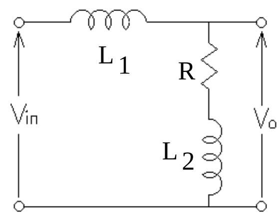
A. Stolp
11/22/09,
1/22/15

Now that we've reviewed Laplace transforms of signals, we can move on to systems, the transfer function, and system block diagrams using blocks which contain transfer functions.

Consider a circuit:



$$\mathbf{H}(s) = \frac{\mathbf{V_o}(s)}{\mathbf{V_{in}}(s)} = \frac{R + L_2 \cdot s}{R + L_1 \cdot s + L_2 \cdot s} = \frac{R + L_2 \cdot s}{R + \left(L_1 + L_2\right) \cdot s}$$

$$= \frac{L_2 \cdot s + R}{\left(L_1 + L_2\right) \cdot s + R}$$

This could be represented in as a block operator:

$$\mathbf{V_{in}}(s) \longrightarrow \boxed{\frac{L_2 \cdot s + R}{\left(L_1 + L_2\right) \cdot s + R}} \longrightarrow \mathbf{V_o}(s) = \mathbf{V_{in}}(s) \cdot \mathbf{H}(s)$$

Transfer functions can be written for all kinds of devices and systems, not just electric circuits and the input and output do not have to be similar. For instance, the potentiometers used to measure angular position in the crude servo of lab 1 can be represented like this:

$$\boldsymbol{\theta}_{\mathbf{in}}(s) \longrightarrow \boxed{Kp = 0.7 \cdot \frac{V}{rad} = 0.012 \bullet \frac{V}{deg}} \longrightarrow \mathbf{V_{out}}(s) = K_p \cdot \boldsymbol{\theta}_{\mathbf{in}}(s)$$

In general:

$$\mathbf{H}(s) = \frac{\text{output}}{\text{input}} = \frac{\mathbf{Y}(s)}{\mathbf{X}(s)}$$

X and **Y** could be anything from small electrical signals to powerful mechanical motions or forces.

$$\mathbf{X}(s) \longrightarrow \boxed{\mathbf{H}(s)} \longrightarrow \mathbf{Y}(s) = \mathbf{X}(s) \cdot \mathbf{H}(s)$$

$$= \frac{\mathbf{N_X}(s)}{\mathbf{D_X}(s)} \cdot \frac{\mathbf{N_H}(s)}{\mathbf{D_H}(s)}$$

The output signal has the poles of both the input AND the transfer function.

Serial - path systems

Two blocks with transfer functions $\mathbf{A}(s)$ and $\mathbf{B}(s)$ in a row would look like this:

$$\mathbf{X}(s) \longrightarrow \boxed{\mathbf{A}(s)} \xrightarrow{\mathbf{X}(s) \cdot \mathbf{A}(s)} \boxed{\mathbf{B}(s)} \longrightarrow \mathbf{Y}(s) = \mathbf{X}(s) \cdot \mathbf{A}(s) \cdot \mathbf{B}(s)$$

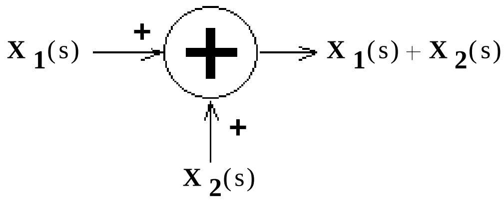
$$= \mathbf{X}(s) \cdot \frac{\mathbf{N_A}(s)}{\mathbf{D_A}(s)} \cdot \frac{\mathbf{N_B}(s)}{\mathbf{D_B}(s)}$$

The two blocks could be replaced by a single equivalent block:

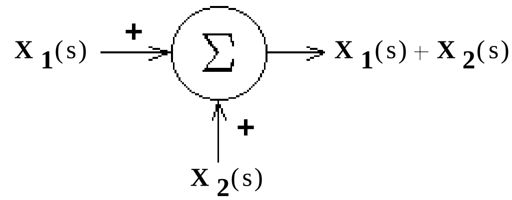
$$\mathbf{X}(s) \longrightarrow \boxed{\mathbf{A}(s) \cdot \mathbf{B}(s)} \longrightarrow \mathbf{Y}(s) = \mathbf{X}(s) \cdot \mathbf{A}(s) \cdot \mathbf{B}(s)$$

The output signal has the poles of the input AND BOTH transfer functions.

