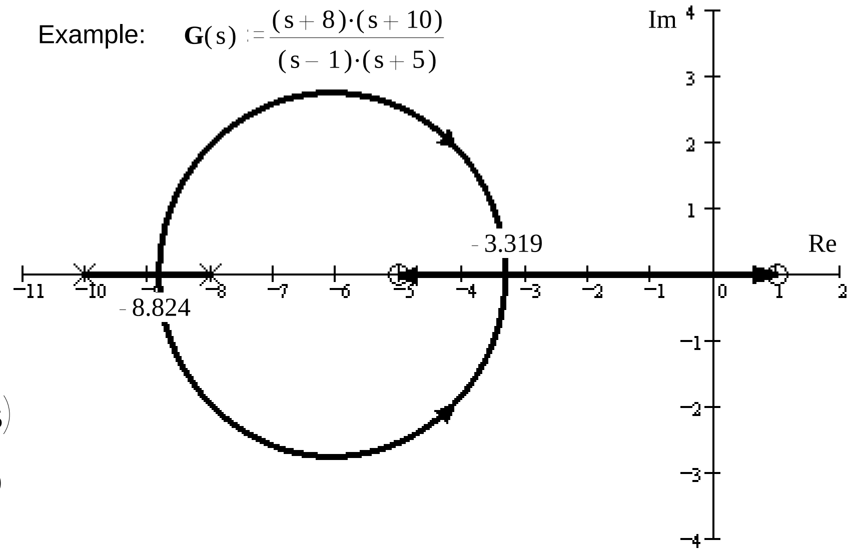
A.Stolp
2/27/11,
2/25/24

ECE 3510 Root Locus Examples continued (Additional rules)

10. The breakaway (and arrival) points are solutions to:

$$\sum_{all} \frac{1}{(s + -p_i)} = \sum_{all} \frac{1}{(s + -z_i)}$$

Example: $\mathbf{G}(s) := \dfrac{(s+8)\cdot(s+10)}{(s-1)\cdot(s+5)}$



Solve: $\dfrac{1}{s+5} + \dfrac{1}{s-1} = \dfrac{1}{s+8} + \dfrac{1}{s+10}$

$$\frac{(s-1)+(s+5)}{(s+5)\cdot(s-1)} = \frac{(s+8)+(s+10)}{(s+8)\cdot(s+10)}$$

$$\frac{2\cdot s+4}{s^2+4\cdot s-5} = \frac{2\cdot s+18}{s^2+18\cdot s+80}$$

$$(2\cdot s+4)\cdot\left(s^2+18\cdot s+80\right) = (2\cdot s+18)\cdot\left(s^2+4\cdot s-5\right)$$

$$2\cdot s^3+40\cdot s^2+232\cdot s+320 = 2\cdot s^3+26\cdot s^2+62\cdot s-90$$

$$40\cdot s^2+232\cdot s+320 = 26\cdot s^2+62\cdot s-90$$

$$14\cdot s^2+170\cdot s+410 = 0$$

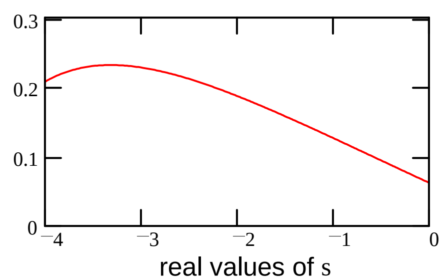
$$\frac{-170-\sqrt{170^2-4\cdot 14\cdot 410}}{2\cdot 14} = -8.824 \qquad \frac{-170+\sqrt{170^2-4\cdot 14\cdot 410}}{2\cdot 14} = -3.319$$

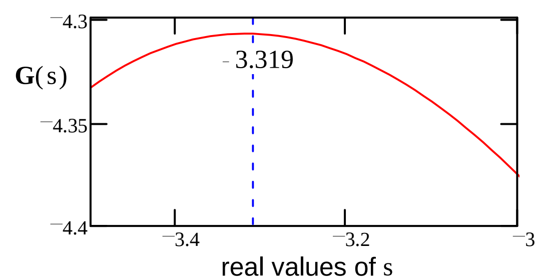
Breakaway (and arrival) points from the real axis (σ_b) are also the solutions to: $\dfrac{d}{ds}\mathbf{G}(s) = 0$ where s is a real number (on the real axis)

