdot\frac{1}{(R+L_2\cdot s)} = \frac{1}{L_1\cdot s+L_1\cdot s\cdot C\cdot s\cdot(R+L_2\cdot s)+(R+L_2\cdot s)}$$

$$\mathbf{H}(j\omega) = \frac{1}{L_1\cdot(j\cdot\omega)+L_1\cdot C\cdot(j\cdot\omega)^2\cdot\left[R+L_2\cdot(j\cdot\omega)\right]+\left[R+L_2\cdot(j\cdot\omega)\right]} = 5.249\cdot 10^{-3}-4.926\cdot 10^{-3}j\quad\cdot\frac{1}{\Omega} = \frac{7.198}{k\Omega}\cdot e^{-j\cdot 43.181\cdot deg}$$

$$\mathbf{I_{L2}} = \mathbf{V_{in}}(\omega)\cdot\mathbf{H}(j\omega) = 12\cdot V\cdot e^{j\cdot 20\cdot deg}\cdot\left(\frac{7.198}{k\Omega}\cdot e^{-j\cdot 43.181\cdot deg}\right) = 12\cdot V\cdot\frac{7.198}{k\Omega}\ \underline{/\ 20 - 43.181^o} = 86.38mA\ \underline{/\ -23.18^o}$$

$$i_{L2}(t) = 86.38\cdot mA\cdot\cos\left(15708\cdot\frac{rad}{sec}\cdot t-23.18\cdot deg\right)$$

Magnitude and Phase of transfer functions

With steady-state sinusoidal inputs which start at $t = 0$

Ex4 a) Find the magnitude and phase of the following transfer function at this frequency: $f := 5\cdot Hz$ $\quad \omega := 2\cdot\pi\cdot f$

$$\omega = 31.42\cdot\frac{rad}{sec}$$

$$\mathbf{H}(s) = \frac{s^2 + \frac{20}{sec}\cdot s + \frac{1000}{sec^2}}{s^2 + \frac{8}{sec}\cdot s + \frac{800}{sec^2}}$$

$$s := j\cdot\omega \;=\; j\cdot 31.42\cdot\frac{rad}{sec} \qquad \mathbf{H}(j\cdot\omega) = \frac{(j\cdot\omega)^2 + \frac{20}{sec}\cdot(j\cdot\omega) + \frac{1000}{sec^2}}{(j\cdot\omega)^2 + \frac{8}{sec}\cdot(j\cdot\omega) + \frac{800}{sec^2}} = \frac{(j\cdot 31.42)^2 + 20\cdot(j\cdot 31.42) + 1000}{(j\cdot 31.42)^2 + 8\cdot(j\cdot 31.42) + 800} \quad \text{without units}$$

$$= \frac{13.04 + 628.319\cdot j}{-187.2164 + 251.36\cdot j} = 1.583 - 1.231j$$

$$|\mathbf{H}(j\cdot\omega)| = M = \sqrt{(1.583)^2 + (1.231)^2} = 2.005 \qquad \underline{/\,\mathbf{H}(j\omega)} = \operatorname{atan}\left(\frac{-1.231}{1.583}\right) = -37.87\cdot deg \qquad \mathbf{H}(j\cdot\omega) = 2.005\ \underline{/-37.87^o}$$

b) Find the steady-state sinusoidal output if the input is: $x(t) = 4\cdot\cos(2\cdot\pi\cdot 5\cdot Hz\cdot t)$ $\qquad \mathbf{X}(\omega) = 4 + 0j$

and then $\mathbf{Y}(\omega) = \mathbf{X}(\omega)\cdot\mathbf{H}(j\cdot\omega)$ $\qquad 4\cdot(1.583 - 1.231\cdot j) = 6.332 - 4.924j$

Note that you can use the rectangular form of $\mathbf{H}(j\cdot\omega)$

$$y_{ss}(t) = \left(6.332\cdot\cos\left(31.42\cdot\frac{rad}{sec}\cdot t\right) + 4.924\cdot\sin\left(31.42\cdot\frac{rad}{sec}\cdot t\right)\right)\cdot u(t)$$

note that the sine carries the opposite sign as the imaginary part of phasor.