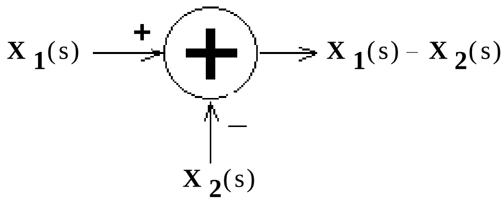
Summer blocks can be used to add signals:



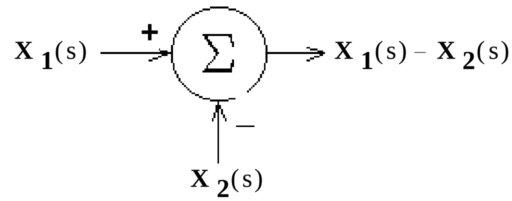
OR



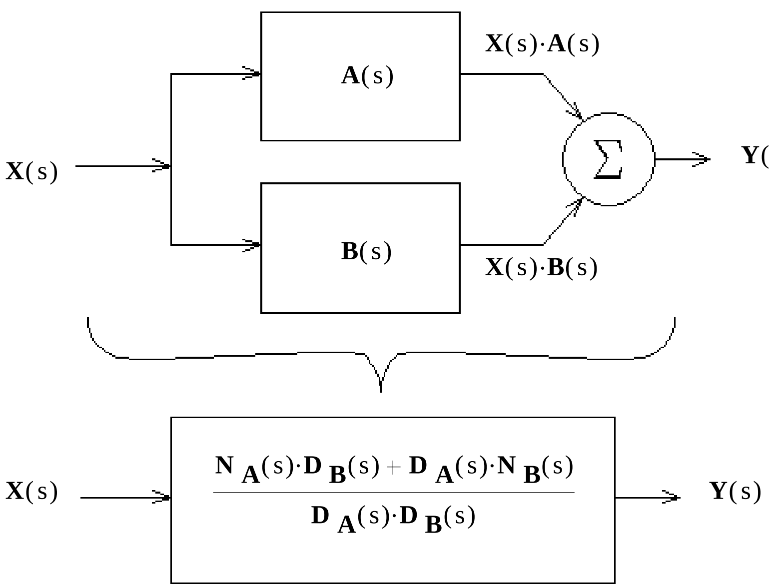
or subtract signals:



OR



Parallel - path systems



$$\mathbf{Y}(s) = \mathbf{X}(s)\cdot(\mathbf{A}(s)+\mathbf{B}(s))$$

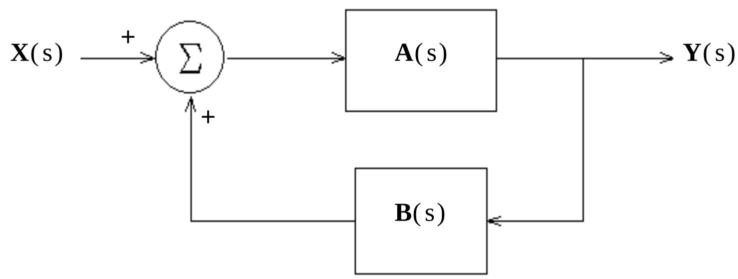
$$= \mathbf{X}(s)\cdot\left(\frac{\mathbf{N}_{\mathbf{A}}(s)}{\mathbf{D}_{\mathbf{A}}(s)}+\frac{\mathbf{N}_{\mathbf{B}}(s)}{\mathbf{D}_{\mathbf{B}}(s)}\right)$$

$$= \mathbf{X}(s)\cdot\left(\frac{\mathbf{N}_{\mathbf{A}}(s)\cdot\mathbf{D}_{\mathbf{B}}(s)+\mathbf{D}_{\mathbf{A}}(s)\cdot\mathbf{N}_{\mathbf{B}}(s)}{\mathbf{D}_{\mathbf{A}}(s)\cdot\mathbf{D}_{\mathbf{B}}(s)}\right)$$

Again, the output signal has the poles of the input AND BOTH transfer functions.

