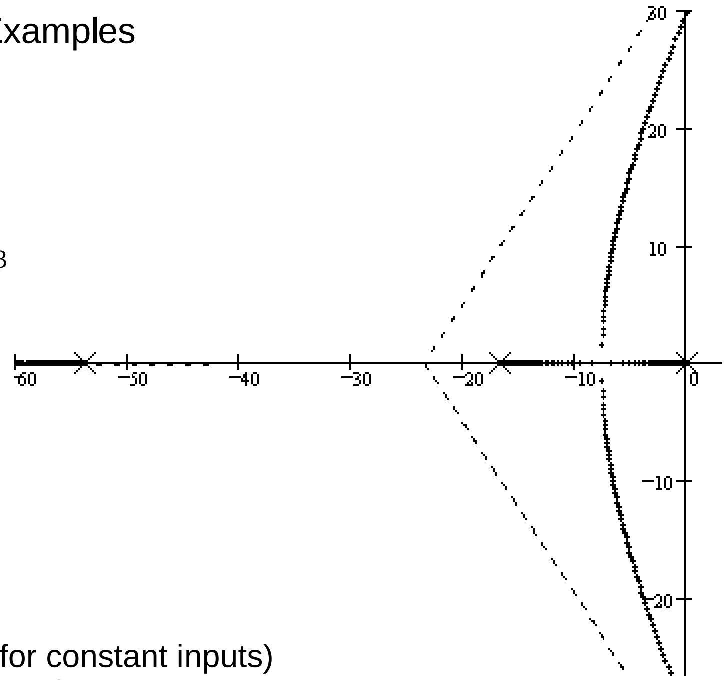
ECE 3510 Root Locus Design Examples

Recall the simple crude servo from lab 1

$$\mathbf{G}(s) := \frac{1643}{s\cdot(s+16.64)\cdot(s+53.78)}$$

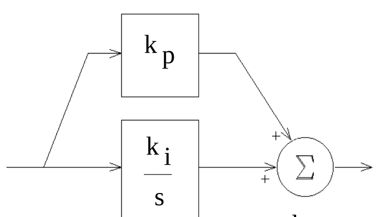
$$\sigma = \frac{-0-16.64-53.78}{3} = -23.473$$

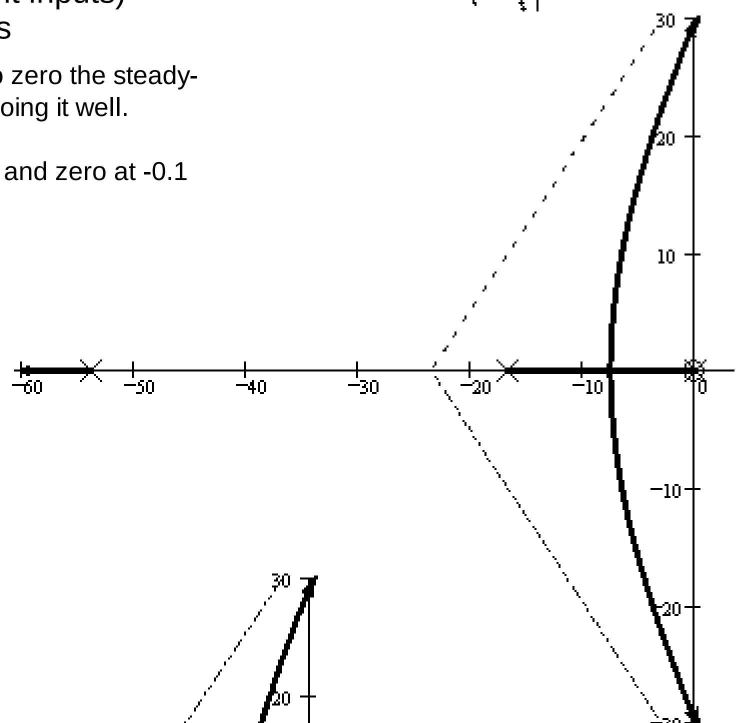


PI To eliminate steady-state error (for constant inputs) & perfect rejection of constant disturbances

Note: The DC motor has a pole at zero and should do zero the steady-state error by itself, but nonlinearities prevent it from doing it well.

$$\mathbf{G}_c(s) := \frac{1643}{s\cdot(s+16.64)\cdot(s+53.78)}\cdot\frac{s+0.1}{s}$$ Add pole at 0 and zero at -0.1