$$\sqrt{6.332^2 + 4.924^2} = 8.021 \qquad \operatorname{atan}\left(\frac{-4.924}{6.332}\right) = -37.87\cdot deg \qquad y_{ss}(t) := 8.021\cdot\cos(31.42\cdot t - 37.87\cdot deg)\cdot u(t)$$

c) Find the transient response for the same input. $x(t) = 4\cdot\cos(2\cdot\pi\cdot 5\cdot Hz\cdot t)$ $\qquad \mathbf{X}(s) = 4\cdot\frac{s}{s^2 + 31.42^2}$

$$\mathbf{Y}_{tr}(s) = \frac{4\cdot s}{s^2 + 31.42^2}\cdot\frac{s^2 + 20\cdot s + 1000}{s^2 + 8\cdot s + 800} \qquad a := -4 \qquad b := \sqrt{800 - 4^2} \qquad b = 28$$

$$= A\cdot\frac{s+4}{s^2 + 8\cdot s + 800} + B\cdot\frac{28}{s^2 + 8\cdot s + 800} + C\cdot\frac{s}{s^2 + 31.42^2} + D\cdot\frac{31.42}{s^2 + 31.42^2}$$

multiply both sides by $\left(s^2 + 8\cdot s + 800\right)$ and let $s := -4 + 28\cdot j$

$$\frac{4\cdot(-4 + 28\cdot j)\cdot\left[(-4 + 28\cdot j)^2 + 20\cdot(-4 + 28\cdot j) + 1000\right]}{(-4 + 28\cdot j)^2 + 31.42^2} = -115.969 - 65.365j = A((-4 + 28\cdot j) - 4) + B\cdot 28$$

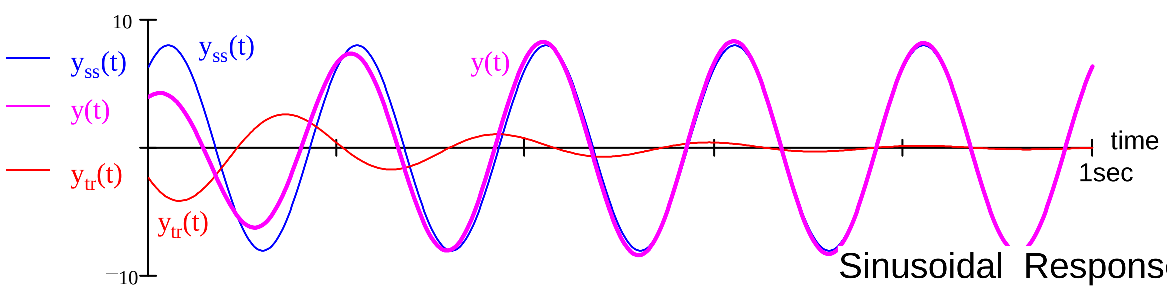
$$\frac{-115.969 - 65.365\cdot j}{28} = -4.142 - 2.334j = A\cdot j + B$$

$$A := -2.334 \qquad B := -4.142$$

$$y_{tr}(t) := \left(-2.334\cdot e^{-4\cdot t}\cdot\cos(28\cdot t) - 4.142\cdot e^{-4\cdot t}\cdot\sin(28\cdot t)\right)\cdot u(t)$$

d) Find the total response for the same input.

$$y(t) := \left(-2.334\cdot e^{-4\cdot t}\cdot\cos(28\cdot t) - 4.142\cdot e^{-4\cdot t}\cdot\sin(28\cdot t) + 8.021\cdot\cos(31.42\cdot t - 37.87\cdot deg)\right)\cdot u(t)$$



Ex4 a) Find the magnitude and phase of the following transfer function at this frequency: $\omega := 2 \cdot \frac{rad}{sec}$

$$\mathbf{H}(s) = \frac{2 \cdot s^2 + 5 \cdot s + 20}{s^2 + 2 \cdot s + 10} = \frac{2 \cdot s^2 + \frac{5}{sec} \cdot s + \frac{20}{sec^2}}{s^2 + \frac{2}{sec} \cdot s + \frac{10}{sec^2}} \cdot \frac{V}{V}$$

Expressed with proper units.
In this case the transfer function is for a circuit where both the input and output are voltages.

$s := j \cdot \omega$

$$\mathbf{H}(j \cdot \omega) = \frac{2 \cdot (j \cdot \omega)^2 + 5 \cdot (j \cdot \omega) + 20}{(j \cdot \omega)^2 + \frac{2}{sec} \cdot (j \cdot \omega) + \frac{10}{sec^2}} = \frac{\left(20 - 2 \cdot \omega^2\right) + (5 \cdot \omega) \cdot j}{\left(10 - \omega^2\right) + (2 \cdot \omega) \cdot j} = \frac{\left(20 - 2 \cdot 2^2\right) + (5 \cdot 2) \cdot j}{\left(10 - 2^2\right) + (2 \cdot 2) \cdot j} = \frac{12 + 10 \cdot j}{6 + 4 \cdot j}$$

