ECE 3510 Electrical Analogies of Mechanical Systems

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2/07/06,
rev,

This method is based on the method of Linear Graphs. In Linear Graphs, all systems are reduced to a universal symbology peculiar to Linear Graphs (system graph) and then analyzed by methods very similar to those used with electric circuits. Since we, as electrical engineers, are already well versed in the symbology of electric circuits, my variation of linear graphs reduces mechanical systems to electric circuits and leaves them in that symbology for ready analysis by methods, tools and computer programs that we are already familiar with.

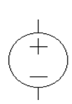
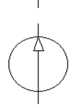
Across and Through Variables

	Across Variable	**Through Variable**
Electrical	V = voltage (volts) or (V)	I = current (Amps) or (A)
Mechanical translational	v = velocity $\left(\frac{m}{sec}\right)$	F = force (newtons) or (N) or $\left(Kg\cdot\frac{m}{sec^2}\right)$
Mechanical rotational	ω = angular velocity $\left(\frac{rad}{sec}\right)$	T = torque $(N\cdot m)$
Fluid	P = pressure $\left(\frac{N}{m^2}\right)$ or (Pa)	Q = flow $\left(\frac{m^3}{sec}\right)$

Elements

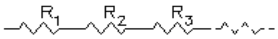
	Dissipation	**Across Variable Energy Storage**	**Through Variable Energy Storage**
Electrical	R = resistance $\left(\frac{V}{A}\right)$ or (Ω)	C = capacitance $\left(\frac{A\cdot sec}{V}\right)$ or (F)	L = inductor $\left(\frac{V\cdot sec}{A}\right)$ or (H)
Mechanical translational	B = damping $\left(\frac{N\cdot sec}{m}\right)$	M = mass (Kg) or $\left(\frac{N\cdot sec^2}{m}\right)$	k = Spring constant $\left(\frac{N}{m}\right)$
Mechanical rotational	B = damping $(N\cdot m\cdot sec)$ or $\left[\frac{N\cdot m}{\left(\frac{rad}{sec}\right)}\right]$	J = moment of inertia $\left(Kg\cdot m^2\right)$ or $\left(\frac{N\cdot m^3}{sec^2}\right)$	k = Spring constant $\left(\frac{N\cdot m}{rad}\right)$
Fluid	R_f = fluid resistance $\left(\frac{N\cdot sec}{m^5}\right)$	C_f = fluid capacitance $\left(\frac{m^5}{N}\right)$	I = fluid inertia $\left(\frac{Kg}{m^4}\right)$

Basic Electric Circuit Analysis

Element	Parts like resistors, capacitors, inductors & transformers
Wires and connections	Direct the current, but do not affect voltage
Circuit	Wires and elements connected to form loops
Voltage	Measured as a difference **across** an element
Current	Flows **through** a wire or element
Kirchhoff's Current Law (KCL)	Current in = current out of all elements, wires & connections
Kirchhoff's Voltage Law (KVL)	Voltage gains = voltage "losses" around any circuit loop
Node	Connected wires and connections which all have the same voltage
Ground	Zero-reference node for all other nodal voltages
Branch	Connected wires and elements which all have the same current
Power $P = V\cdot I$	Power = Across variable x Through variable
Voltage Source 	Constant voltage regardless of current in or out
Current Source 	Constant current regardless of voltage + or -

Passive Electrical Elements

Resistors



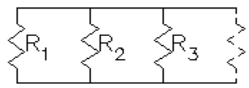
series: $R_{eq} = R_1 + R_2 + R_3 + \ldots$

Exactly the **same current** through each resistor

voltage divider:

$$V_{Rn} = V_{total} \cdot \frac{R_n}{R_1 + R_2 + R_3 + \ldots}$$

parallel: $R_{eq} = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \ldots}$



Exactly the **same voltage** across each resistor

current divider:

$$I_{Rn} = I_{total} \cdot \frac{\dfrac{1}{R_n}}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \ldots}$$

Resistors dissipate power $P = V \cdot I = I^2 \cdot R = \dfrac{V^2}{R}$

Capacitors

$C = \dfrac{Q}{V}$ $\quad$ $\text{farad} = \dfrac{\text{coul}}{\text{volt}} = \dfrac{\text{amp} \cdot \text{sec}}{\text{volt}}$

$$v_C = \frac{1}{C} \cdot \int_{-\infty}^{t} i_C \, dt = \frac{1}{C} \cdot \int_{0}^{t} i_C \, dt + v_C(0)$$