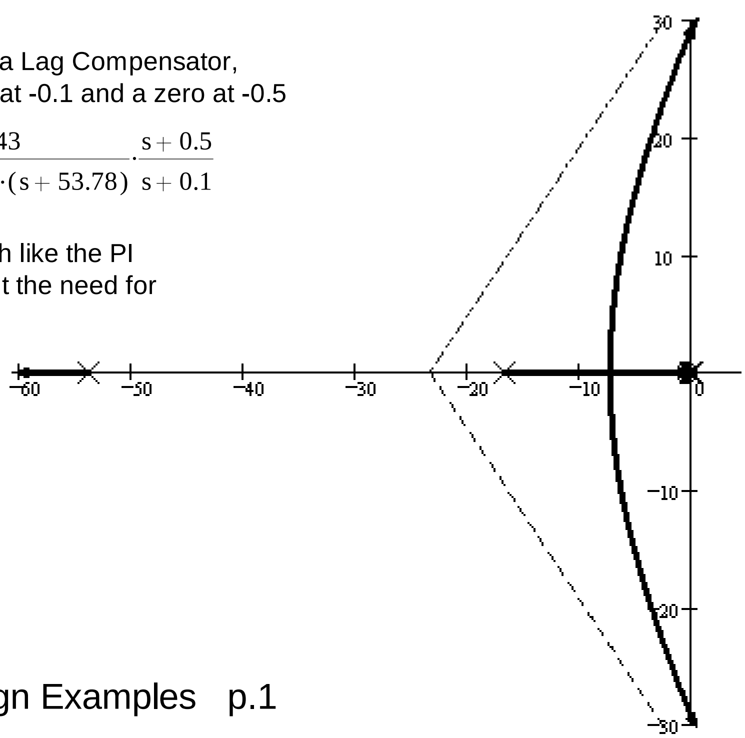


$$\mathbf{C}(s) = k_p + \frac{k_i}{s} = k_p\cdot\frac{s+\frac{k_i}{k_p}}{s}$$

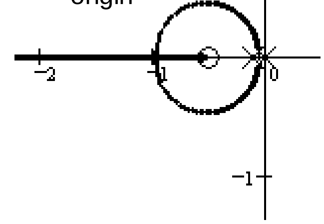
LAG An alternative is a Lag Compensator, here with a pole at -0.1 and a zero at -0.5

$$\mathbf{G}_c(s = \frac{1643}{s\cdot(s+16.64)\cdot(s+53.78)}\cdot\frac{s+0.5}{s+0.1}$$

This works very much like the PI controller, but without the need for active components.



The area near the origin



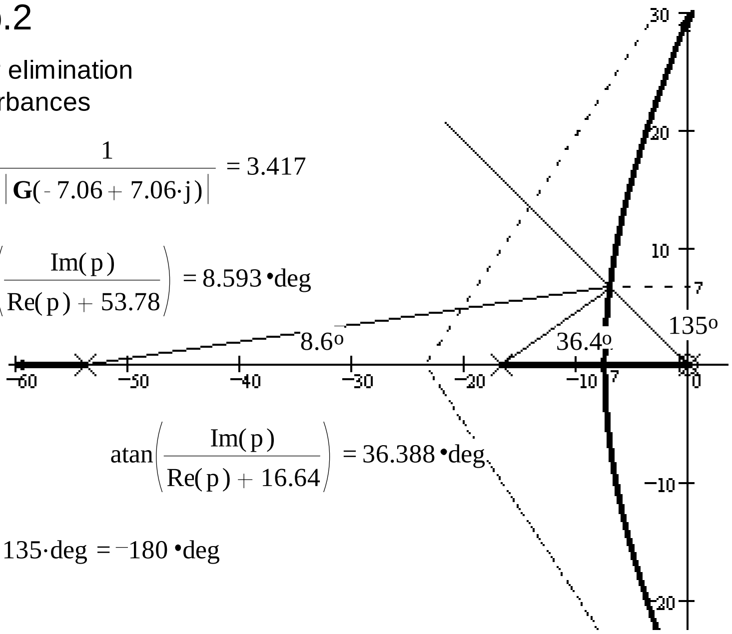
Root Locus Design Example p.2

Let's keep the pole at 0 and zero at -0.1 for elimination of steady-state errors and rejection of disturbances

CL poles at $p := -7.06 + 7.06 \cdot j$ and $-7.06 - 7.06 \cdot j$

$$k = \frac{1}{\left|\mathbf{G}(-7.06 + 7.06 \cdot j)\right|} = 3.417$$

At gain of 3.44



$$\operatorname{atan}\left(\frac{\operatorname{Im}(p)}{\operatorname{Re}(p) + 53.78}\right) = 8.593 \cdot deg$$

$$\operatorname{atan}\left(\frac{\operatorname{Im}(p)}{\operatorname{Re}(p) + 16.64}\right) = 36.388 \cdot deg$$

