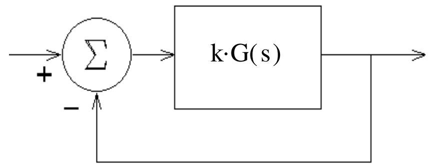
# ECE 3510 Root-Locus Plots

A. Stolp
2/16/06,
rev 2/22/06




$G(s) = \frac{N_G}{D_G}$ = the Open-Loop (O-L) transfer function

## The Rules

1. Root-locus plots are symmetric about the real axis.

2. On the real axis, spaces left of an odd number of O-L poles and zeros are always part of the locus.
(Essentially, every other space on the real axis (counting leftward) is part of the plot.)

3. Each O-L pole originates ($k = 0$) one branch. $(n)$

   Each O-L zero terminates ($k = \infty$) one branch. $(m)$

   All remaining branches go to ∞. $(n - m)$

   These remaining branches approach asymptotes as they go to ∞.

4. The origin of the asymptotes is the *centroid*.

$$\text{centroid} = \sigma = \frac{\sum_{\text{all}} \text{OLpoles} - \sum_{\text{all}} \text{OLzeros}}{n - m}$$

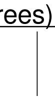
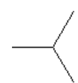
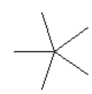
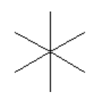
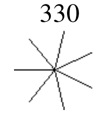
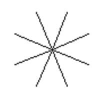
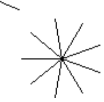
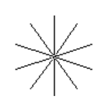
( # poles - # zeros)

5. The angles of the asymptotes are:

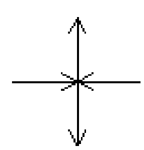
$$i \cdot \frac{180}{n - m}$$

where $i = 1, 3, 5, 7, 9, \ldots$ full circle

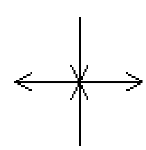
Or figure for half circle and mirror around the real axis.

| n - m | angles (degrees) | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 90 | 270 |  | | | | |
| 3 | 60 | 180 | 300 |  | | | |
| 4 | 45 | 135 | 225 | 315 |  | | |
| 5 | 36 | 108 | 180 | 252 | 324 |  | |
| 6 | 30 | 90 | 150 | 210 | 270 | 330 |  |
| 7 | $\frac{180}{7}$ | $3 \cdot \frac{180}{7}$ | $5 \cdot \frac{180}{7}$ | $7 \cdot \frac{180}{7}$ | . . . |  | |
| 8 | 22.5 | 67.5 | 112.5 | 157.5 | . . . |  | |
| 9 | 20 | 60 | 100 | 140 | 180 | . . . |  |
| 10 | 18 | 54 | 90 | 126 | 162 | . . . |  |

6. The angles of departure (and arrival) of the locus are almost always:

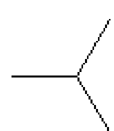


OR:

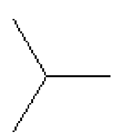


Only multiple poles result in different departure angles:
(or zeros)

triple poles:

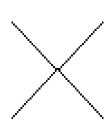


OR:

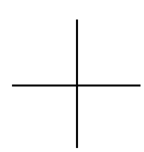


Check real-axis rule, above

Quadruple poles:



OR:



Check real-axis rule, above

Multiple zeros attract branches from these same angles

## Additional Rules

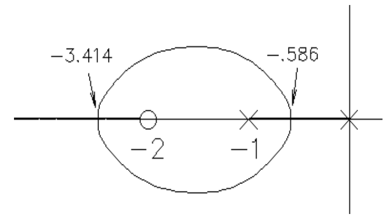
7. Breakaway points from the real axis ($\sigma_b$) are the solutions to: $\frac{d}{ds}G(s) = 0$
(and arrival)

The breakaway points are also solutions to: $$\sum_{\text{all}} \frac{1}{(s+-p_i)} = \sum_{\text{all}} \frac{1}{(s+-z_i)}$$

IE: $$\frac{1}{(s+-p_1)} + \frac{1}{(s+-p_2)} + \frac{1}{(s+-p_3)} + \dots = \frac{1}{(s+-z_1)} + \frac{1}{(s+-z_2)} + \frac{1}{(s+-z_3)} + \dots$$

Not all of these solutions will actually be breakaway points, some may be eliminated by the real-axis rule, and some may be arrival points. Both of these methods can be messy and result in high-order equations to solve. The second example will show an iterative way to deal with the complexity.