Why? Because gain = $k = -\dfrac{1}{\mathbf{G}(s)}$ The breakaway point will be the point between -5 and +1 with the highest gain. That is also the point with the lowest $\mathbf{G}(s)$ and highest - $\mathbf{G}(s)$. Either way $\dfrac{d}{ds}\mathbf{G}(s)$

Make some plots $s := -4, -3.99 .. 0$

gain = $k = -\dfrac{1}{\mathbf{G}(s)}$

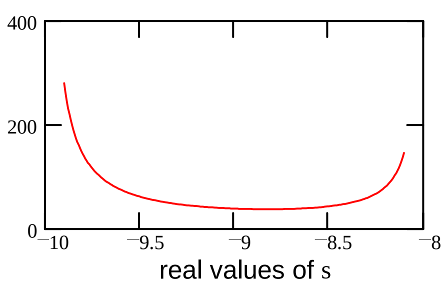


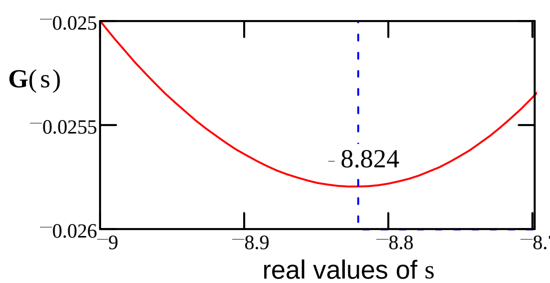


The breakin point will be the point between -10 and -8 with the lowest gain.

Make some plots $s := -9.9, -9.893 .. -8.1$

gain = $k = -\dfrac{1}{\mathbf{G}(s)}$





Root Locus Examples, continued p2

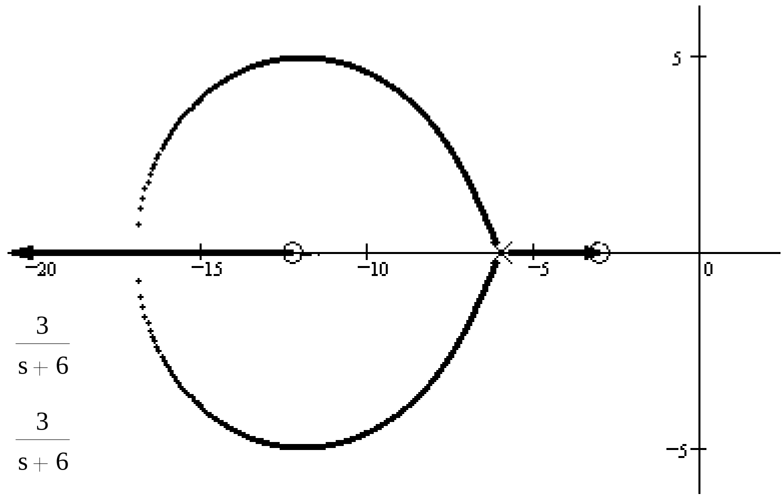
Find the Break-in points for Basic Root Locus Examples, Example 11

11 $\mathbf{G}(s) = \dfrac{(s+3)\cdot(s+12)}{(s+6)^3}$ $m := 2$

$n := 3$

$n - m = 1$

no asymptotes



Break-away points

$$\frac{1}{(s+3)} + \frac{1}{(s+12)} = \frac{1}{s+6} + \frac{1}{s+6} + \frac{1}{s+6} = \frac{3}{s+6}$$

$$\frac{(s+12)+(s+3)}{(s+3)\cdot(s+12)} = \frac{(2\cdot s+15)}{(s+3)\cdot(s+12)} = \frac{3}{s+6}$$

$$(2\cdot s+15)\cdot(s+6) = 3\cdot(s+3)\cdot(s+12)$$

$$0 = (2\cdot s+15)\cdot(s+6) - 3\cdot(s+3)\cdot(s+12)$$

$s^2 + 18\cdot s + 18$ Solve: $\begin{pmatrix} -9+3\cdot\sqrt{7} \\ -9-3\cdot\sqrt{7} \end{pmatrix} = \begin{pmatrix} -1.063 \\ -16.937 \end{pmatrix}$ Useless solution

Breaks in at -16.937

Finding the jω crossing point using rule 9:

Rule 9. Phase angle of G(s) at any point s on the root locus: $\arg(G(s)) = \arg(N(s)) - \arg(D(s)) = \pm 180^o$ $\pm 540^o$ ...