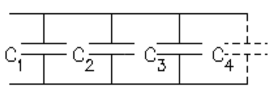
(initial voltage: $v_C(0)$)

$$i_C = C \cdot \frac{d}{dt} v_C$$

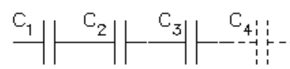
Energy stored in electric field: $E_C = \dfrac{1}{2} \cdot C \cdot V_C^{\,2}$

Capacitor voltage **cannot** change instantaneously

parallel: $C_{eq} = C_1 + C_2 + C_3 + \ldots$



series: $C_{eq} = \dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2} + \dfrac{1}{C_3} + \ldots}$



Steady-state sinusoids:
Impedance: $\mathbf{Z_C} = \dfrac{1}{j \cdot \omega \cdot C} = \dfrac{-j}{\omega \cdot C}$

Current leads voltage by 90 deg

Laplace:
Impedance: $\mathbf{Z_C} = \dfrac{1}{C \cdot s}$

Inductors

$\text{henry} = \dfrac{\text{volt} \cdot \text{sec}}{\text{amp}}$

$$i_L = \frac{1}{L} \cdot \int_{-\infty}^{t} v_L \, dt = \frac{1}{L} \cdot \int_{0}^{t} v_L \, dt + i_L(0)$$

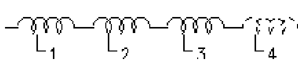
(initial current: $i_L(0)$)

$$v_L = L \cdot \frac{d}{dt} i_L$$

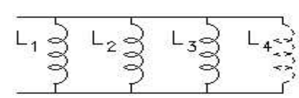
Energy stored in magnetic field: $E_L = \dfrac{1}{2} \cdot L \cdot I_L^{\,2}$

Inductor current **cannot** change instantaneously

series: $L_{eq} = L_1 + L_2 + L_3 + \ldots$



parallel: $L_{eq} = \dfrac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2} + \dfrac{1}{L_3} + \ldots}$

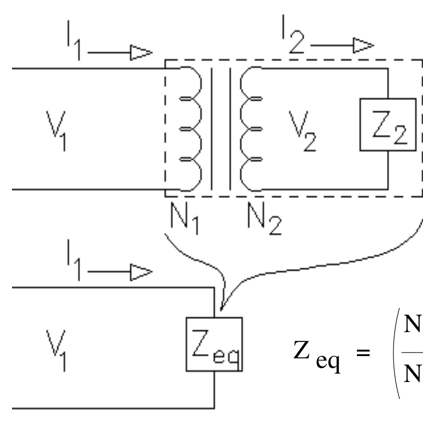


Steady-state sinusoids:
Impedance: $\mathbf{Z_L} = j \cdot \omega \cdot L$

Current lags voltage by 90 deg

Laplace:
Impedance: $\mathbf{Z_L} = L \cdot s$

Transformers (ideal)



$Z_{eq} = \left(\dfrac{N_1}{N_2}\right)^2 \cdot Z_2$

Ideal: $P_1 = P_2$ $\quad$ power in = power out

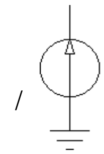
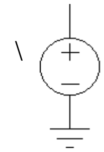
Turns ratio $= N = \dfrac{N_1}{N_2} = \dfrac{V_1}{V_2} = \dfrac{I_2}{I_1}$

Note: some books define the turns ratio as N_2/N_1

Equivalent impedance in primary: $\mathbf{Z_{eq}} = N^2 \cdot \mathbf{Z_2} = \left(\dfrac{N_1}{N_2}\right)^2 \cdot \mathbf{Z_2}$

You can replace the entire transformer and load with ($\mathbf{Z_{eq}}$).
This "impedance transformation" can work across systems.

Mechanical system with linear motion (translational)

	Mechanical translational	Electrical	
Through Variable:	F = Force (N)	I = current (A)	Source: 
Across Variable:	v = velocity $\left(\frac{m}{sec}\right)$	V = voltage (V)	Source: 
$\int v\,dt$	x = displacement (m)		
$\frac{V(s)}{s}$	$X(s)$ = displacement $(m \cdot sec)$ (in freq domain)		