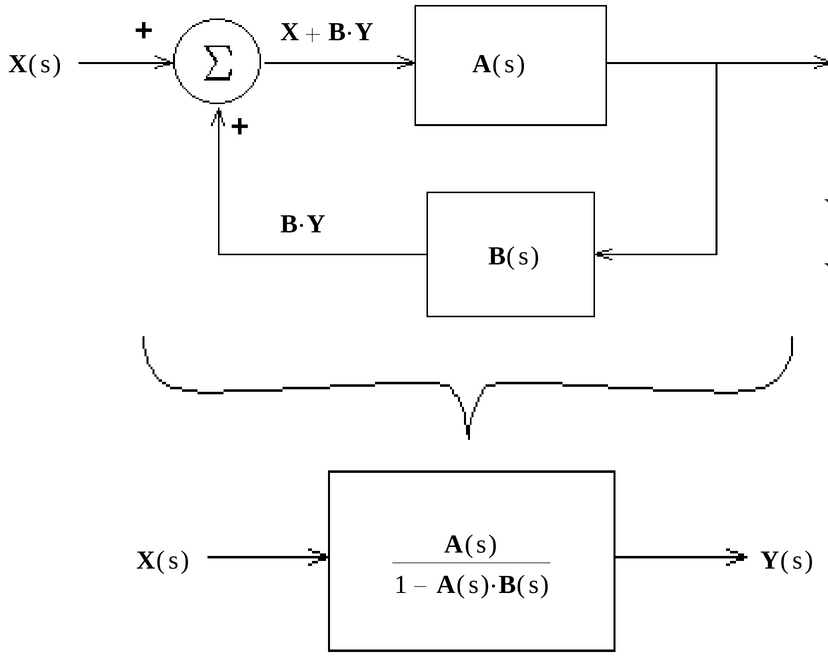
The two blocks could be replaced by a single equivalent block:

A feedback loop system is particularly interesting and useful:



The entire loop can be replaced by a single equivalent block:

Note that I've begun to drop the (s)



$$\mathbf{Y}(s) = \mathbf{A}\cdot(\mathbf{X} + \mathbf{B}\cdot\mathbf{Y})$$

$$= \mathbf{A}\cdot\mathbf{X} + \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{Y}$$

$$\mathbf{Y} - \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{Y} = \mathbf{A}\cdot\mathbf{X}$$

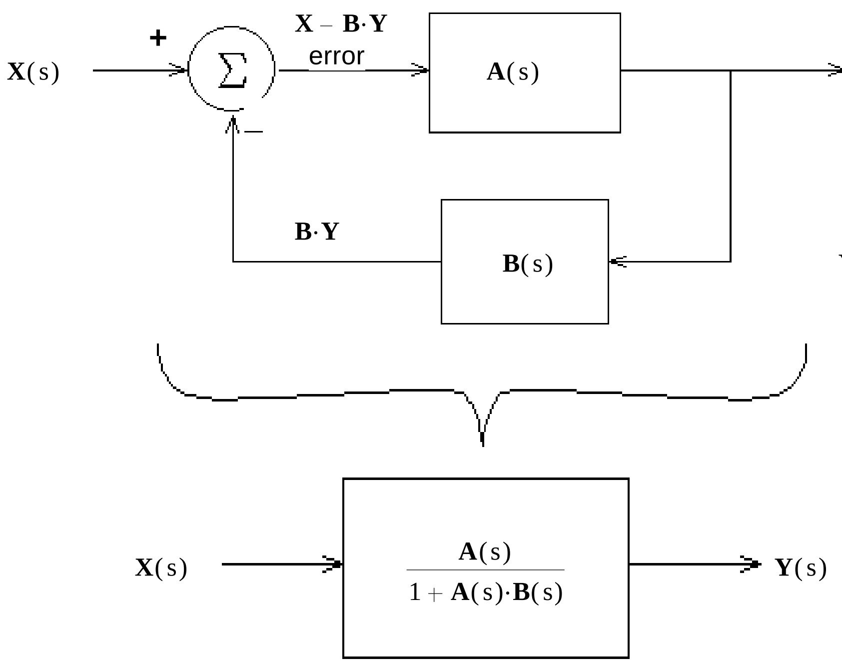
$$\mathbf{Y}\cdot(1 - \mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{X}$$

$$\frac{\mathbf{Y}}{\mathbf{X}} = \frac{\mathbf{A}}{1 - \mathbf{A}\cdot\mathbf{B}} = \mathbf{H}(s)$$

The equivalent transfer function

$\mathbf{A}(s)\cdot\mathbf{B}(s)$ is called the "loop gain" or "open loop gain"

Negative feedback is more common and is used as a control system:



$$\mathbf{Y}(s) = \mathbf{A}\cdot(\mathbf{X} - \mathbf{B}\cdot\mathbf{Y})$$

$$= \mathbf{A}\cdot\mathbf{X} - \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{Y}$$

$$\mathbf{Y} + \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{Y} = \mathbf{A}\cdot\mathbf{X}$$

$$\mathbf{Y}\cdot(1 + \mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{X}$$