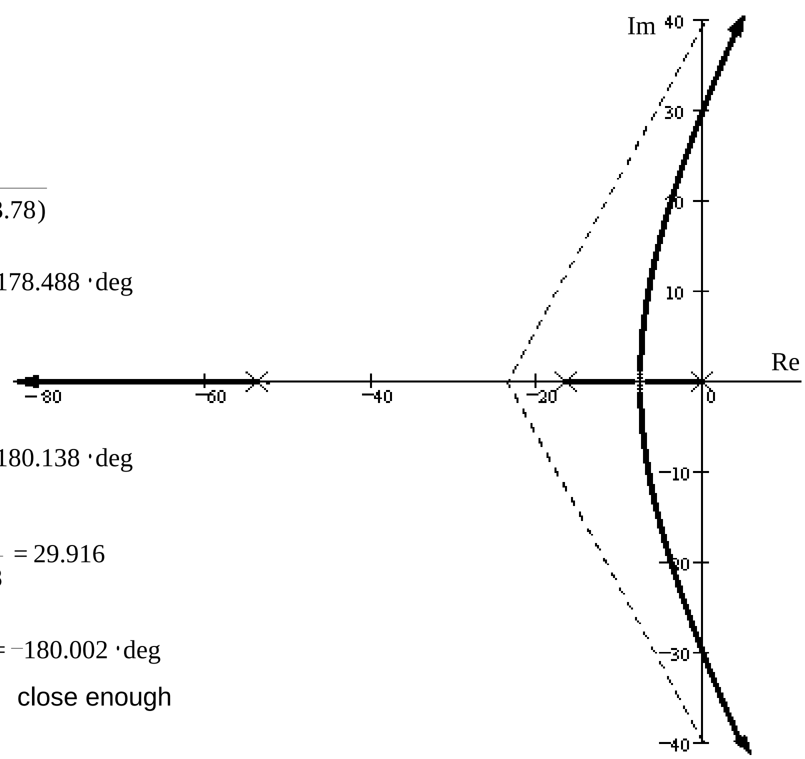
3. Crude servo: $\mathbf{G}(s) = \dfrac{1643}{s\cdot(s+16.64)\cdot(s+53.78)}$

Like example 3 from Basic Examples

I think it crosses the imaginary axis at 29j

$s := 29\cdot j$ $\mathbf{G}(s) = \dfrac{1643}{29\cdot j\cdot(29\cdot j+16.64)\cdot(29\cdot j+53.78)}$

$$\angle\mathbf{G}(s) = -90\cdot deg - atan\left(\frac{29}{16.64}\right) - atan\left(\frac{29}{53.78}\right) = -178.488\cdot deg$$



Try 30j:

$$\angle\mathbf{G}(s) = -90\cdot deg - atan\left(\frac{30}{16.64}\right) - atan\left(\frac{30}{53.78}\right) = -180.138\cdot deg$$

linear interpolation $30 - \dfrac{180.138 - 180}{180.138 - 178.488} = 29.916$

$$\angle\mathbf{G}(s) = -90\cdot deg - atan\left(\frac{29.916}{16.64}\right) - atan\left(\frac{29.916}{53.78}\right) = -180.002\cdot deg$$

close enough

Finding the jω crossing gain using rule 8:

Gain: $\dfrac{1}{|\mathbf{G}(s)|} = \dfrac{29.916\cdot\sqrt{29.916^2+16.64^2}\cdot\sqrt{29.916^2+53.78^2}}{1643} = 38$