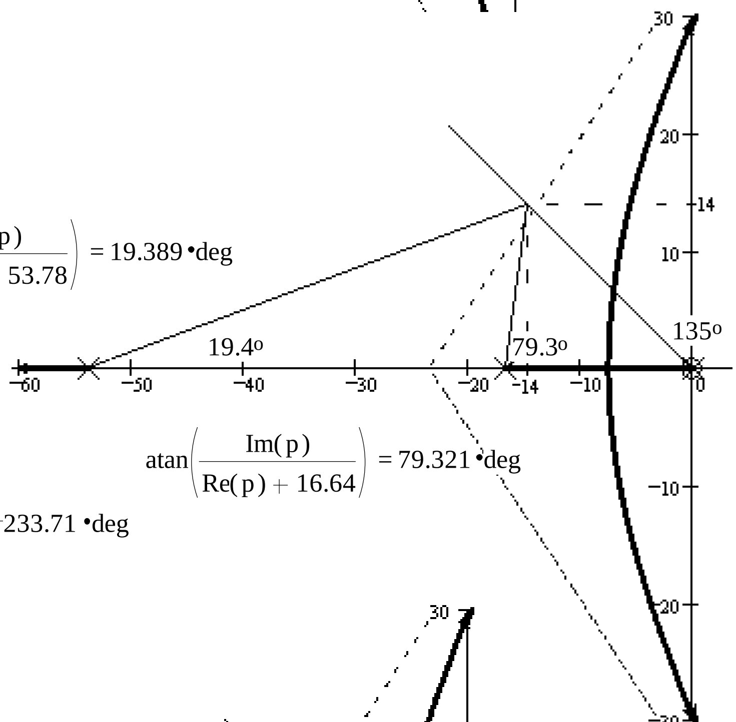
This is a point in the root locus because:

$$-8.6 \cdot deg - 36.4 \cdot deg - 135 \cdot deg - 135 \cdot deg + 135 \cdot deg = -180 \cdot deg$$

PD or PID To Improve the dynamic response

Want to double the speed

Want poles to move to: $p := -14 + 14 \cdot j$, $-14 - 14 \cdot j$



$$\operatorname{atan}\left(\frac{\operatorname{Im}(p)}{\operatorname{Re}(p) + 53.78}\right) = 19.389 \cdot deg$$

Unfortunately, this point in NOT on the root locus

$$\operatorname{atan}\left(\frac{\operatorname{Im}(p)}{\operatorname{Re}(p) + 16.64}\right) = 79.321 \cdot deg$$

$$-\operatorname{atan}\left(\frac{\operatorname{Im}(p)}{\operatorname{Re}(p) + 53.78}\right) - \operatorname{atan}\left(\frac{\operatorname{Im}(p)}{\operatorname{Re}(p) + 16.64}\right) - 135 \cdot deg = -233.71 \cdot deg$$

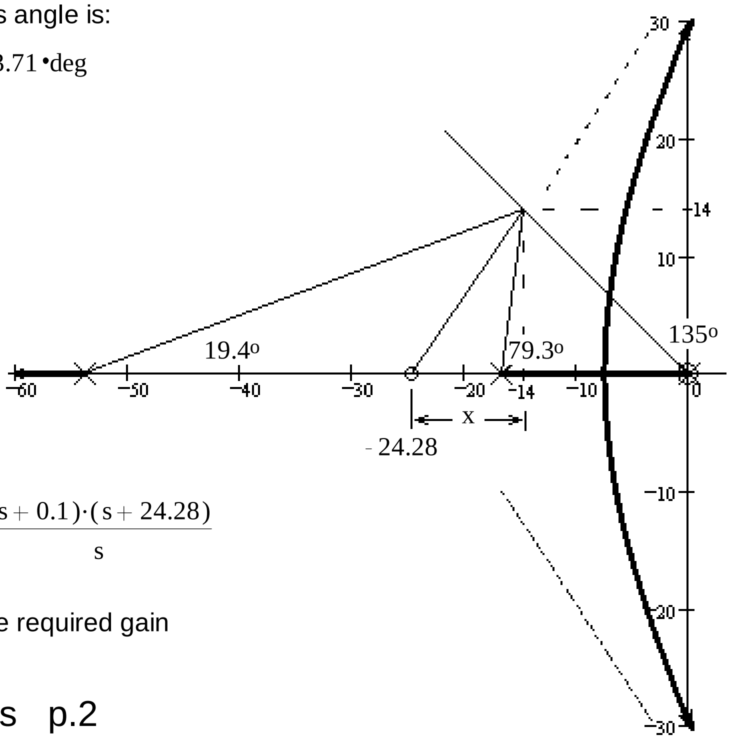
Maybe we could add a zero so that it's angle is:

$$\theta_z := 233.71 \cdot deg - 180 \cdot deg \qquad \theta_z = 53.71 \cdot deg$$

$$x = \operatorname{Im}(p) \cdot \frac{1}{\tan(\theta_z)} = 10.28$$

$$z := \operatorname{Re}(p) - \operatorname{Im}(p) \cdot \frac{1}{\tan(\theta_z)}$$

