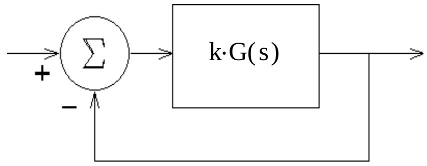
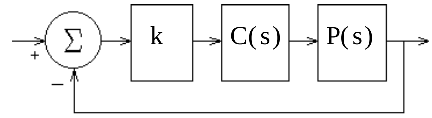
A. Stolp 2/16/06, rev. 2/26/23

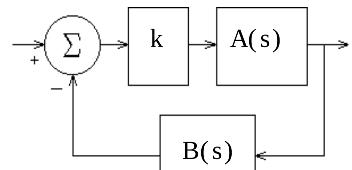
ECE 3510 Root-Locus Plots

$G(s) = \dfrac{N_G}{D_G}$ = the Open-Loop (O-L) transfer function





$G(s) = C(s)\cdot P(s)$



$G(s) = A(s)\cdot B(s)$

The poles of the C-L transfer function solve: $1 + k\cdot G(s) = 0$

Any s that makes $\underline{/\,G(s)} = 180^o$ will work for some k and be a part of the Root Locus.

The Rules ($k \geq 0$)

1. Root-locus plots are symmetric about the real axis.
2. On the real axis, spaces left of an odd number of O-L poles and zeros are always part of the locus.
   (Essentially, every other space on the real axis (counting leftward) is part of the plot.)
3. Each O-L pole originates ($k = 0$) one branch. (n)
   Each O-L zero terminates ($k = \infty$) one branch. (m)
   All remaining branches go to ∞, one per asymptote. $(n - m)$
   They each approach their asymptotes as they go to ∞.
4. The origin of the asymptotes is the *centroid*.

$$\text{centroid} = \sigma = \frac{\sum_{\text{all}} \text{OLpoles} - \sum_{\text{all}} \text{OLzeros}}{n - m}$$

(# poles - # zeros)

5. The angles of the asymptotes are:

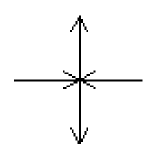
$$i\cdot\frac{180}{n - m}$$

where $i = 1, 3, 5, 7, 9, ...$ full circle

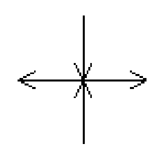
Or figure for half circle and mirror around the real axis.

n - m	angles (degrees)					
2	90	270				
3	60	180	300			
4	45	135	225	315		
5	36	108	180	252	324	
6	30	90	150	210	270	330
7	$\frac{180}{7}$	$3\cdot\frac{180}{7}$	$5\cdot\frac{180}{7}$	$7\cdot\frac{180}{7}$	. . .	

6. The angles of departure (and arrival) of the locus are almost always:

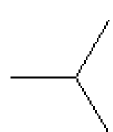


OR:



Only multiple poles result in different departure angles:
(or zeros)

triple poles:



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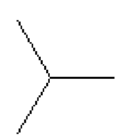
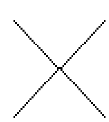


Check real-axis rule, above

Quadruple poles:



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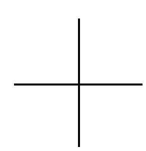


Check real-axis rule, above

Multiple zeros attract branches from these same angles

Good guesses: Draw your break-outs midway between poles, and your break-ins midway between zeroes.
Draw circles centered approximately midway between poles and zeroes.

7. Gain at any point on the root locus: $k = -\dfrac{1}{G(s)} = \dfrac{1}{|G(s)|} = \dfrac{|D(s)|}{|N(s)|}$

8. Phase angle of $G(s)$ at any point s on the root locus: $\arg(G(s)) = \arg(N(s)) - \arg(D(s)) = \pm 180^o \quad \pm 540^o \ldots$

Note: arg(x) is /(x)

Or: $\arg\left(\dfrac{1}{G(s)}\right) = \arg(D(s)) - \arg(N(s)) = \pm 180^o \quad \pm 540^o \ldots$

Or: $\arg(-G(s)) = 0^o \quad \pm 360^o \ldots$

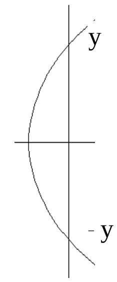
$$\sum_{\text{all zeroes}} (\text{angle of point } s \text{ relative to zero}) - \sum_{\text{all poles}} (\text{angle of point } s \text{ relative to pole}) = \pm 180^o \quad \pm 540^o \ldots$$

9. Gain at $j\omega$ crossing: Use Routh-Hurwitz test.

OR: a) Get a rough s (say y) value from your plot,

b) Check it (evaluate the angle of $G(jy)$) and iterate using rule 9,

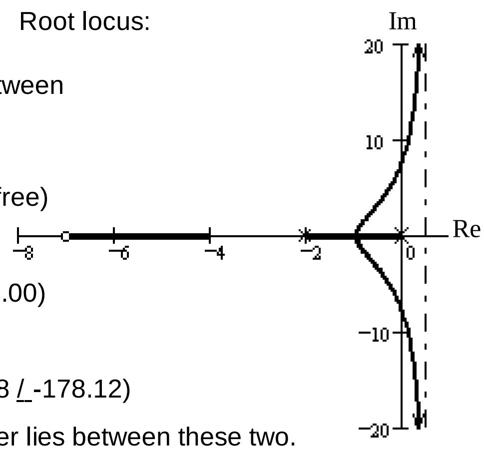
c) Find k using rule 8.