Root Locus Examples, continued p3

Find the Break-in points for Basic Root Locus Examples, Example 7

7. $\mathbf{G}(s) = \dfrac{(s+5)\cdot(s+8)}{s^2 - 6\cdot s + 13}$

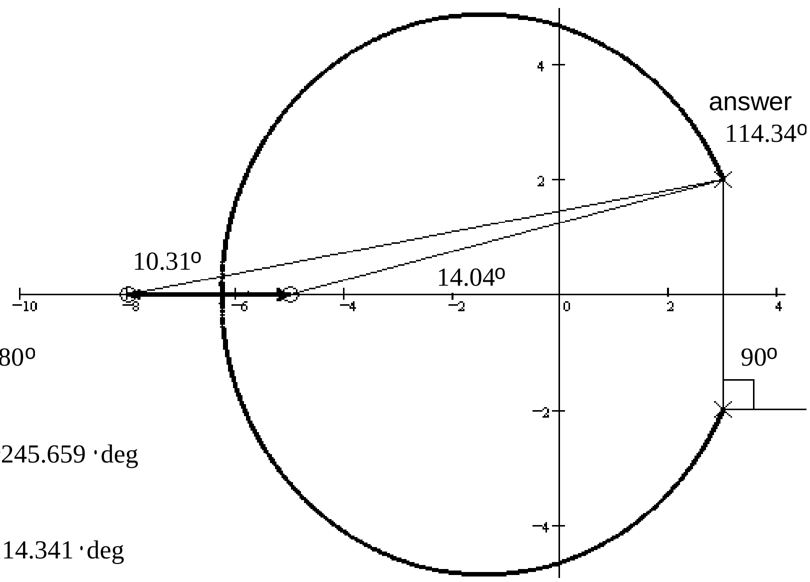
Break-away points $\dfrac{1}{(s+5)} + \dfrac{1}{(s+8)} = \dfrac{1}{s-3-2\cdot j} + \dfrac{1}{s-3+2\cdot j} = \dfrac{2\cdot s - 6}{s^2 - 6\cdot s + 13}$ **NOT** $\neq$ $\dfrac{1}{s^2 - 6\cdot s + 13}$

Note the way these poles are expressed

Guess -6.3 Use this guess in all but the closest poles and zeroes

$\dfrac{1}{s-3-2\cdot j} + \dfrac{1}{s-3+2\cdot j} - \dfrac{2\cdot(-6.3)-6}{s^2 - 6\cdot(-6.3)+13} = 0$ Solutions: $\begin{pmatrix} 2.57 \\ -6.3 \end{pmatrix}$ guess was good

Find the Departure angles from complex poles for Basic Root Locus Examples, Example 7



$\operatorname{atan}\left(\dfrac{2}{8+3}\right) = 10.305\cdot\deg$

$\operatorname{atan}\left(\dfrac{2}{5+3}\right) = 14.036\cdot\deg$

$\operatorname{atan}\left(\dfrac{2}{8+3}\right) + \operatorname{atan}\left(\dfrac{2}{5+3}\right) - 90\cdot\deg - \theta = \pm 180^\circ$

$\operatorname{atan}\left(\dfrac{2}{8+3}\right) + \operatorname{atan}\left(\dfrac{2}{5+3}\right) - 90\cdot\deg - 180\cdot\deg = -245.659\cdot\deg$

$\operatorname{atan}\left(\dfrac{2}{8+3}\right) + \operatorname{atan}\left(\dfrac{2}{5+3}\right) - 90\cdot\deg + 180\cdot\deg = 114.341\cdot\deg$

better answer

Finding the jω crossing point using rule 9:

$\mathbf{G}(s) := \dfrac{(s+5)\cdot(s+8)}{s^2 - 6\cdot s + 13}$

Try: $s := 5\cdot j$ $\angle\mathbf{G}(s) = \arg(\mathbf{G}(5\cdot j)) = -171.193\cdot\deg$

Try: $s := 4.5\cdot j$ $\angle\mathbf{G}(s) = \arg(\mathbf{G}(s)) = 176.375\cdot\deg$ $\arg(\mathbf{G}(s)) - 360\cdot\deg = -183.625\cdot\deg$

linear interpolation $4.5 + \dfrac{183.625 - 180}{183.625 - 171.193}\cdot(5 - 4.5) = 4.646$

Try: $s := 4.646\cdot j$ $\angle\mathbf{G}(s) = \arg(\mathbf{G}(s)) = -179.838\cdot\deg$

linear interpolation $4.5 + \dfrac{183.625 - 180}{183.625 - 179.838}\cdot(4.646 - 4.5) = 4.64$

Try: $s := 4.64\cdot j$ $\angle\mathbf{G}(s) = \arg(\mathbf{G}(s)) = -179.991\cdot\deg$ close enough

Finding the jω crossing gain using rule 8:

Gain: $\dfrac{1}{|\mathbf{G}(s)|} = \dfrac{(4.64\cdot j)^2 - 6\cdot(4.64\cdot j) + 13}{(4.64\cdot j + 5)\cdot(4.64\cdot j + 8)} = \dfrac{\sqrt{\left[13 - (4.64)^2\right]^2 + (6\cdot(4.64))^2}}{\sqrt{4.64^2 + 5^2}\cdot\sqrt{4.64^2 + 8^2}} = 0.462$ to be stable: $k > 0.462$

Find the Break-in points for Basic Root Locus Examples, Example 9

9 $\mathbf{G}(s) = \dfrac{s+12}{\left(s^2+4\cdot s+13\right)\cdot(s+1)\cdot(s+5)}$

Break-away points

$$\frac{1}{(s+12)} = \frac{1}{(s+2+3\cdot j)} + \frac{1}{(s+2-3\cdot j)} + \frac{1}{(s+1)} + \frac{1}{(s+5)} \qquad = \frac{(s+2-3\cdot j)+(s+2+3\cdot j)}{s^2+4\cdot s+13} + \frac{1}{(s+1)} + \frac{1}{(s+5)}$$

$$\frac{1}{(s+12)} = \frac{2\cdot s+4}{s^2+4\cdot s+13} + \frac{1}{(s+1)} + \frac{1}{(s+5)}$$

Guess -4 Use this guess in all but the closest poles and zeroes

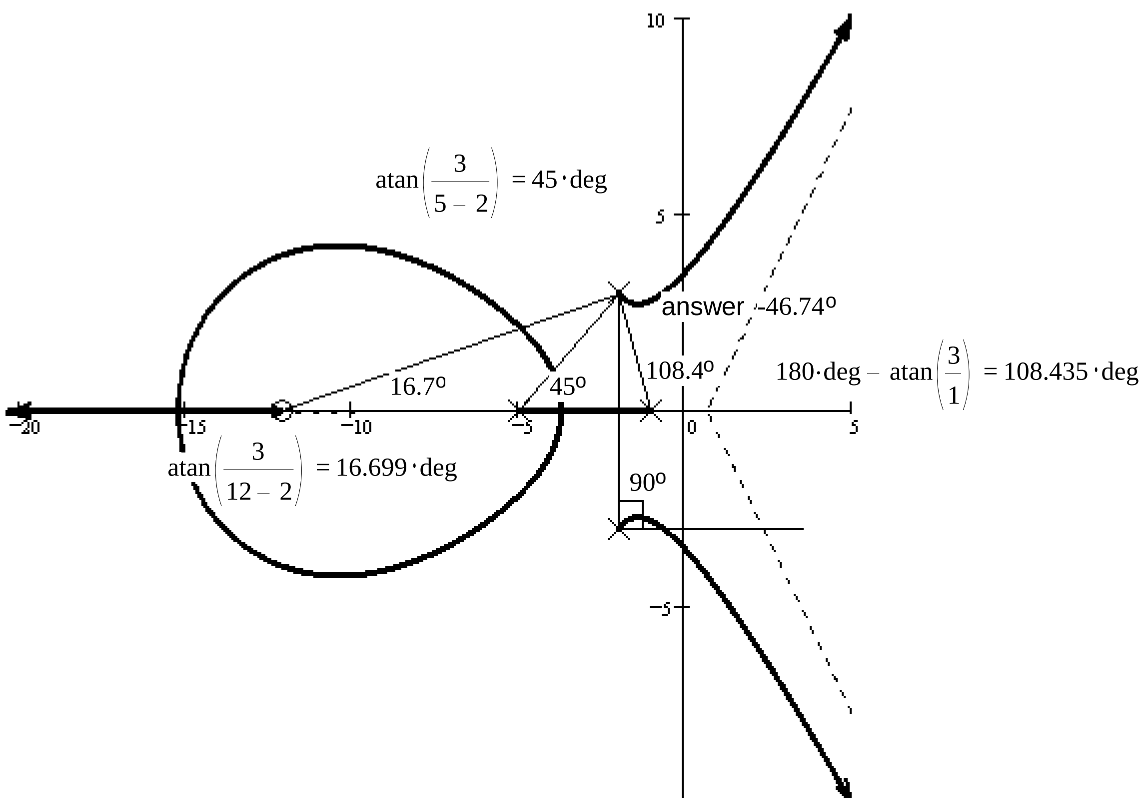
$$\frac{1}{(-4+12)} = \frac{2\cdot s+4}{s^2+4\cdot s+13} + \frac{1}{(-4+1)} + \frac{1}{(s+5)}$$