$$z = -24.28$$



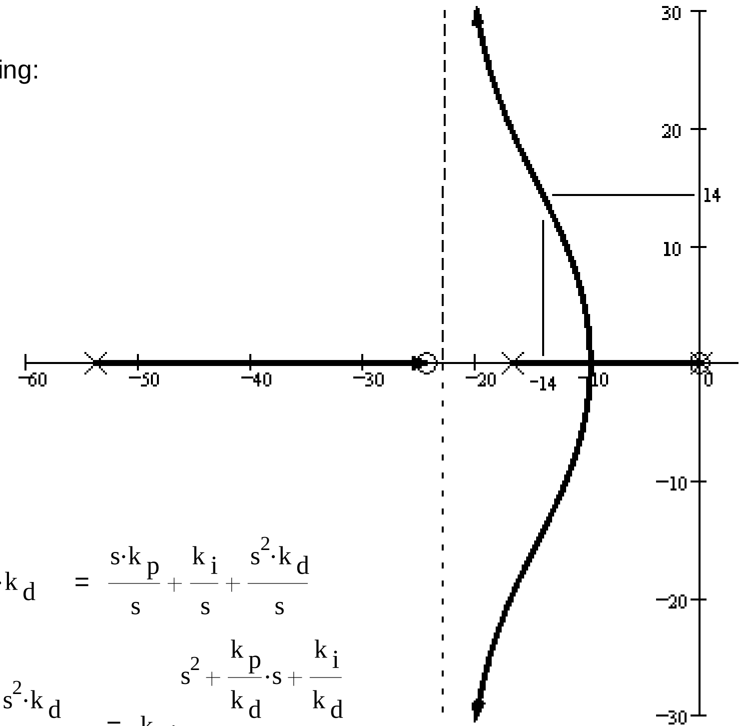
$$\mathbf{G}_c(s) := \frac{1643}{s \cdot (s + 16.64) \cdot (s + 53.78)} \cdot \frac{(s + 0.1) \cdot (s + 24.28)}{s}$$

$$k = \frac{1}{\left|\mathbf{G}(-14 + 14 \cdot j)\right|} = 7.24$$ is the required gain

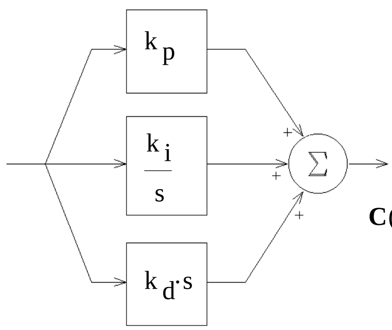
Root Locus Design Examples p.3

We have designed a our compensation with the following:

- A pole at the origin
- A zero at -0.1
- A zero at -24.28
- Gain of 0.418



Find the k_p, k_i, & k_d of a PID controller.



$$\mathbf{C}(s) = k_p + \frac{k_i}{s} + s \cdot k_d \quad = \frac{s \cdot k_p}{s} + \frac{k_i}{s} + \frac{s^2 \cdot k_d}{s}$$

$$= \frac{s \cdot k_p + k_i + s^2 \cdot k_d}{s} \quad = k_d \cdot \frac{s^2 + \frac{k_p}{k_d} \cdot s + \frac{k_i}{k_d}}{s}$$

gain = $k_d := 0.418$

$$(s + 0.1) \cdot (s + 24.28) = s^2 + 24.38 \cdot s + 2.43$$

$$= s^2 + \frac{k_p}{k_d} \cdot s + \frac{k_i}{k_d}$$

$$\frac{k_i}{k_d} = 2.43 \qquad k_i := k_d \cdot 2.43 \qquad k_i = 1.016$$

$$\frac{k_p}{k_d} = 24.38 \qquad k_p := k_d \cdot 24.38 \qquad k_p = 10.191$$

Notice: $\frac{k_i}{k_p} = 0.1 \quad \simeq \; 0.1$

Notice that the proportional gain is actually almost 3 times higher than it was before. $3 \cdot 3.44 = 10.32$

LEAD An alternative to the differentiator is a Lead Compensator.

Instead of a single zero with: $\theta_z = 53.71 \cdot deg$

How about a zero with $\theta_z := 70 \cdot deg$ And a pole with $\theta_p := 70 \cdot deg - 53.71 \cdot deg$