Calculator example: $G(s) = \dfrac{s+7}{s\cdot(s+2)\cdot(s+4)}$

Find the gain at $j\omega$ crossing:

Root locus:



Let's assume that the root locus crosses the $j\omega$ axis somewhere between 5 and 10. I first try 5, evaluating $\dfrac{1}{G(5j)}$ on my calculator

Note: I'm evaluating $1/G(s)$ so I'll end up with the gain value for free)

In a TI-86, I enter the following:

5.000->S:((0,S)*(2,S)*(4,S))/((7,S)) It returns: (20.04 /174.00)

Next I try:

10.00->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (89.98 /-178.12)

The first was a positive angle, and this is negative, yep, the answer lies between these two.

The first was 6^o under 180^o and the second is 2^o over, interpolate: $5 + \dfrac{6}{6+2}\cdot 5 = 8.75$

Try: 8.75 8.750->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (67.43 /-178.78)

$$8.75 - \frac{180 - 178.78}{180 - 178.12}\cdot(10 - 8.75) = 7.939$$

Try: 7.9 7.900->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (54.01 /-179.52)

$$7.9 - \frac{.48}{1.22}\cdot(8.75 - 7.9) = 7.566$$

Try: 7.5 7.500->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (48.23 /-179.97)

$$7.5 - \frac{.03}{.48}\cdot(7.9 - 7.5) = 7.475$$

Try: 7.475 7.475->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (47.88 /179.99)

The root locus crosses at $\pm 7.475j$ and the gain is 48. $k = 48$

 Additional Rules

10. Breakaway points from the real axis (σ_b) are the solutions to: $\frac{d}{ds}G(s) = 0$
(and arrival)

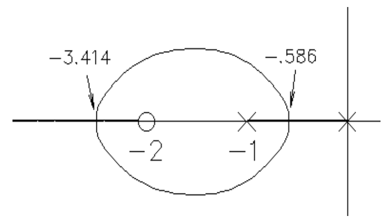
The breakaway points are also solutions to: $$\sum_{all}\frac{1}{(s+-p_i)} = \sum_{all}\frac{1}{(s+-z_i)}$$

IE: $$\frac{1}{(s+-p_1)}+\frac{1}{(s+-p_2)}+\frac{1}{(s+-p_3)} + \dots = \frac{1}{(s+-z_1)}+\frac{1}{(s+-z_2)}+\frac{1}{(s+-z_3)} + \dots$$

Not all of these solutions will actually be breakaway points, some may be eliminated by the real-axis rule, and some may be arrival points. Both of these methods can be messy and result in high-order equations to solve. The second example will show an iterative way to deal with the complexity.

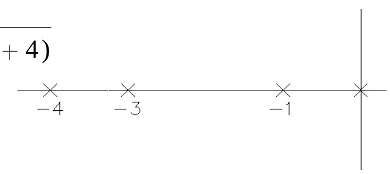
Example 1 $G(s) = \frac{s+2}{s\cdot(s+1)}$ Solve: $\frac{1}{s}+\frac{1}{s+1} = \frac{1}{s+2}$

$$\frac{(s+1)+s}{s\cdot(s+1)} = \frac{1}{s+2}$$



$(2\cdot s+1)\cdot(s+2) = s\cdot(s+1)$ $s^2+4\cdot s+2 = 0$ $s = -3.414$ $s = -0.586$

Example 2 Iterative process, best shown by example: $G(s) = \frac{1}{s\cdot(s+1)\cdot(s+3)\cdot(s+4)}$



Find the breakaway point between 0 and -1.

Must solve: $\frac{1}{s}+\frac{1}{(s+1)}+\frac{1}{(s+3)}+\frac{1}{(s+4)} = 0$

Guess $s = -0.4$ and use that for all the s's except those closest to the breakaway you want to find.