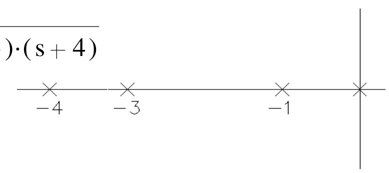
Example 1 $G(s) = \frac{s+2}{s\cdot(s+1)}$ Solve: $\frac{1}{s} + \frac{1}{s+1} = \frac{1}{s+2}$




$$\frac{(s+1)+s}{s\cdot(s+1)} = \frac{1}{s+2}$$

$$(2\cdot s+1)\cdot(s+2) = s\cdot(s+1) \qquad s^2 + 4\cdot s + 2 = 0 \qquad s = -3.414 \qquad s = -0.586$$

Example 2 Iterative process, best shown by example: $G(s) = \frac{1}{s\cdot(s+1)\cdot(s+3)\cdot(s+4)}$




Find the breakaway point between 0 and -1.

Must solve: $$\frac{1}{s} + \frac{1}{(s+1)} + \frac{1}{(s+3)} + \frac{1}{(s+4)} = 0$$

Guess $s = -0.4$ and use that for all the s's except those closest to the breakaway you want to find.

Solve this instead: $$\frac{1}{s} + \frac{1}{(s+1)} + \frac{1}{(-0.4+3)} + \frac{1}{(-0.4+4)} = 0$$

$$\frac{1}{s} + \frac{1}{(s+1)} + \left(\frac{1}{2.6} + \frac{1}{3.6}\right) = 0$$

multiply by $s(s+1)$: $$\frac{s+1}{1} + \frac{s}{1} + s\cdot(s+1)\cdot\left(\frac{1}{2.6} + \frac{1}{3.6}\right) = 0$$

$$s^2 + 4.0194\cdot s + 1.5097 = 0$$

$$s = \frac{-4.0194 + \sqrt{4.0194^2 - 4\cdot 1.5097}}{2} = -0.419$$ Use this answer to try again

ignore the -3.6 solution for this answer.

$$\frac{1}{s} + \frac{1}{(s+1)} + \left(\frac{1}{2.581} + \frac{1}{3.581}\right) = 0$$

$$s^2 + 4\cdot s + 1.5 = 0$$

$$s = \frac{-4 + \sqrt{4^2 - 4\cdot 1.5}}{2} = -0.419$$ No significant change, so this is the breakaway point

To find the breakaway point between -3 and -4: Guess $s = -3.6$

$$\frac{1}{-3.6} + \frac{1}{(-3.6+1)} + \frac{1}{(s+3)} + \frac{1}{(s+4)} = 0$$

solve for s: $s = -3.58$ Not much change, so this is the breakaway point

Actually, it usually doesn't matter that much just where the breakaway points are.

8. Gain at any point on the root locus: $k = -\dfrac{1}{G(s)} = \dfrac{1}{|G(s)|} = \dfrac{|D(s)|}{|N(s)|}$

9. Phase angle of G(s) at any point on the root locus: $\arg(G(s)) = \arg(N(s)) - \arg(D(s)) = \pm 180^o \quad \pm 360^o \ldots$

Or: $\arg\left(\dfrac{1}{G(s)}\right) = \arg(D(s)) - \arg(N(s)) = \pm 180^o \quad \pm 360^o \ldots$

Note: arg(x) is $\angle(x)$ (This is the form you' ll use the most.)