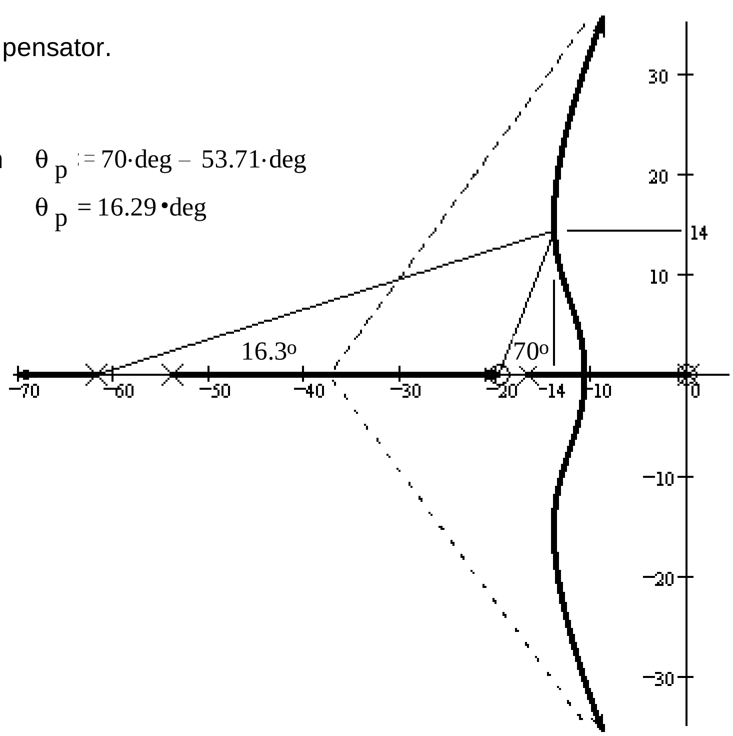
$\theta_p = 16.29 \cdot deg$

$$x = \operatorname{Im}(p) \cdot \frac{1}{\tan(\theta_z)} = 5.096$$

$$z := \operatorname{Re}(p) - \operatorname{Im}(p) \cdot \frac{1}{\tan(\theta_z)} \qquad z = -19.096$$

$$xp = \operatorname{Im}(p) \cdot \frac{1}{\tan(\theta_p)} = 47.907$$

$$p := \operatorname{Re}(p) - \operatorname{Im}(p) \cdot \frac{1}{\tan(\theta_p)} \qquad p = -61.907$$



This example is actually a PI-Lead controller

Problems with the differentiator

1. Tries to differentiate a step input into an impulse -- not likely.
   You'll have to consider how your differentiator will actually handle a step input and how your amplifier will saturate.

   If the differentiator and amplifiers saturate in such a way the the "area under the curve" approximates the impulse "area under the curve", then this may not be such a problem. It may not be as fast as predicted from the linear model, but it may be as fast as the system limits allow. (Pedal-to-the-metal.)

2. It's a high-pass filter and can accentuate noise.
   This is actually common to all compensators that speed up the response.

3. Requires active components and a power supply to build.
   Usually no big deal since your amplifier (source of gain) does too.

4. Is never perfect (always has higher-order poles), but then neither is anything else. Especially in mechanical systems, these poles usually are well beyond where they could cause problems.

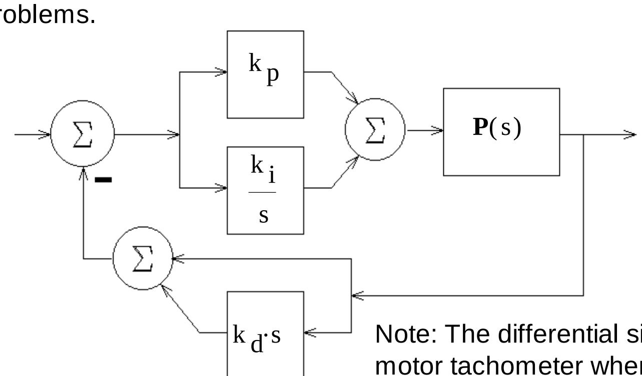
Alternatives:

1. Lag-Lead or PI-Lead compensation. This eliminates the differentiator, but it is still a high-pass filter that can be a noise problem and it could still saturate the amplifier if the input changes too rapidly.

   Be sure to check for saturation problems.

2. Place the differentiator in the feedback loop. The output of the plant is much less likely to be a step or to change so rapidly that it causes problems.

   Differentiation in the feedback



Note: The differential signal is often taken from a motor tachometer when the output is a position. Then you don't need a separate differentiator circuit, just a separate gain for that signal.

Find the k_p, k_i, & k_d of this controller.

$$\mathbf{F}(s)\cdot\mathbf{C}(s) \;=\; \underbrace{\left(k_p + \frac{k_i}{s}\right)}_{\mathbf{C}(s)}\cdot\underbrace{\left(1 + k_d\cdot s\right)}_{\mathbf{F}(s)} \;=\; k_p\cdot k_d\cdot\left[\frac{s + \frac{k_i}{k_p}}{s}\right]\cdot\left(\frac{1}{k_d} + s\right) \;=\; k_p\cdot k_d\cdot\frac{\left(s + \frac{k_i}{k_p}\right)\cdot\left(s + \frac{1}{k_d}\right)}{s}$$

For our example:

$$= 0.418\cdot\frac{(s + 0.1)\cdot(s + 24.28)}{s}$$

$$k_d := \frac{1}{24.38} \qquad k_d = 0.041$$

$$k_p := \frac{0.418}{k_d} \qquad k_p = 10.191$$

$$k_i := k_p\cdot 0.1 \qquad k_i = 1.019$$

In this case the open-loop zero in the feedback loop IS NOT in the closed-loop. This turns out to make the step response slower than predicted by the second-order approximation, but try a simulation, you may be able to use significantly more gain with no more overshoot. The differentiator in this position inhibits overshoot.

PI and PID Design Examples p.5

Ex.2, from S16 Exam 3 Consider the transfer function: $\mathbf{G}(s) := \dfrac{s+5}{(s+1)\cdot(s^2+4\cdot s+20)}$

a) Find the departure angle from a complex pole.

