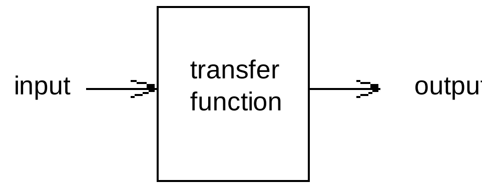
A. Stolp
1/8/08,
1/8/13

Yesterday we drew a block diagram on the board.
Let's examine those blocks a little more closely

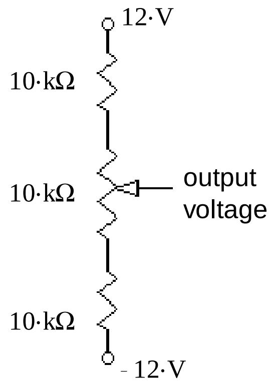


What's inside?

How are the input and output related? If you know the input, how do you find the output? Sometimes we can just multiply the input by the expression in the box to get the output. Then the expression in the box is called a transfer function.

In that case, the transfer function = $\dfrac{\text{output}}{\text{input}}$

A very simple case, a potentiometer



the input is θ ,
the angle of the shaft

We measure the voltage over its range of motion to find Kp.

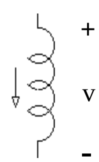
$$K_p = \frac{\text{output}}{\text{input}} = \frac{4\cdot V - -4\cdot V}{270\cdot deg} = 29.63\cdot\frac{mV}{deg} = 1.7\cdot\frac{V}{rad}$$

range of motion

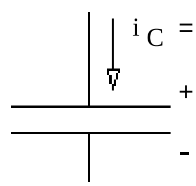
In this case, "zero" must be in the center of the range of motion

$$\theta_{in}(t) \longrightarrow \boxed{K_p = 1.7\cdot\frac{V}{rad} = 0.03\cdot\frac{V}{deg}} \longrightarrow V_{out}(t) = K_p\cdot\theta_{in}(t)$$

Nice... too bad it works for so few things in the time domain! Simple voltage dividers, amplifiers, and not much else.
All real electrical systems also have inductors and capacitors.



$$\frac{1}{L}\cdot\int v_L\,dt = i_L \qquad v_L = L\cdot\frac{d}{dt}i_L$$



$$i_C = C\cdot\frac{d}{dt}v_C \qquad v_C = \frac{1}{C}\cdot\int i_C\,dt$$

We'll have to avoid capacitors and inductors-- they're too complicated... You can't just multiply when there are differentials involved

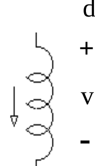
How about the mechanical world? $F = ma$, Great, no differentials... uh, except... $F = m\cdot a = m\cdot\frac{d}{dt}v = m\cdot\frac{d^2}{dt^2}x$

And then there are springs: $F = k\cdot x = k\cdot\int v\,dt = k\cdot\int\int a\,dt\,dt$

Isn't there some way that we could possibly replace all this differentiation and integration with multiplication and division?

Laplace transforms $\frac{d}{dt}$ operation can be replaced with s, and $\int \blacksquare\, dt$ can be replaced by $\frac{1}{s}$

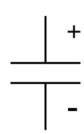
Then...



$$v_L = L\cdot\frac{d}{dt}i_L$$

$$\mathbf{V_L}(s) = L\cdot s\cdot\mathbf{I_L}(s)$$

Inductive impedance: $\mathbf{Z_L} = L\cdot s$



$$v_C = \frac{1}{C}\cdot\int i_C\,dt$$

$$\mathbf{V_C}(s) = \frac{1}{C}\cdot\frac{1}{s}\cdot\mathbf{I_C}(s) \qquad \mathbf{Z_C} = \frac{1}{C\cdot s}$$

Recall from your Ordinary Differential Equations class, the Laplace transform method of solving differential equations. The Laplace transform allowed you to change time-domain functions to frequency-domain functions.