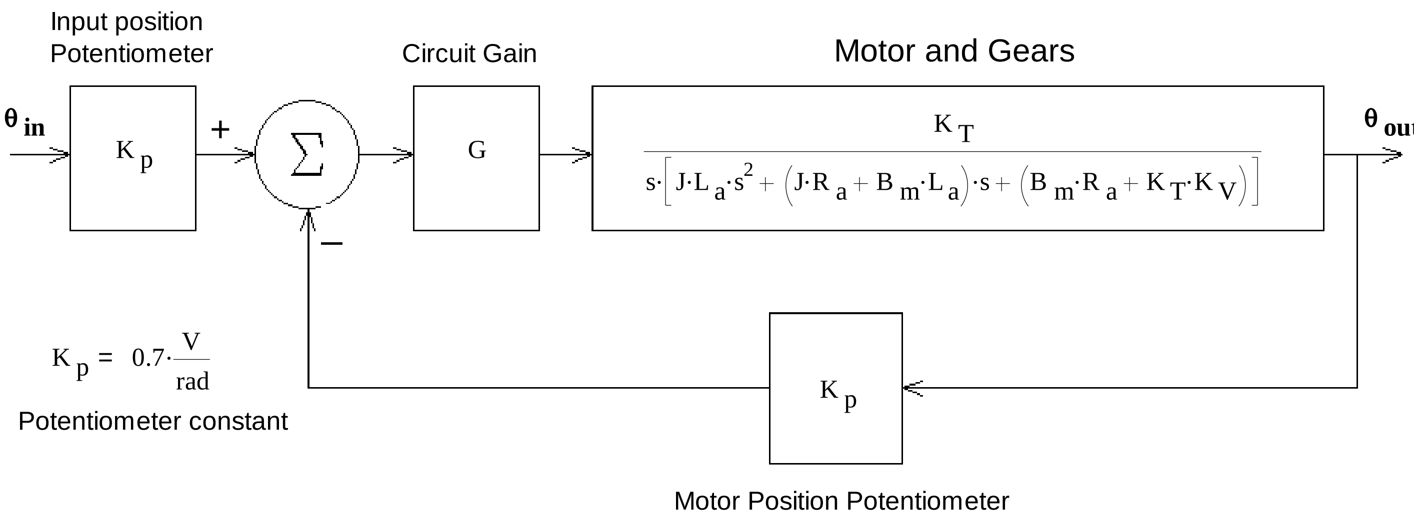
$$\frac{\mathbf{Y}}{\mathbf{X}} = \frac{\mathbf{A}}{1 + \mathbf{A}\cdot\mathbf{B}} = \mathbf{H}(s)$$

The equivalent transfer function

This is called a "closed loop" system, whereas a a system without feedback is called "open loop".
The term "open loop" is often used to describe a system that is out of control.

The output signal poles are <u>different</u> than either the poles of the input or the transfer functions.
Different poles means different characteristics! This implies that you might start with a stable system and make an unstable system or (more productively) start with an unstable system and make a stable system.

The servo used in our lab can be represented by:



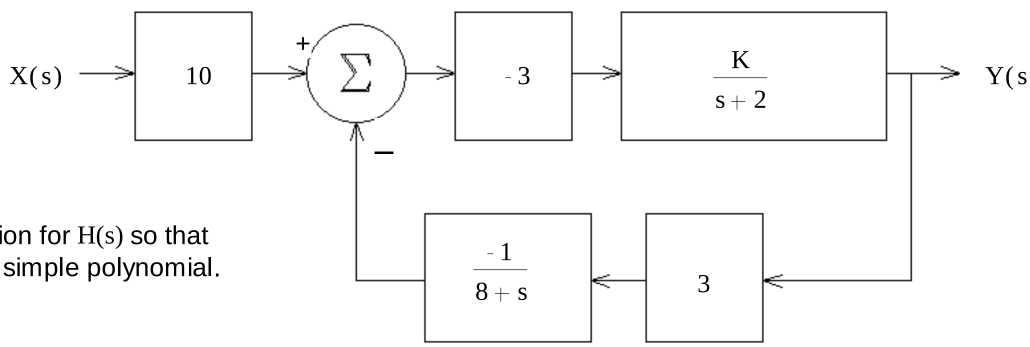
$K_p = 0.7\cdot\dfrac{V}{rad}$

Potentiometer constant

$$\mathbf{H}(s) = \frac{\boldsymbol{\theta}_{\mathbf{out}}(s)}{\boldsymbol{\theta}_{\mathbf{in}}(s)} = \frac{G\cdot K_T\cdot K_p}{s\cdot\left[J\cdot L_a\cdot s^2+\left(J\cdot R_a+B_m\cdot L_a\right)\cdot s+\left(B_m\cdot R_a+K_T\cdot K_V\right)\right]+K_p\cdot G\cdot K_T}$$

See the appendix to lab 1 for the complete analysis

Ex. 1 a) A feedback system is shown in the figure. What is the transfer function of the whole system, with feedback.