without units

$$|\mathbf{H}(j \cdot \omega)| = M = \frac{\sqrt{12^2 + 10^2}}{\sqrt{6^2 + 4^2}} = 2.166 \qquad \angle \mathbf{H}(j\omega) = \operatorname{atan}\left(\frac{10}{12}\right) - \operatorname{atan}\left(\frac{4}{6}\right) = 6.12 \cdot deg \qquad \mathbf{H}(j \cdot \omega) = 2.166 \angle 6.12^o$$

b) Find the steady-state response if the input is: $v_{in}(t) = 3.2 \cdot \cos(2 \cdot t + 15 \cdot deg)$ $\quad \mathbf{V_{in}} := 3.2 \cdot V \cdot e^{j \cdot 15 \cdot deg} = 3.2V \angle 15^o$

$$\mathbf{V_{out\_ss}}(\omega) = \mathbf{V_{in}}(\omega) \cdot \mathbf{H}(j \cdot \omega) = \left(3.2 \cdot V \cdot e^{j \cdot 15 \cdot deg}\right) \cdot \left(2.166 \cdot e^{j \cdot 6.12 \cdot deg}\right) = 3.2 \cdot V \cdot 2.166 \cdot e^{j \cdot (15 \cdot deg + 6.12 \cdot deg)}$$

$$= 6.931 \cdot V \cdot e^{j \cdot 21.12 \cdot deg} = 6.465 + 2.497j \cdot V$$

$$v_{out\_ss}(t) = 6.931 \cdot V \cdot \cos(2 \cdot t + 21.12 \cdot deg) \cdot u(t) = (6.465 \cdot \cos(2 \cdot t) - 2.497 \cdot \sin(2 \cdot t)) \cdot u(t)$$

Note that the sine carries the opposite sign as the imaginary part of phasor.

c) Find the transient response for the same input.

Phasor: $\mathbf{V_{in}} = 3.091 + 0.828j \cdot V$ $\qquad \mathbf{V_{in}}(s) = 3.091 \cdot \frac{s}{s^2 + 2^2} - 0.828 \cdot \frac{2}{s^2 + 2^2} = \frac{3.091 \cdot s - 0.828 \cdot 2}{s^2 + 2^2}$

Note the sign change for the sine.

$$\mathbf{V_{out\_tr}}(s) = \frac{(3.091 \cdot s - 0.828 \cdot 2)}{s^2 + 2^2} \cdot \frac{\left(2 \cdot s^2 + 5 \cdot s + 20\right)}{s^2 + 2 \cdot s + 10} = \frac{6.182 \cdot s^3 + 12.143 \cdot s^2 + 53.54 \cdot s - 33.12}{\left(s^2 + 4\right) \cdot \left(s^2 + 2 \cdot s + 10\right)}$$

$$= A \cdot \frac{s + 1}{s^2 + 2 \cdot s + 10} + B \cdot \frac{3}{s^2 + 2 \cdot s + 10} + C \cdot \frac{s}{s^2 + 4} + D \cdot \frac{2}{s^2 + 4}$$

multiply both sides by $\left(s^2 + 2 \cdot s + 10\right)$ and let $s := -1 + 3 \cdot j$

$$\frac{6.182 \cdot s^3 + 12.143 \cdot s^2 + 53.54 \cdot s - 33.12}{\left(s^2 + 4\right)} = A \cdot ((-1 + 3 \cdot j) + 1) + B \cdot 3$$

$$\frac{6.182 \cdot (-1 + 3 \cdot j)^3 + 12.143 \cdot (-1 + 3 \cdot j)^2 + 53.54 \cdot (-1 + 3 \cdot j) - 33.12}{\left[(-1 + 3 \cdot j)^2 + 4\right]} = \frac{-23.072 - 23.514 \cdot j}{-4 - 6 \cdot j} = 4.488 - 0.853j = A \cdot 3 \cdot j + B \cdot 3$$

$$\frac{4.488 - 0.853 \cdot j}{3} = 1.496 - 0.284j \qquad A := -0.284 \qquad B := 1.496$$

$$v_{out\_tr}(t) = \left(-0.281 \cdot V \cdot e^{-1 \cdot t} \cdot \cos(3 \cdot t) + 1.496 \cdot V \cdot e^{-1 \cdot t} \cdot \sin(3 \cdot t)\right) \cdot u(t)$$

d) Find the total response for the same input.

$$v_{out}(t) = \left(-0.281 \cdot V \cdot e^{-1 \cdot t} \cdot \cos(3 \cdot t) + 1.496 \cdot V \cdot e^{-1 \cdot t} \cdot \sin(3 \cdot t) + 6.931 \cdot V \cdot \cos(2 \cdot t + 21.12 \cdot deg)\right) \cdot u(t)$$