1) Transform your signals into the frequency domain with the Laplace transform.

$$F(s) = \int_0^{\infty} f(t)\cdot e^{-s\cdot t}\,dt \qquad \text{Unilateral Laplace transform}$$

2) Solve your differential equations with plain old algebra, where:

$$\frac{d}{dt} \text{ operation can be replaced with s,} \qquad \text{and} \qquad \int \blacksquare\, dt \text{ can be replaced by } \frac{1}{s}$$

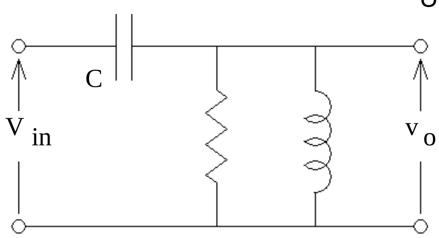
3) Transform your result back to the time domain with the inverse Laplace transform.

$$f(t) = \frac{1}{2\cdot\pi\cdot j}\cdot\int_{c-j\infty}^{c+j\infty} F(s)\cdot e^{s\cdot t}\,ds$$

OK, truth be told, we never actually use the inverse Laplace transform. We use tables instead.

Then our nice, linear, blocks could contain Laplace transfer functions, like this:

Consider a circuit:



Using the impedances in a voltage divider:

$$\mathbf{H}(s) = \frac{\mathbf{V_o}(s)}{\mathbf{V_{in}}(s)} = \frac{\dfrac{1}{\dfrac{1}{R}+\dfrac{1}{L\cdot s}}}{\dfrac{1}{\dfrac{1}{R}+\dfrac{1}{L\cdot s}}+\dfrac{1}{C\cdot s}}\cdot\frac{\left(\dfrac{1}{R}+\dfrac{1}{L\cdot s}\right)}{\left(\dfrac{1}{R}+\dfrac{1}{L\cdot s}\right)} = \frac{1}{1+\dfrac{1}{C\cdot s\cdot R}+\dfrac{1}{C\cdot s\cdot L\cdot s}}\cdot\frac{\left(s^2\right)}{\left(s^2\right)}$$

$$= \frac{s^2}{s^2+\dfrac{1}{C\cdot R}\cdot s+\dfrac{1}{C\cdot L}}$$

This could now be represented as a block operator:

$$\mathbf{V_{in}}(s) \longrightarrow \boxed{\frac{s^2}{s^2+\dfrac{1}{C\cdot R}\cdot s+\dfrac{1}{C\cdot L}}} \longrightarrow \mathbf{V_o}(s) = \mathbf{V_{in}}(s)\cdot\mathbf{H}(s)$$

Transfer functions can be written for all kinds of devices and systems, not just electric circuits and the input and output do not have to be similar. For instance, the potentiometers used to measure angular position in the servo have an angle as input and a voltage as output.

Laplace transforms will be important!!

BUT, remember, the first step is to transform the signals into the frequency domain with the Laplace transform. Maybe we ought to deal with the signals first...

FIRST: Laplace transforms of <u>signals</u>

Let's evaluate some of these and see if we can make a table

Ex. 1 $f(t) = \delta(t)$ The Impulse or "Dirac" function, not a very likely signal in real life.

$$\mathbf{F}(s) = \int_0^{\infty} \delta(t)\cdot e^{-s\cdot t}\,dt$$

but: $\delta(t)\cdot g(t) = \delta(t)\cdot g(0)$ so:

any function

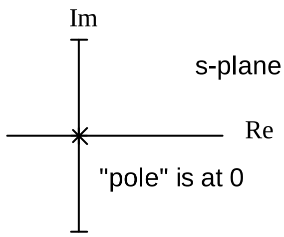
$$= \int_0^{\infty} \delta(t)\cdot e^{-s\cdot 0}\,dt = \int_0^{\infty} \delta(t)\cdot 1\,dt = 1$$

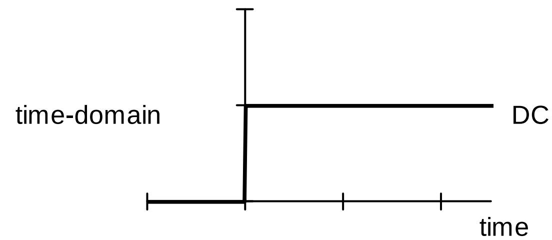
Ex. 2 $f(t) = u(t)$ The unit-step function, a constant value (DC) signal

$$\mathbf{F}(s) = \int_0^\infty u(t)\cdot e^{-s\cdot t}\,dt \qquad \text{but:} \quad u(t) = 1 \quad \text{from 0 to } \infty$$