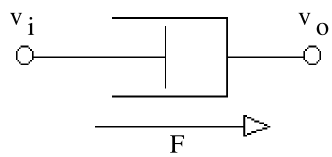
Dissipation element:

power

$$P = v \cdot F = \frac{F^2}{B}$$
$$= v^2 \cdot B$$

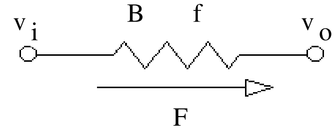
Damper or friction

B or f



Resistor

$\frac{1}{B}$ or $\frac{1}{f}$



Impedance

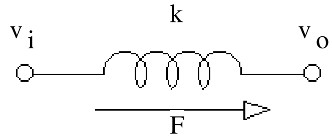
$\frac{1}{B}$ or $\frac{1}{f}$

Through variable energy storage:

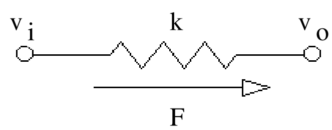
$$E = \frac{1}{2} \cdot \frac{1}{k} \cdot F^2$$

Spring

k

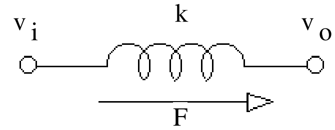


Springs are Sometimes shown like this:



Inductor

$\frac{1}{k}$

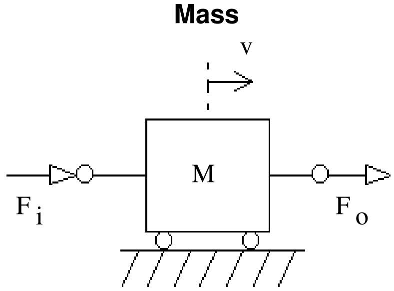


$\frac{s}{k}$

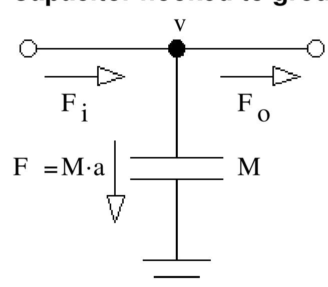
Through variable energy storage:

$$E = \frac{1}{2} \cdot M \cdot v^2$$

Mass

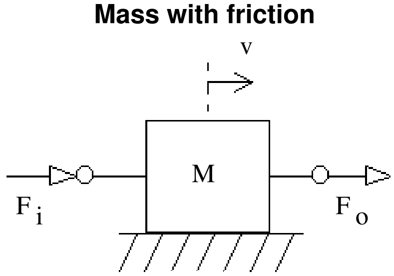


Capacitor hooked to ground



$\frac{1}{M \cdot s}$

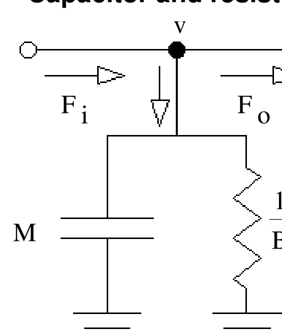
Mass with friction



B or f

sliding friction

Capacitor and resistor

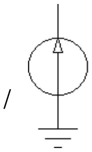
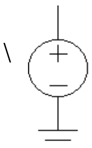


$\frac{1}{B}$ or $\frac{1}{f}$

$$\frac{1}{\frac{1}{\left(\frac{1}{M \cdot s}\right)} + \frac{1}{\left(\frac{1}{B}\right)}}$$

$$\frac{1}{M \cdot s + B}$$

Mechanical system with circular motion (rotational)

	Mechanical rotational	**Electrical**	
Through Variable:	T = Torque $(N \cdot m)$	I = current (A)	Source: 
Across Variable:	ω = angular velocity $\left(\frac{rad}{sec}\right)$	V = voltage (V)	Source: 
$\int \omega\, dt$	θ = angular displacement (rad)		
$\frac{\omega(s)}{s}$	$\theta(s)$ = angular displacement $(rad \cdot sec)$ (in freq domain)		

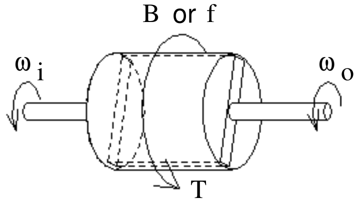
Dissipation element:

power

$$P = v \cdot T = \frac{T^2}{B} = \omega^2 \cdot B$$

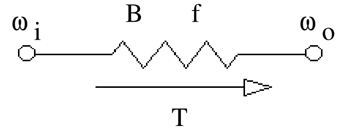
Damper or friction

B or f



Resistor

$\frac{1}{B}$ or $\frac{1}{f}$



Impedance

$\frac{1}{B}$ or $\frac{1}{f}$

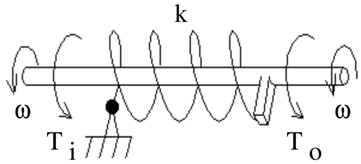
Through variable energy storage:

$$E = \frac{1}{2} \cdot \frac{1}{k} \cdot T^2$$

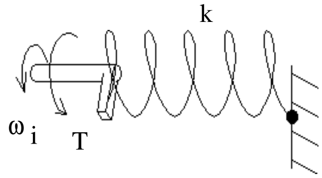
Springs

k

k

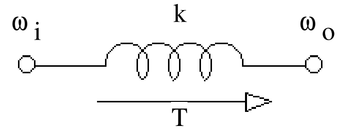


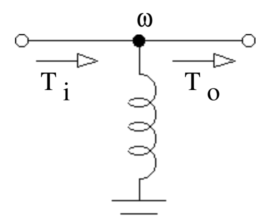
k

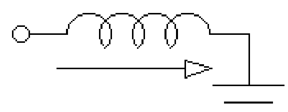


Inductor

$\frac{1}{k}$





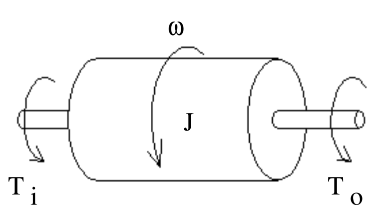


Impedance: $\frac{s}{k}$

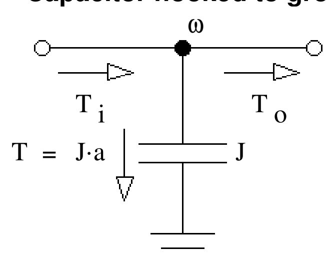
Through variable energy storage:

$$E = \frac{1}{2} \cdot J \cdot \omega^2$$

Moment of Inertia, J

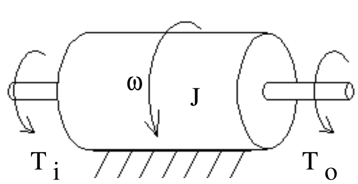


Capacitor hooked to ground



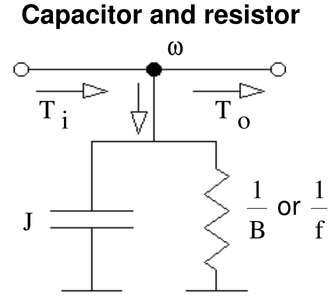
Impedance: $\frac{1}{J \cdot s}$

J with friction



B or f
sliding friction

Capacitor and resistor



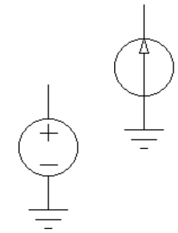
Impedance:

$$\frac{1}{\frac{1}{\left(\frac{1}{J \cdot s}\right)} + \frac{1}{\left(\frac{1}{B}\right)}}$$

$$\frac{1}{J \cdot s + B}$$

Fluid (hydraulic) system

Sources

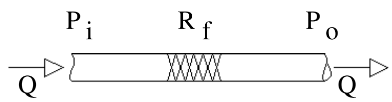


	Fluid	**Electrical**
Through Variable:	Q = volumetric flow rate $\left(\frac{m^3}{sec}\right)$	I = current (A)
Across Variable:	P = Pressure $\left(\frac{N}{m^2}\right)$ or (Pa)	V = voltage (V)

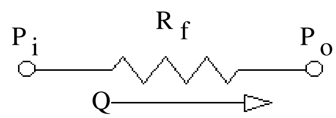
Dissipation element: power

$$P = P\cdot Q = \frac{Q^2}{R_f} = P^2\cdot R_f$$

Fluid resistance



Resistor

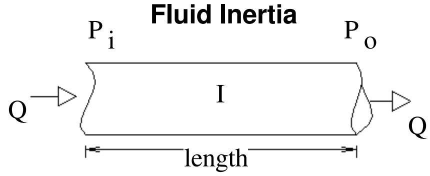


Impedance: R_f

Through variable energy storage:

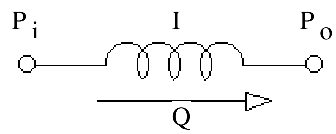
$$E = \frac{1}{2}\cdot I\cdot Q^2$$

Fluid Inertia



$$I = \frac{M}{A^2} = \frac{\rho\cdot length}{A}$$

Inductor

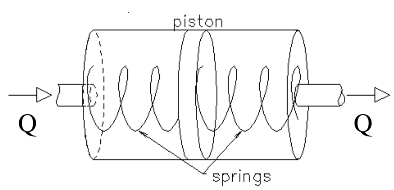


Impedance: $I\cdot s$

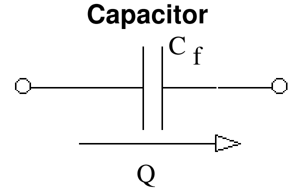
Through variable energy storage:

$$E = \frac{1}{2}\cdot C_f\cdot P^2$$

Fluid Capacitors

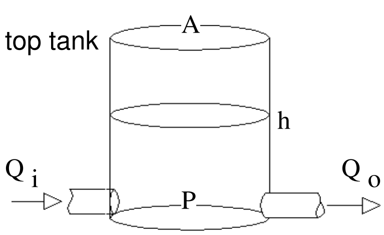


Capacitor

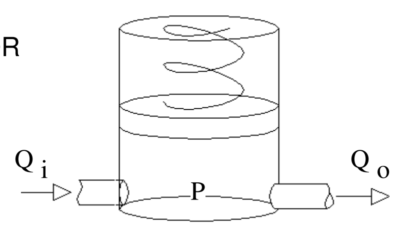


Impedance: $\frac{1}{C_f\cdot s}$

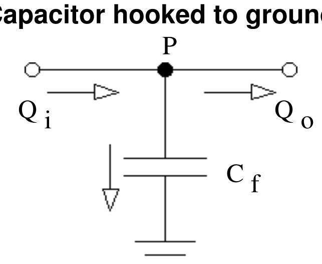
open top tank



OR



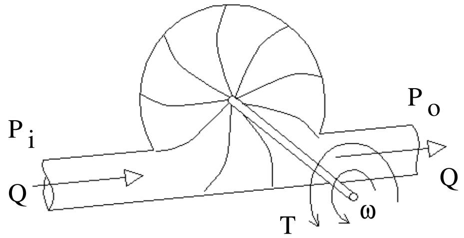
Capacitor hooked to ground



$$C_f = \frac{\Delta volume}{\Delta pressure} = \frac{\Delta h\cdot A}{\Delta h\cdot p\cdot g} = \frac{A}{p\cdot g}$$

For open top tank

Turbine or Pump



$$T = K\left(P_o - P_i\right)$$

$$\omega = \frac{1}{K}\cdot Q$$

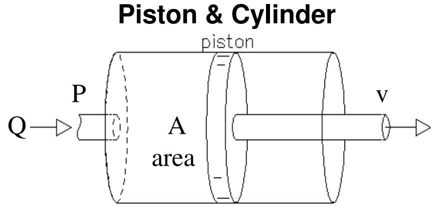
Turbines & pistons convert through variables to across variables & vice versa, so there are no good electrical analogies.

Yet you can still transform an impedance from a mechanical system into the fluid system. But you will find that capacitors become inductors, inductors become capacitors and parallel swaps with series.

Piston & Cylinder

$$P = \frac{F}{A}$$

$$Q = A\cdot v$$



$$v = \frac{Q}{A}$$

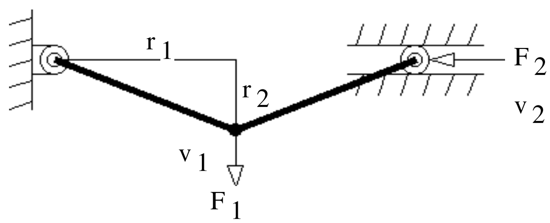
$$F = A\cdot P$$

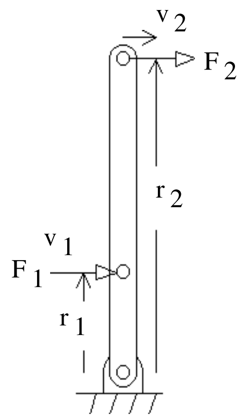
$$\mathbf{Z}_{eq} = \frac{\left(\frac{F}{A}\right)}{A\cdot v} = \frac{1}{A^2}\cdot\frac{F}{v} = \frac{1}{A^2}\cdot\frac{1}{\mathbf{Z}_2} = \frac{1}{A^2\cdot\mathbf{Z}_2}$$

Transducers and Transformers

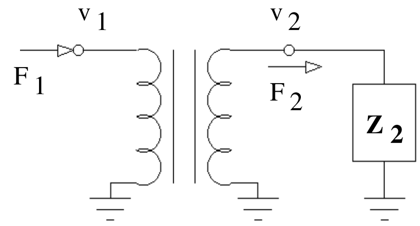
A transducer converts power from one type to another. We can model many of them with transformers.
Transformers increase the through variable and correspondingly decrease the across variable or vice-versa.