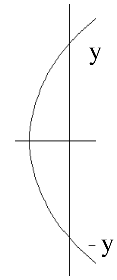
Or: $\arg(-G(s)) = 0^o \quad \pm 360^o \ldots$

10. Gain at jω crossing: Use Routh-Hurwitz test.

OR: a) Get a rough s (say y) value from your plot,

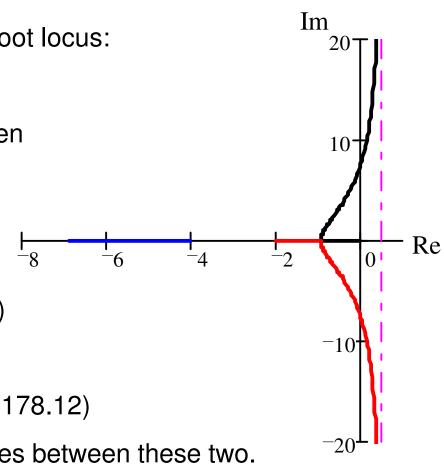
b) Check it (evaluate the angle of G(jy)) and iterate using rule 9,

c) Find k using rule 8.




Calculator example: $G(s) = \dfrac{s+7}{s\cdot(s+2)\cdot(s+4)}$

Root locus:




Find the gain at jω crossing:

Let' s assume that the root locus crosses the jω axis somewhere between 5 and 10. I first try 5, evaluating $\dfrac{1}{G(5j)}$ on my calculator

In a TI-86, I enter the following:

5.000->S:((0,S)*(2,S)*(4,S))/((7,S)) It returns: (20.04 ∠174.00)

Next I try:

10.00->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (89.98 ∠-178.12)

The first was a positive angle, and this is negative, yep, the answer lies between these two.

The first was 6 under 180º and the second is 2 over, interpolate: $5 + \dfrac{6}{6+2}\cdot 5 = 8.75$

Try: $8.75$ 8.750->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (67.43 ∠-178.78)

$$8.75 - \frac{1.22}{1.88}\cdot 1.25 = 7.939$$

Try: $7.9$ 7.900->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (54.01 ∠-179.52)

$$7.9 - \frac{.48}{1.22}\cdot(8.75 - 7.9) = 7.566$$

Try: $7.5$ 7.500->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (48.23 ∠-179.97)

$$7.5 - \frac{.03}{.48}\cdot(7.9 - 7.5) = 7.475$$

Try: $7.475$ 7.475->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (47.88 ∠179.99)

The root locus crosses at $\pm 7.475j$ and the gain is $48$. $k = 48$

11. Departure angles from complex poles.

$$\theta_D = 180\cdot deg + \arg\left[G(p_i)\cdot(s+p_i)\right]$$

The O-L phase angle computed at the complex pole, but ignoring the effect of that complex pole.

For multiple (r) poles:

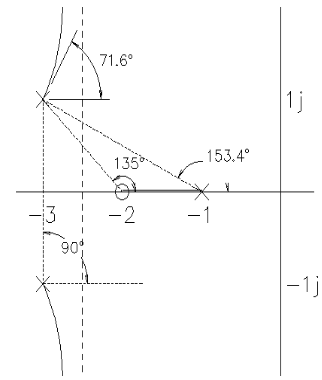
Divide the circle into r divisions: $\dfrac{360\cdot deg}{r}$

and rotate all by $\dfrac{\theta_D}{r}$

OR: $$\theta_D = 90\cdot deg + \arg\left[G(p_i)\cdot(s+a_i-b_i\cdot j)\cdot(s+a_i+b_i\cdot j)\right]$$ ignoring both poles

OR: $$\theta_D = 90\cdot deg + \arg\left[G(p_i)\cdot\left[(s+a_i)^2+(b_i)^2\right]\right]$$ ignoring both poles