$$= \int_0^\infty 1\cdot e^{-s\cdot t}\,dt = \frac{1}{-s}\cdot e^{-s\cdot t}\Big|_0^\infty = \frac{1}{-s}\cdot \overset{0}{e^{-s\cdot\infty}} - \frac{1}{-s}\cdot \overset{1}{e^{-s\cdot 0}} = 0 - \frac{1}{-s}\cdot(1) = \frac{1}{s}$$

if $s > 0$





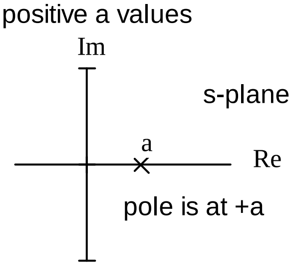
Ex. 3 $f(t) = u(t)\cdot e^{at}$

$$\mathbf{F}(s) = \int_0^\infty e^{at}\cdot e^{-s\cdot t}\,dt = \int_0^\infty e^{(a-s)\cdot t}\,dt = \frac{1}{(a-s)}\cdot e^{(a-s)\cdot t}\Big|_0^\infty$$

$$= \frac{1}{(a-s)}\cdot \overset{0}{e^{(a-s)\cdot\infty}} - \frac{1}{(a-s)}\cdot \overset{1}{e^{(a-s)\cdot 0}} = 0 - \frac{1}{(a-s)}\cdot(1) = \frac{1}{s-a}$$

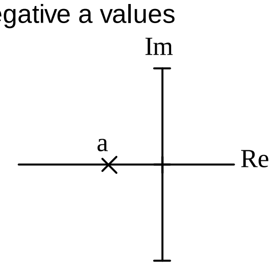
if $s > a$ "pole" is at +a

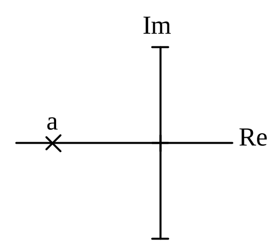
for positive a values

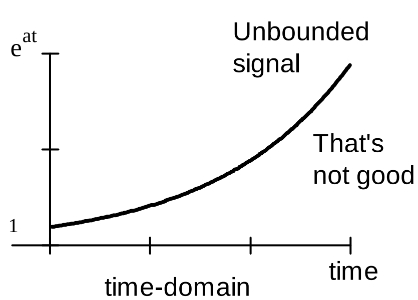


$$\frac{A}{s-a}$$

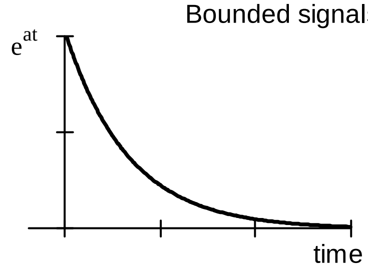
for negative a values

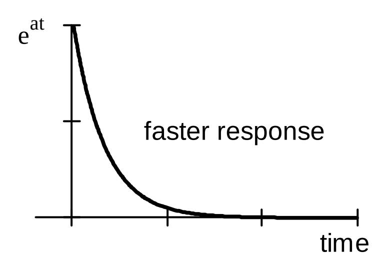






Bounded signals





This is the single most-important Laplace transform case. In fact we really don't need any others. Ex.1 can be thought of as this case with $a = -\infty$. Ex. 2 can be thought of as $a = 0$. And finally, all sinusoids can be made from exponentials if you let the poles (a) be complex. Remember Euler's equations...

Euler's equations $e^{j\cdot\omega\cdot t} = \cos(\omega t) + j\cdot\sin(\omega t)$ $\qquad e^{(\alpha\cdot t + j\cdot\omega\cdot t)} = e^{\alpha\cdot t}\cdot(\cos(\omega t) + j\cdot\sin(\omega t))$

Pole Location(s) correspond to the type of signal.

Euler's equations $\quad \cos(\omega \cdot t) = \dfrac{e^{j\cdot\omega\cdot t} + e^{-j\cdot\omega\cdot t}}{2} \qquad \sin(\omega \cdot t) = \dfrac{e^{j\cdot\omega\cdot t} - e^{-j\cdot\omega\cdot t}}{2\cdot j}$

Ex. 4 $\quad f(t) = u(t)\cdot\cos(\omega\cdot t)$

$$\mathbf{F}(s) = \int_0^\infty \cos(\omega\cdot t)\cdot e^{-s\cdot t}\,dt = \int_0^\infty \left(\frac{e^{j\cdot\omega\cdot t} + e^{-j\cdot\omega\cdot t}}{2}\right)\cdot e^{-s\cdot t}\,dt = \int_0^\infty \frac{e^{(j\cdot\omega - s)\cdot t} + e^{-(j\cdot\omega + s)\cdot t}}{2}\,dt$$