$$0 = \frac{2\cdot s+4}{s^2+4\cdot s+13} + \frac{1}{(s+5)} + \frac{1}{(-4+1)} - \frac{1}{(-4+12)} \qquad \text{Solve: } \begin{pmatrix} 2.105 \\ -3.648 \\ -0.912 \end{pmatrix}$$

$$0 = \frac{2\cdot s+4}{s^2+4\cdot s+13} + \frac{1}{(s+5)} + \frac{1}{(-3.648+1)} - \frac{1}{(-3.648+12)} \qquad \text{Solve: } \begin{pmatrix} 1.091 \\ -3.727 \\ -0.332 \end{pmatrix}$$

Close to actual answer of -3.712

Find the Departure angles from complex poles for Basic Root Locus Examples, Example 9

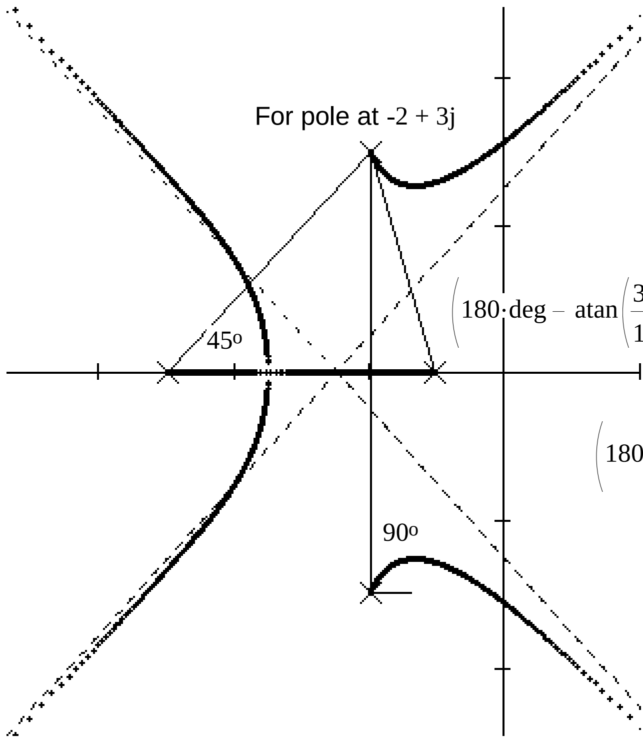


$$\operatorname{atan}\left(\frac{3}{12-2}\right) - \operatorname{atan}\left(\frac{3}{5-2}\right) - \left(180\cdot deg - \operatorname{atan}\left(\frac{3}{1}\right)\right) - 90\cdot deg - \theta \;=\; \underline{+}180^{o}$$

$$\theta = 180\cdot deg + 16.699\cdot deg - 45\cdot deg - 108.435\cdot deg - 90\cdot deg = -46.736\cdot deg$$

ECE 3510 Root Locus Departure and Arrival Angles

$$G(s) := \frac{1}{\left(s^2 + 4\cdot s + 13\right)\cdot(s+1)\cdot(s+5)} \quad = \quad \frac{1}{(s+2+3\cdot j)\cdot(s+2-3\cdot j)\cdot(s+1)\cdot(s+5)}$$



$$s := -2 + 3\cdot j$$

$$\frac{1}{(s+2+3\cdot j)\cdot(s+1)\cdot(s+5)} = -5.5556\cdot 10^{-3} + 0.0111i$$

$$180\cdot deg - \arg\left(-5.5556\cdot 10^{-3} + 0.0111i\right) = 63.412\cdot deg$$

$$\left(180\cdot deg - \operatorname{atan}\left(\frac{3}{1}\right)\right) = 108.435\cdot deg$$

$$\left(180\cdot deg - \operatorname{atan}\left(\frac{3}{1}\right)\right) + 90\cdot deg + 45\cdot deg = 243.435\cdot deg \quad + \theta = \pm\ 180^o$$