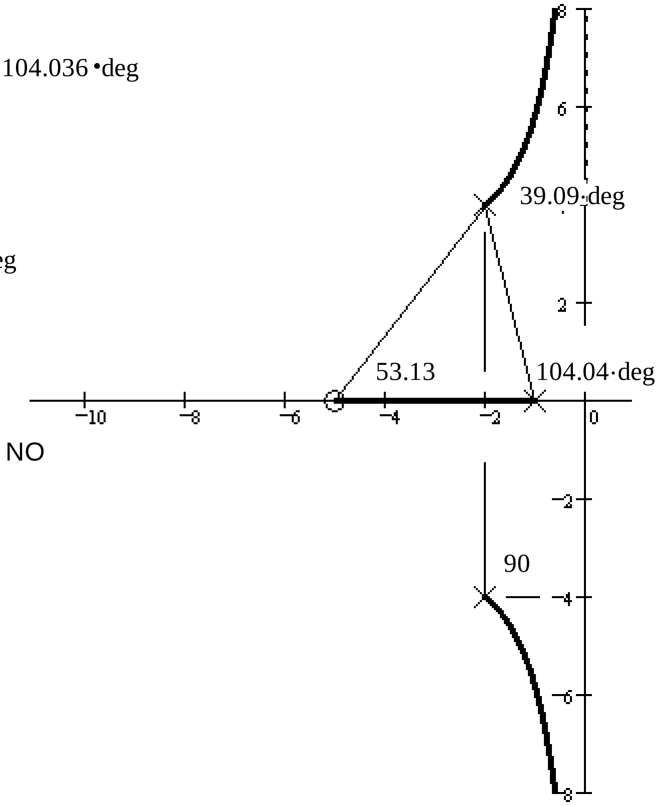
Angles:

from pole at -1 $\theta_{p1} := 180\cdot deg - atan\left(\frac{4}{1}\right)$ $\theta_{p1} = 104.036\cdot deg$

from pole at -2-4j $\theta_{p2} := 90\cdot deg$ $\theta_{p2} = 90\cdot deg$

from zero at -5 $\theta_z := atan\left(\frac{4}{3}\right)$ $\theta_z = 53.13\cdot deg$

$\theta = 53.13\cdot deg - 90\cdot deg - 104.036\cdot deg + 180\cdot deg = 39.094\cdot deg$



b) Draw a root locus plot. Calculate the centroid and accurately draw the departure angle.

$\sigma := \dfrac{5-1-2-2}{2}$ $\sigma = 0$

c) Is there any decent place to locate the closed-loop poles? NO

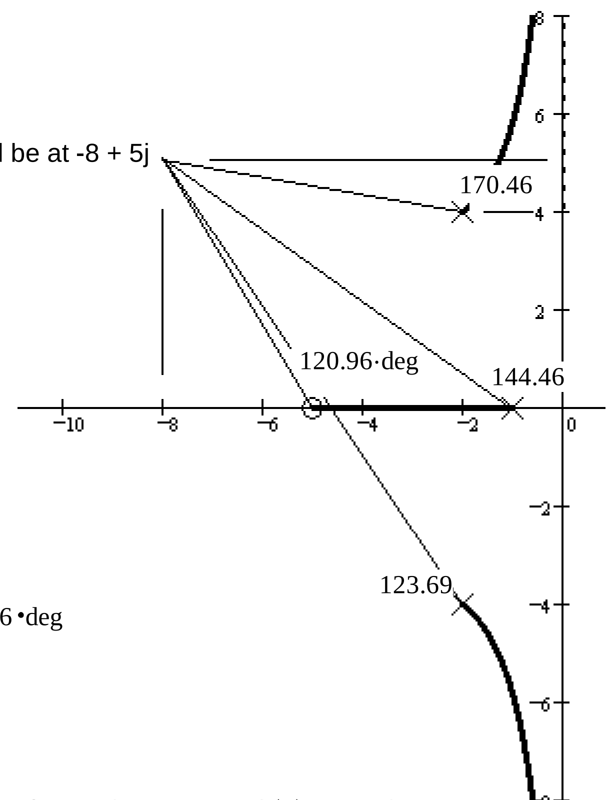
d) You would like to place your closed-loop poles to get a settling time of 1/2 sec and 0.656% overshoot. Add the simplest possible compensator to accomplish this and calculate what the compensator should be.