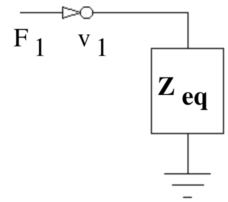
Levers





$$N = \frac{r_1}{r_2} = \frac{v_1}{v_2} = \frac{F_2}{F_1}$$

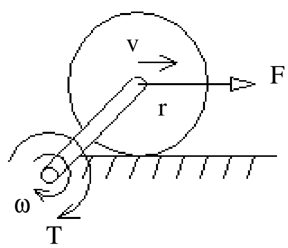


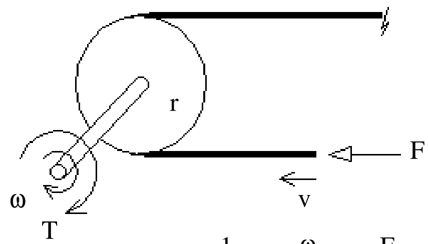


$$\mathbf{Z}_{eq} = N^2 \cdot \mathbf{Z}_2 = \left(\frac{r_1}{r_2}\right)^2 \cdot \mathbf{Z}_2$$

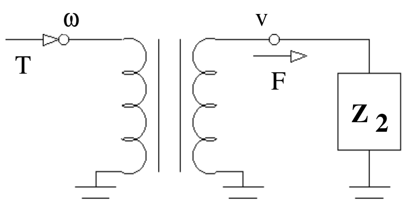
Tires, racks, & conveyors

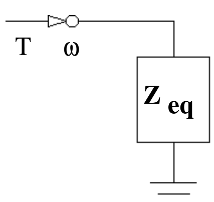
r = radius of wheel or pitch radius of pinion gear





$$N = \frac{1}{r} = \frac{\omega}{v} = \frac{F}{T}$$

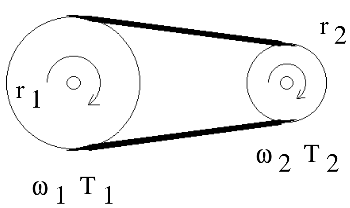


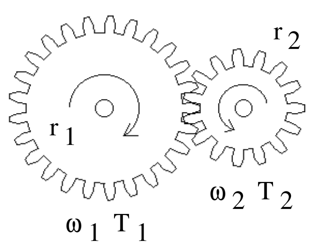


$$\mathbf{Z}_{eq} = N^2 \cdot \mathbf{Z}_2 = \left(\frac{1}{r}\right)^2 \cdot \mathbf{Z}_2$$

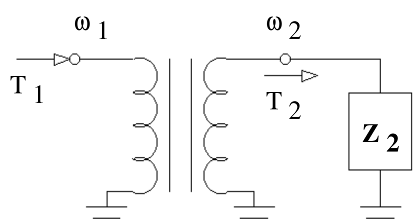
Belts, chains, & gears

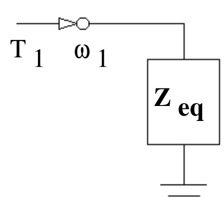
r = radius of wheel or pitch radius of gears





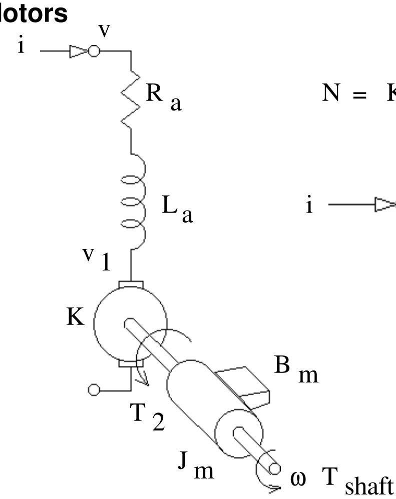
$$N = \frac{r_2}{r_1} = \frac{\omega_1}{\omega_2} = \frac{T_2}{T_1}$$





$$\mathbf{Z}_{eq} = N^2 \cdot \mathbf{Z}_2 = \left(\frac{r_2}{r_1}\right)^2 \cdot \mathbf{Z}_2$$

DC Motors



$$N = K = \frac{v_1}{\omega} = \frac{T_2}{i}$$