$H(s) = \dfrac{Y(s)}{X(s)} = ?$

Simplify your expression for $H(s)$ so that the denominator is a simple polynomial.

Feedback loop:

Loop gain: $L = \left(\dfrac{-3\cdot K}{s+2}\right)\cdot\left(\dfrac{-1\cdot 3}{8+s}\right)$

Simplification:

$$A_f = \frac{\left(\dfrac{-3\cdot K}{s+2}\right)}{1+\left(\dfrac{-3\cdot K}{s+2}\right)\cdot\left(\dfrac{-3}{8+s}\right)}$$

$$A_f = \frac{\left(\dfrac{-3\cdot K}{s+2}\right)}{1+\left(\dfrac{-3\cdot K}{s+2}\right)\cdot\left(\dfrac{-3}{8+s}\right)}\cdot\left[\frac{(s+2)\cdot(8+s)}{(s+2)\cdot(8+s)}\right] = \frac{(-3\cdot K)\cdot(s+8)}{(s+2)\cdot(8+s)+(3\cdot K)\cdot 3}$$

$$= \frac{(-3\cdot K)\cdot s - K\cdot 24}{s^2+10\cdot s+16+9\cdot K}$$

Whole system:

$$H(s) = 10\cdot\frac{-3\cdot K\cdot s - 24\cdot K}{s^2+10\cdot s+16+9\cdot K} = \frac{-30\cdot K\cdot(s+8)}{s^2+10\cdot s+16+9\cdot K}$$

b) Find the value of K to make the transfer function critically damped. Answer may be left as a fraction.

characteristic eq.: $0 = s^2 + 10\cdot s + 16 + 9\cdot K$

to solve for the poles: $s = \dfrac{-10 + \sqrt{10^2 - 4\cdot(16 + 9\cdot K)}}{2}$ at critical damping, the part under the radical is zero.

thus: $10^2 = 4\cdot(16 + 9\cdot K)$

$$100 = 64 + 36\cdot K$$

$$K = \frac{100 - 64}{36} = \frac{36}{36} = 1$$ solve for K

c) If K is less than the value found in part b), will the system be under-, critical-, or overdamped?

$10^2 - 4\cdot(16 + 9\cdot K) > 0$ so it will be overdamped

d) If $K := 5$, find the pole(s) of the transfer function:

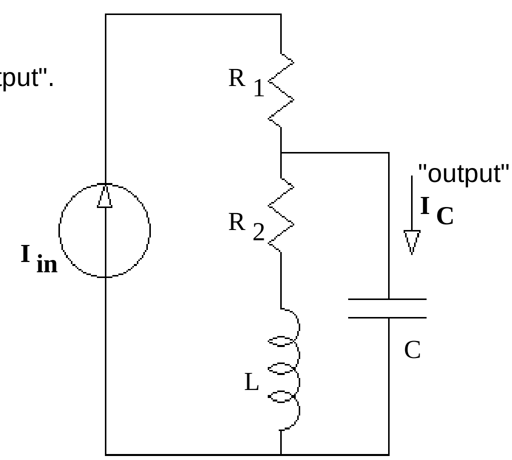
characteristic eq.: $0 = s^2 + 10\cdot s + 16 + 9\cdot K = s^2 + 10\cdot s + 61$

$$\frac{-10 + \sqrt{10^2 - 4\cdot 61}}{2} = -5 + 6j \qquad \frac{-10 - \sqrt{10^2 - 4\cdot 61}}{2} = -5 - 6j$$

e) If $K := 5$, find the zero(s) of the transfer function: $s + 8 = 0$ $s = -8$

Ex.2 a) Find the transfer function of the circuit shown. Consider I_C as the "output".