2% settling time: $T_s = \dfrac{4}{a}$ $a = \dfrac{4}{\left(\frac{1}{2}\right)} = 8$

Overshoot: $OS = e^{-\pi\cdot\frac{a}{b}}$ $\%OS = 100\%\cdot e^{-\pi\cdot\frac{a}{b}}$

$\dfrac{a}{b} = \dfrac{\ln(OS)}{-\pi} = \dfrac{\ln(0.00656)}{-\pi} = 1.6$ $b = \dfrac{8}{1.6} = 5$

Pole should be at -8 + 5j



Angles:

from pole at -1 $180\cdot deg - atan\left(\frac{5}{7}\right) = 144.462\cdot deg$

from pole at -2+4j $180\cdot deg - atan\left(\frac{1}{6}\right) = 170.538\cdot deg$

from pole at -2-4j $180\cdot deg - atan\left(\frac{9}{6}\right) = 123.69\cdot deg$

from zero at -5 $180\cdot deg - atan\left(\frac{5}{3}\right) = 120.964\cdot deg$

$144.462\cdot deg + 170.538\cdot deg + 123.69\cdot deg - 120.964\cdot deg = 317.726\cdot deg$

$\theta_z := 317.726\cdot deg - 180\cdot deg$ $\theta_z = 137.726\cdot deg$

$\tan(137.726\cdot deg - 90\cdot deg) = 1.1 = \dfrac{x}{5}$ $x := 5\cdot 1.1$ $8 - x = 2.5$ $\mathbf{C}(s) = s + 2.5$

$\mathbf{G}_c(s) := \dfrac{(s+5)\cdot(s+2.5)}{(s+1)\cdot(s^2+4\cdot s+20)}$ $s := -8 + 5\cdot j$ Check: $\arg\left[\dfrac{(s+5)\cdot(s+2.5)}{(s+1)\cdot(s^2+4\cdot s+20)}\right] = 180\cdot deg$

e) What is the gain? $k := \dfrac{1}{|G_c(s)|} = \left|\dfrac{(-8+5\cdot j+1)\cdot\left[(-8+5\cdot j)^2+4\cdot(-8+5\cdot j)+20\right]}{(-8+5\cdot j+5)\cdot(-8+5\cdot j+2.5)}\right| = 13.059$

f) What is the steady-state error for a unit-step input?

$$G_c(s) := \frac{(s+5)\cdot(s+2.5)}{(s+1)\cdot(s^2+4\cdot s+20)} \qquad G_c(0) = \frac{(0+5)\cdot(0+2.5)}{(0+1)\cdot(0^2+4\cdot 0+20)} = \frac{(5)\cdot(2.5)}{(1)\cdot(20)} = 0.625$$

$$G_c(0) = 0.625 \qquad e_{step} = \frac{1}{1+k\cdot 0.625} = 10.91\cdot\%$$

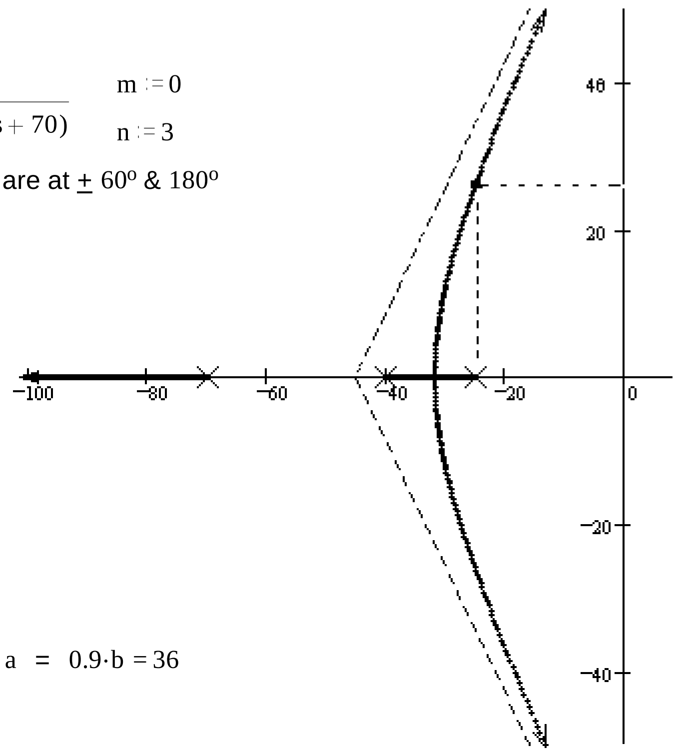
g) If this steady-state error was a little too big, what would be the very simplest way to reduce it?

turn up the gain

Ex.3, from S17 Exam 3

a) Sketch the root locus plot of, $G(s) := \dfrac{100}{(s+25)\cdot(s+40)\cdot(s+70)}$ $\quad m := 0 \quad n := 3$

$\sigma_C = \dfrac{-25+-40+-70}{n-m} = -45 \quad n-m=3$ so asymptotes are at ± 60º & 180º



The gain is set at 452, so that one of the closed-loop poles is at,
$s := -24.48 + 27.2\cdot j$

Further calculations yield:
- Settling time: $0.163\cdot sec$
- % overshoot: $5.92\cdot\%$
- Steady-state error to a unit-step input: 60.8%

b) You wish to increase the frequency of ringing to 40 rad/sec without changing the % overshoot at all. Where should the closed-loop pole be located?