Example: $G(s) := \dfrac{s+2}{(s+1)\cdot\left[(s+3)^2+1^2\right]}$ $\qquad s := -3+1\cdot j$




$$\theta_D = 180\cdot deg + \arg\left[\frac{s+2}{(s+1)\cdot(s+3+1\cdot j)}\right]$$

$$\theta_D = 180\cdot deg + \arg\left[\frac{(-3+1\cdot j)+2}{((-3+1\cdot j)+1)\cdot((-3+1\cdot j)+3+1\cdot j)}\right] = 71.6\cdot deg$$

OR: $$\theta_D = 90\cdot deg + \arg\left[\frac{s+2}{(s+1)}\right] \quad = 90\cdot deg + \arg\left[\frac{(-3+1\cdot j)+2}{((-3+1\cdot j)+1)}\right] = 71.6\cdot deg$$

Or, the method from the text: $180-90-153.4+135 = 71.6 \quad deg$

(probably the best method of all)

12. Arrival angles to complex zeros.

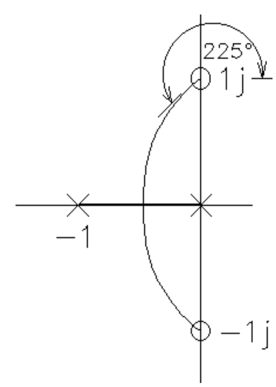
$$\theta_A = 180\cdot deg - \arg\left[\frac{G(z_i)}{(s+z_i)}\right]$$

The O-L phase angle computed at the complex zero, but ignoring the effect of that complex zero.

OR: $$\theta_A = 90\cdot deg - \arg\left[\frac{G(z_i)}{(s+a_i-b_i\cdot j)\cdot(s+a_i+b_i\cdot j)}\right]$$ ignoring both zeros

OR: $$\theta_A = 90\cdot deg - \arg\left[\frac{G(z_i)}{(s+a_i)^2+(b_i)^2}\right]$$ ignoring both zeros

Example: $G(s) := \dfrac{s^2+1^2}{s\cdot(s+1)} \quad = \dfrac{(s-1\cdot j)\cdot(s+1\cdot j)}{s\cdot(s+1)}$ $\qquad s := 1\cdot j$




$$\theta_A = 180\cdot deg - \arg\left[\frac{G(z_i)}{(s+z_i)}\right] \quad = 180\cdot deg - \arg\left[\frac{1\cdot j+1\cdot j}{1\cdot j\cdot(1\cdot j+1)}\right] = 225\cdot deg$$

OR: $$\theta_A = 90\cdot deg - \arg\left[\frac{1}{s\cdot(s+1)}\right] \quad = 90\cdot deg - \arg\left[\frac{1}{1\cdot j\cdot(1\cdot j+1)}\right] = 225\cdot deg$$

Or, the method from the text: $-(180-90-45+90) = -135 \quad = 225\cdot deg$

Example: $G(s) := \dfrac{\left[(s+3)^2+1^2\right]}{s\cdot(s+1)} \quad = \dfrac{(s+3-1\cdot j)\cdot(s+3+1\cdot j)}{s\cdot(s+1)}$ $\qquad s := -3+1\cdot j$

$$\theta_A = 180\cdot deg - \arg\left[\frac{G(z_i)}{(s+z_i)}\right] \quad = 180\cdot deg - \arg\left[\frac{(s+3+1\cdot j)}{s\cdot(s+1)}\right]$$

$$= 180\cdot deg - \arg\left[\frac{((-3+1\cdot j)+3+1\cdot j)}{(-3+1\cdot j)\cdot((-3+1\cdot j)+1)}\right] = 45\cdot deg$$

OR: $$\theta_A = 90\cdot deg - \arg\left[\frac{1}{s\cdot(s+1)}\right] \quad = 90\cdot deg - \arg\left[\frac{1}{(-3+1\cdot j)\cdot((-3+1\cdot j)+1)}\right] = 45\cdot deg$$