Properly simplify all your expressions for $\mathbf{H}(s)$. By this I mean that he numerator and denominator should both be simple polynomials or factored polynomials. There should be no $1/s^n$ terms in either the numerator or denominator. Also, there should be no coefficient on the highest-order term in the denominator



$$\mathbf{H}(s) = \frac{\mathbf{I_C}(s)}{\mathbf{I_{in}}(s)} = ?$$

Current divider:

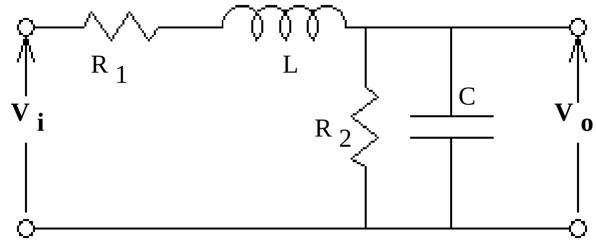
$$\mathbf{I_C}(s) = \mathbf{I_{in}}(s)\cdot\frac{\dfrac{1}{\left(\dfrac{1}{C\cdot s}\right)}}{\dfrac{1}{\left(\dfrac{1}{C\cdot s}\right)} + \dfrac{1}{R_2 + L\cdot s}} = \mathbf{I_{in}}(s)\cdot\frac{C\cdot s}{C\cdot s + \dfrac{1}{R_2 + L\cdot s}}$$

$$\mathbf{H}(s) = \frac{\mathbf{I_C}(s)}{\mathbf{I_{in}}(s)} = \frac{C\cdot s}{C\cdot s + \dfrac{1}{R_2 + L\cdot s}}\cdot\frac{(R_2 + L\cdot s)}{(R_2 + L\cdot s)} = \frac{C\cdot R_2 + L\cdot C\cdot s^2}{C\cdot R_2\cdot s + L\cdot C\cdot s^2 + 1}\cdot\frac{\left(\dfrac{1}{L\cdot C}\right)}{\left(\dfrac{1}{L\cdot C}\right)} = \frac{\dfrac{R_2}{L}\cdot s + s^2}{\dfrac{R_2}{L}\cdot s + s^2 + \dfrac{1}{L\cdot C}}$$

$$= \frac{s\cdot\left(s + \dfrac{R_2}{L}\right)}{s^2 + \dfrac{R_2}{L}\cdot s + \dfrac{1}{L\cdot C}}$$

b) How many zeroes does this transfer function have? 2 , 0 and $-R_2/L$

c) How many poles does this transfer function have? 2 at: $-\dfrac{R_2}{2\cdot L} \pm \dfrac{1}{2}\cdot\sqrt{\left(\dfrac{R_2}{L}\right)^2 - \dfrac{4}{L\cdot C}}$



Ex.3 a) Find the transfer function of the circuit shown.
$\mathbf{V_i}$ is the input and $\mathbf{V_O}$ is the output.

$$\mathbf{H}(s) = \frac{\mathbf{V_o}(s)}{\mathbf{V_i}(s)} = \frac{\dfrac{1}{\dfrac{1}{R_2}+C\cdot s}}{R_1+L\cdot s+\dfrac{1}{\dfrac{1}{R_2}+C\cdot s}}\quad\frac{\dfrac{1}{R_2}+C\cdot s}{\dfrac{1}{R_2}+C\cdot s} = \frac{1}{R_1\cdot\left(\dfrac{1}{R_2}+C\cdot s\right)+L\cdot s\cdot\left(\dfrac{1}{R_2}+C\cdot s\right)+1}$$

$$= \frac{1}{\dfrac{R_1}{R_2}+R_1\cdot C\cdot s+\dfrac{L\cdot s}{R_2}+L\cdot s\cdot C\cdot s+1}\quad\frac{\dfrac{1}{L\cdot C}}{\dfrac{1}{L\cdot C}} = \frac{\dfrac{1}{L\cdot C}}{\dfrac{R_1}{R_2}\cdot\dfrac{1}{L\cdot C}+\dfrac{R_1\cdot C}{L\cdot C}\cdot s+\dfrac{L\cdot s}{R_2}\cdot\dfrac{1}{L\cdot C}+s^2+\dfrac{1}{L\cdot C}}$$