$\dfrac{a}{b} = \dfrac{24.48}{27.2} = 0.9$ new $b := 40$ new $a = 0.9\cdot b = 36$

New location: $s := -36 + 40\cdot j$

c) Add a LEAD compensator so that you will be able to place the closed-loop pole at the location found in b).
Add the new zero at -30. Find the location of the new pole.

Angles:

from pole at -25

$\theta_{25} := 180\cdot deg - atan\left(\dfrac{40}{36-25}\right) \qquad \theta_{25} = 105.376\cdot deg$

from pole at -40

$\theta_{40} := atan\left(\dfrac{40}{40-36}\right) \qquad \theta_{40} = 84.289\cdot deg$

from pole at -70

$\theta_{70} := atan\left(\dfrac{40}{70-36}\right) \qquad \theta_{70} = 49.635\cdot deg$

from new zero at -30

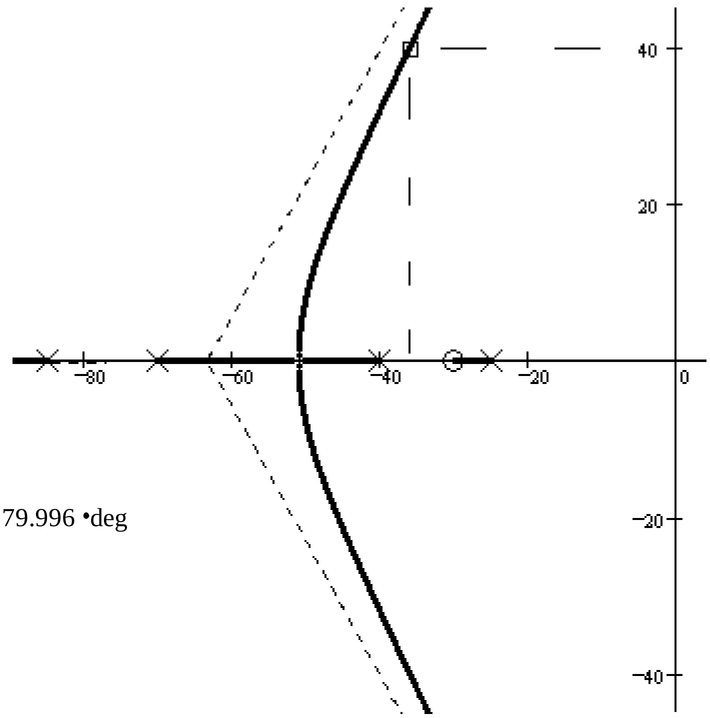
$\theta_{30} := 180\cdot deg - atan\left(\dfrac{40}{36-30}\right) \qquad \theta_{30} = 98.531\cdot deg$

$$\theta_{25} + \theta_{40} + \theta_{70} - \theta_{30} + \theta_p = 180 \cdot deg$$

$$\theta_p := 180 \cdot deg - \theta_{25} - \theta_{40} - \theta_{70} + \theta_{30}$$

$$\theta_p = 39.23 \cdot deg$$

$$p := 36 + \frac{40}{\tan(\theta_p)} \qquad p = 84.993 = 85$$



$$G_c(s) := \frac{100 \cdot (s+30)}{(s+25)\cdot(s+40)\cdot(s+70)\cdot(s+85)}$$

Check: $\arg\left[\frac{100 \cdot (s+30)}{(s+25)\cdot(s+40)\cdot(s+70)\cdot(s+85)}\right] = -179.996 \cdot deg$

d) With the compensator in place and a closed-loop pole at the location desired in part b)

i) What is the gain? $k := \frac{1}{|G_c(s)|}$ $k = 1369$

ii) What is the 2% settling time? Use the second-order approximation.

$$T_s = \frac{4}{36} = 0.111 \text{ sec}$$

iii) What is the steady-state error to a unit-step input?

$$G_c(0) = \frac{100 \cdot (0+30)}{(0+25)\cdot(0+40)\cdot(0+70)\cdot(0+85)} = 5.042 \cdot 10^{-4} \qquad e_{step} = \frac{1}{1 + k \cdot G_c(0)} = 59.161 \cdot \%$$

e) Add another compensator: $C_2(s) := \frac{s+2}{s}$ and maintain the gain of part d)

i) What is this type of compensator called and what is its purpose?

PI, used to eliminate steady-state error

ii) Calculate what you need to to show that this compensator achieved its purpose.

$$G_c(s) := \frac{100 \cdot (s+30)}{(s+25)\cdot(s+40)\cdot(s+70)\cdot(s+85)} \cdot \frac{(s+2)}{s}$$

$$G_c(0) = \infty \qquad e_{step} = \frac{1}{1 + k \cdot \infty} = 0 \cdot \%$$

f) With both compensators in place, is there possibility for improvement (quicker settling time speed and/or lower ringing)? If yes, what would be the simplest thing to do? Justify your answer.

A quick sketch of the new root-locus shows that simply decreasing the gain would improve the system