Solve this instead: $$\frac{1}{s}+\frac{1}{(s+1)}+\frac{1}{(-0.4+3)}+\frac{1}{(-0.4+4)} = 0$$

$$\frac{1}{s}+\frac{1}{(s+1)}+\left(\frac{1}{2.6}+\frac{1}{3.6}\right) = 0$$

multiply by $s(s+1)$: $$\frac{s+1}{1}+\frac{s}{1}+s\cdot(s+1)\cdot\left(\frac{1}{2.6}+\frac{1}{3.6}\right) = 0$$

$$s^2+4.0194\cdot s+1.5097 = 0$$

$$s = \frac{-4.0194+\sqrt{4.0194^2-4\cdot 1.5097}}{2} = -0.419$$ Use this answer to try again

ignore the -3.6 solution for this answer.

$$\frac{1}{s}+\frac{1}{(s+1)}+\left(\frac{1}{2.581}+\frac{1}{3.581}\right) = 0$$

$$s^2+4\cdot s+1.5 = 0$$

$$s = \frac{-4+\sqrt{4^2-4\cdot 1.5}}{2} = -0.419$$ No significant change,
so this is the breakaway point

To find the breakaway point between -3 and -4: Guess $s = -3.6$

$$\frac{1}{-3.6}+\frac{1}{(-3.6+1)}+\frac{1}{(s+3)}+\frac{1}{(s+4)} = 0$$

solve for s: $s = -3.58$ Not much change,
so this is the breakaway point

Actually, it usually doesn't matter that much just where the breakaway points are.

11. Departure angle (θ_D) from a complex pole (p_c).

Recall rule 9 (one of the most important rules):

for any point s on the root locus:

$$\arg(G(s)) = \arg(N(s)) - \arg(D(s)) = \pm 180^o \quad \pm 360^o \ ...$$

Note: arg(x) is /(x)

For multiple (r) poles:

Divide the circle into r divisions: $\frac{360 \cdot deg}{r}$

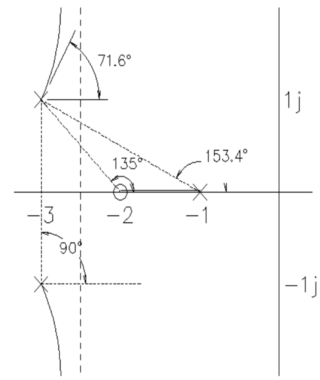
and rotate all by $\frac{\theta_D}{r}$

Now imagine a point ε-distance away from the complex pole. That point would have an angle of θ_D with respect to the complex pole, but it's angle relative to all the other poles and zeros would be essentially the same as the complex pole.

$$\sum_{\text{all zeroes}} (\text{angle of point } s \text{ relative to zero}) - \sum_{\text{all poles but } p_c} (\text{angle of point } s \text{ relative to pole}) - \theta_D = \pm 180^o \quad \pm 540^o \ ...$$

Example: $G(s) := \dfrac{s+2}{(s+1)\cdot\left[(s+3)^2 + 1^2\right]}$

Find the departure angle from the pole at: $p_c := -3 + 1\cdot j$



$$135 - 153.4 - 90 - \theta_D = \pm 180^o \quad \pm 540^o \ ...$$

rearrange: $\theta_D = 180 - 90 - 153.4 + 135 = 71.6 \ \deg$

Mathmatically: $\theta_D = 180\cdot deg + \arg\left[G(p_c)\cdot(s + p_c)\right]$

The O-L phase angle computed at the complex pole, but ignoring the effect of that complex pole.

Our example: $\theta_D = 180\cdot deg + \arg\left[\dfrac{p_c + 2}{(p_c + 1)\cdot(p_c + 3 + 1\cdot j)}\right] = 71.6 \cdot deg$

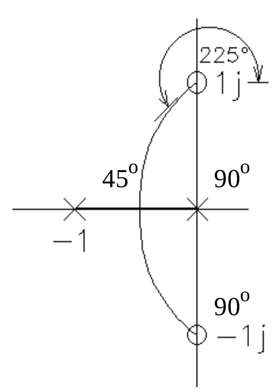
12. Arrival angle (θ_A) to complex zero (z_c).

Exactly the same idea.

$$\sum_{\text{all zeroes but } z_c} (\text{angle of point } s \text{ relative to zero}) + \theta_A - \sum_{\text{all poles}} (\text{angle of point } s \text{ relative to pole}) = \pm 180^o \quad \pm 540^o \ ...$$

Example: $G(s) := \dfrac{s^2 + 1^2}{s\cdot(s+1)} = \dfrac{(s - 1\cdot j)\cdot(s + 1\cdot j)}{s\cdot(s+1)}$

Find the departure angle from the pole at: $z_c := 1\cdot j$



$$90 + \theta_A - 90 - 45 = \pm 180^o \quad \pm 540^o \ ...$$

rearrange: $\theta_A = 180 - 90 + 90 + 45 = 225 \quad \deg$

Mathmatically: $\theta_A = 180\cdot deg - \arg\left[\dfrac{G(z_c)}{(s + z_c)}\right]$

The O-L phase angle computed at the complex zero, but ignoring the effect of that complex zero.

Our example: $\theta_A = 180\cdot deg - \arg\left[\dfrac{1\cdot j + 1\cdot j}{1\cdot j\cdot(1\cdot j + 1)}\right] = 225 \cdot deg$