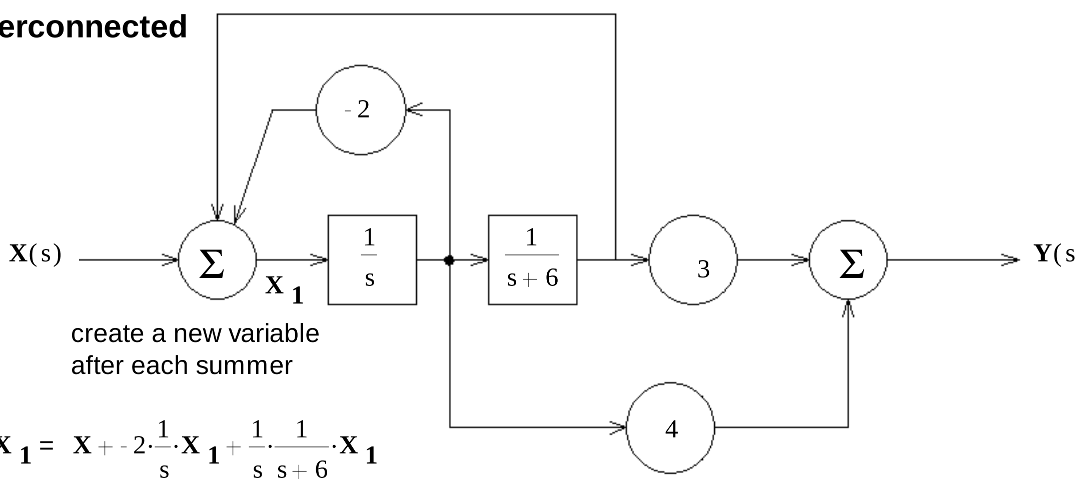
$$= \frac{\dfrac{1}{L\cdot C}}{s^2+\left(\dfrac{R_1}{L}+\dfrac{1}{R_2\cdot C}\right)\cdot s+\left(1+\dfrac{R_1}{R_2}\right)\cdot\dfrac{1}{L\cdot C}}$$

b) Find the characteristic equation of the circuit shown. $\quad 0 = s^2+\left(\dfrac{R_1}{L}+\dfrac{1}{R_2\cdot C}\right)\cdot s+\left(1+\dfrac{R_1}{R_2}\right)\cdot\dfrac{1}{L\cdot C}$

c) The solutions to the characteristic equation are called the _______ of the transfer function. Poles

d) Does the transfer function have one or more zeros? If yes, express it (them) in terms of R_1, R_2, C, & L. NO

General Interconnected System



create a new variable
after each summer

$$\mathbf{X_1} = \mathbf{X} + -2\cdot\frac{1}{s}\cdot\mathbf{X_1} + \frac{1}{s}\cdot\frac{1}{s+6}\cdot\mathbf{X_1}$$

$$\mathbf{X_1}\cdot\left[1+\frac{2}{s}-\frac{1}{s\cdot(s+6)}\right] = \mathbf{X} = \mathbf{X_1}\cdot\left[\frac{s\cdot(s+6)}{s\cdot(s+6)}+\frac{2\cdot(s+6)}{s\cdot(s+6)}-\frac{1}{s\cdot(s+6)}\right] = \mathbf{X_1}\cdot\left[\frac{s^2+8\cdot s+11}{s\cdot(s+6)}\right]$$

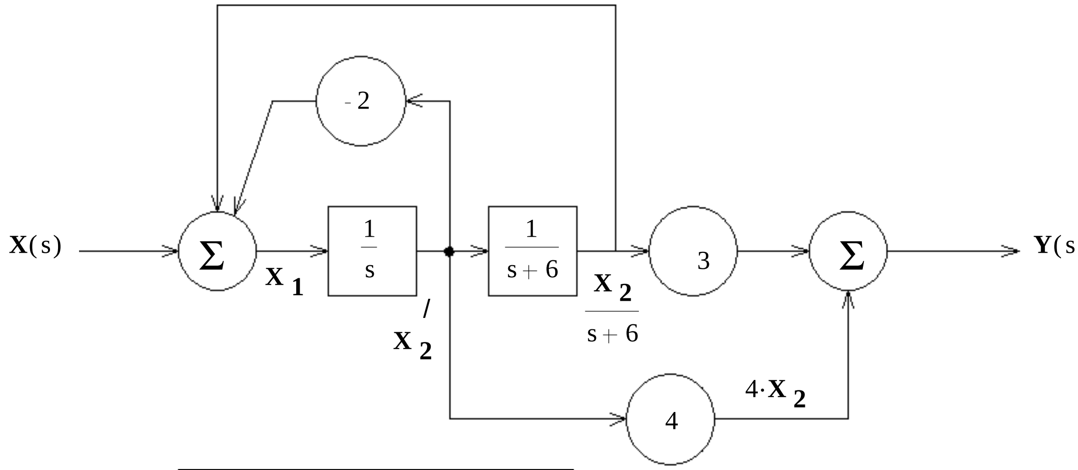
$$\mathbf{X_1} = \frac{s\cdot(s+6)}{s^2+8\cdot s+11}\cdot\mathbf{X}$$

$$\mathbf{Y} = 3\cdot\left(\frac{1}{s}\cdot\frac{1}{s+6}\right)\cdot\mathbf{X_1}+4\cdot\frac{1}{s}\cdot\mathbf{X_1} = 3\cdot\left[\frac{1}{s\cdot(s+6)}\right]\cdot\left[\frac{s\cdot(s+6)}{s^2+8\cdot s+11}\cdot\mathbf{X}\right]+4\cdot\frac{1}{s}\cdot\left[\frac{s\cdot(s+6)}{s^2+8\cdot s+11}\cdot\mathbf{X}\right]$$