$$180 - 243.4 = -63.4 \ \ deg$$

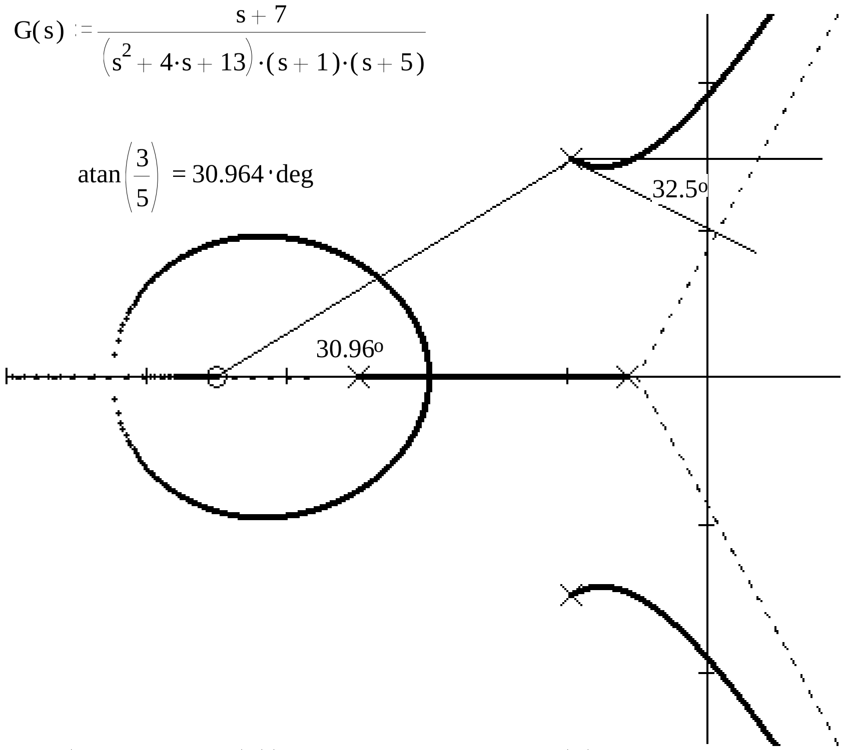
If you leave out 180º (not recommended)

$$\left(-\operatorname{atan}\left(\frac{3}{1}\right)\right) + 90\cdot deg + 45\cdot deg = 63.435\cdot deg \quad + \theta = 0^o,\ \pm\ 360^o$$

$$\theta \ = \ -63.435\cdot deg$$

$$G(s) := \frac{s+7}{\left(s^2 + 4\cdot s + 13\right)\cdot(s+1)\cdot(s+5)}$$

$$\operatorname{atan}\left(\frac{3}{5}\right) = 30.964\cdot deg$$



$$s := -2 + 3\cdot j$$

$$\frac{s+7}{(s+2+3\cdot j)\cdot(s+1)\cdot(s+5)} = -0.0611 + 0.0389i$$

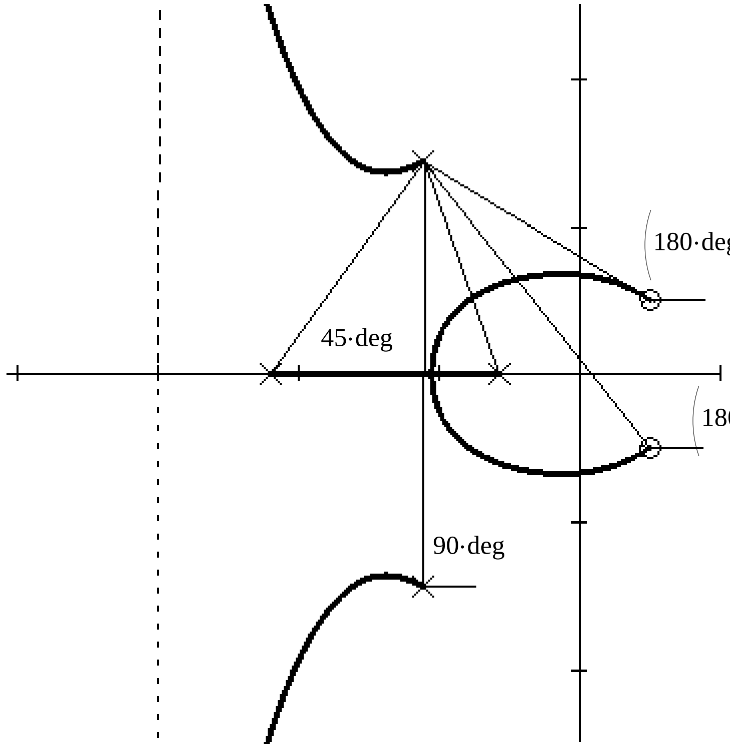
$$180\cdot deg - \arg(-0.0611 + 0.0389i) = 32.483\cdot deg$$

$$\left(180\cdot deg - \operatorname{atan}\left(\frac{3}{1}\right)\right) + 90\cdot deg + 45\cdot deg - \operatorname{atan}\left(\frac{3}{5}\right) = 212.471\cdot deg \quad + \theta = \pm\ 180^o$$

$$180 - 212.5 = -32.5 \ \ deg$$

ECE 3510 Root Locus Departure and Arrival Angles p2

$$G(s) := \frac{s^2 - 2\cdot s + 2}{\left(s^2 + 4\cdot s + 13\right)\cdot(s+1)\cdot(s+5)}$$



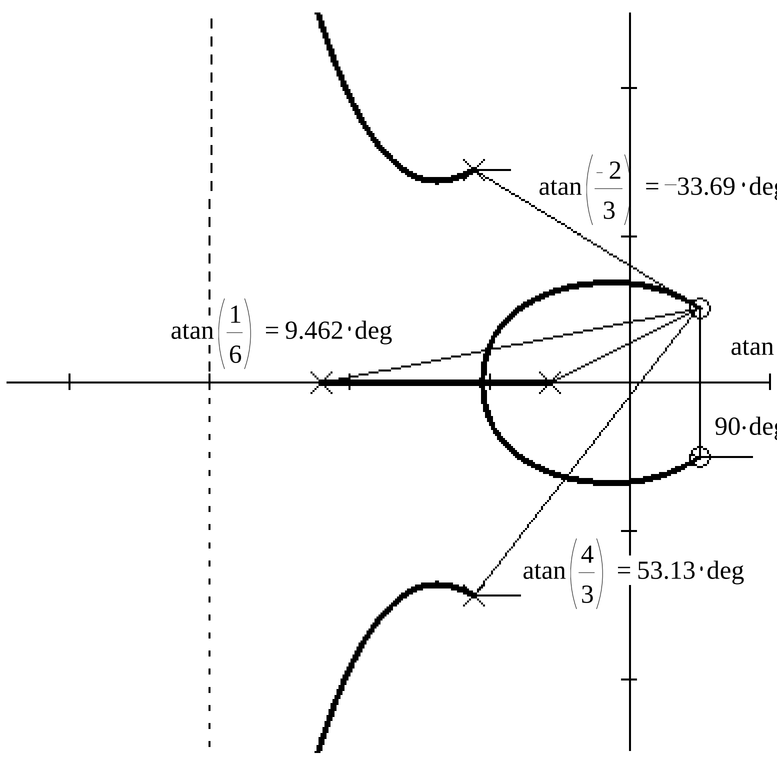
$$\left(180\cdot deg - atan\left(\frac{2}{3}\right)\right) = 146.31\cdot deg$$

$$\left(180\cdot deg - atan\left(\frac{3}{1}\right)\right) = 108.435\cdot deg$$

$$\left(180\cdot deg - atan\left(\frac{4}{3}\right)\right) = 126.87\cdot deg$$

$$\left(180\cdot deg - atan\left(\frac{3}{1}\right)\right) + 90\cdot deg + 45\cdot deg - \left(180\cdot deg - atan\left(\frac{2}{3}\right)\right) - \left(180\cdot deg - atan\left(\frac{4}{3}\right)\right) = -29.745\cdot deg \ + \theta = \pm 180^o$$

$$-(180 - 29.75) = -150.25 \ \ deg$$



$$atan\left(\frac{-2}{3}\right) = -33.69\cdot deg$$

$$atan\left(\frac{1}{6}\right) = 9.462\cdot deg$$

$$atan\left(\frac{1}{2}\right) = 26.565\cdot deg$$

$$atan\left(\frac{4}{3}\right) = 53.13\cdot deg$$

$$atan\left(\frac{1}{2}\right) + atan\left(\frac{-2}{3}\right) + atan\left(\frac{1}{6}\right) + atan\left(\frac{4}{3}\right) - 90\cdot deg = -34.533\cdot deg \quad - \theta = \pm 180^o$$

$$(180 - 34.5) = 145.5 \ \ deg$$

ECE 3510 Root Locus Departure and Arrival Angles p2