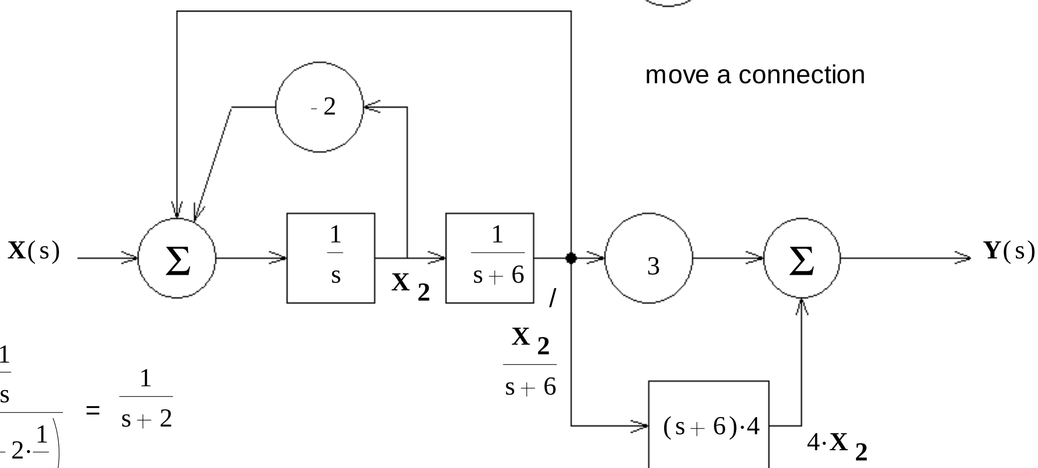
$$= \frac{3+(4\cdot s+24)}{s^2+8\cdot s+11}\cdot\mathbf{X} = \frac{4\cdot s+27}{s^2+8\cdot s+11}\cdot\mathbf{X}$$

$$\mathbf{H}(s) = \frac{\mathbf{Y}}{\mathbf{X}} = \frac{4\cdot s+27}{s^2+8\cdot s+11}$$

General Interconnected System, Another approach



move a connection



parallel paths $3+4\cdot(s+6)$

inner loop $\dfrac{\frac{1}{s}}{1-\left(-2\cdot\frac{1}{s}\right)} = \dfrac{1}{s+2}$

outer loop $\dfrac{\frac{1}{(s+2)}\cdot\frac{1}{(s+6)}}{1-1\cdot\frac{1}{(s+2)}\cdot\frac{1}{(s+6)}} = \dfrac{1}{(s+2)\cdot(s+6)-1} = \dfrac{1}{s^2+8\cdot s+12-1}$

$= \dfrac{1}{s^2+8\cdot s+11}$

whole transfer function $\mathbf{H}(s) = \dfrac{1}{s^2+8\cdot s+11}\cdot(3+4\cdot(s+6)) = \dfrac{4\cdot s+27}{s^2+8\cdot s+11}$

Bounded-Input Bounded-Output (BIBO) Stable

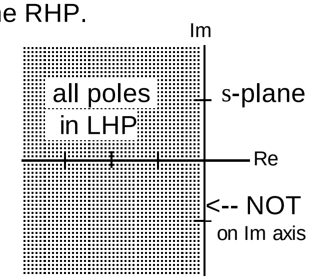
A system is considered BIBO stable if the output in bounded for any bounded input.

A bounded input could have single poles on the imaginary axis at any location.
A bounded output may not have double poles on the imaginary axis or any poles in the RHP (Right-half-plane).
The output will have all the poles of the input plus all the poles of the system. (except in rare pole-zero cancellations.)

Therefore: A BIBO system may not have any poles on the imaginary axis or any poles in the RHP.

Examples of systems with poles on the imaginary axis: If the output of a DC motor is angular position of the shaft then it has a pole at the origin. The response to a DC input is a shaft that keeps turning and the position grows without bounds. This system is not BIBO stable. (If the output is shaft speed, then it would be BIBO stable.)

If a system has a pair of imaginary poles at $\pm j\omega$, then it has a resonant frequency of ω. If the input also had a pair of imaginary poles at $\pm j\omega$ then it would excite that resonance and the output would grow without bounds.



system transfer function poles