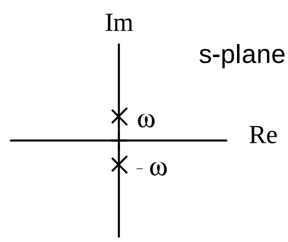
$$= \frac{1}{2}\cdot\int_0^\infty e^{(j\cdot\omega - s)\cdot t}\,dt + \frac{1}{2}\cdot\int_0^\infty e^{-(j\cdot\omega + s)\cdot t}\,dt$$

$$= \frac{1}{2}\cdot\left(\frac{1}{j\cdot\omega - s}\right)\cdot e^{(j\cdot\omega - s)\cdot t}\Bigg|_0^\infty + \frac{1}{2}\cdot\left[\frac{1}{-(j\cdot\omega + s)}\right]\cdot e^{-(j\cdot\omega + s)\cdot t}\Bigg|_0^\infty$$

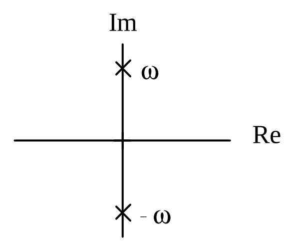
$$= 0 - \frac{1}{2}\cdot\left(\frac{1}{j\cdot\omega - s}\right)\cdot(1) + 0 - \frac{1}{2}\cdot\left[\frac{1}{-(j\cdot\omega + s)}\right]\cdot(1) = \frac{-1}{-2\cdot j\cdot\omega - 2\cdot s} + \frac{-1}{2\cdot j\cdot\omega - 2\cdot s}$$

$$= \frac{1}{2\cdot j\cdot\omega + 2\cdot s} + \frac{-1}{2\cdot j\omega - 2\cdot s} = \frac{(2\cdot j\cdot\omega - 2\cdot s) - (2\cdot j\cdot\omega + 2\cdot s)}{(2\cdot j\cdot\omega + 2\cdot s)\cdot(2\cdot j\cdot\omega - 2\cdot s)}$$

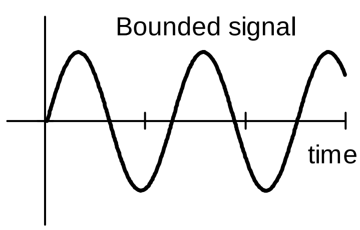
$$= \frac{-4\cdot s}{(2\cdot j\cdot\omega + 2\cdot s)\cdot(2\cdot j\cdot\omega - 2\cdot s)} = \frac{-4\cdot s}{4\cdot j^2\cdot\omega^2 - 4\cdot s^2} = \frac{-s}{-\omega^2 - s^2} = \frac{s}{\omega^2 + s^2}$$



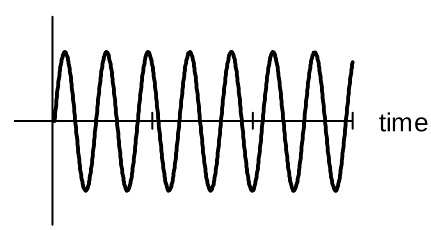
$$\frac{A\cdot s + B}{\left(s^2 + \omega^2\right)}$$



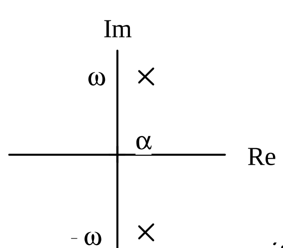
time-domain



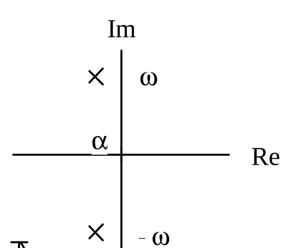
Doesn't converge



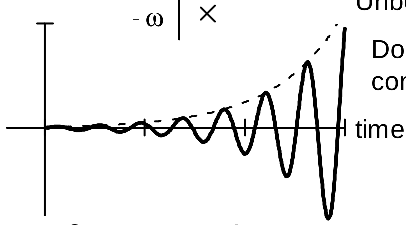
What if the poles have a real component? $\quad f(t) = u(t)\cdot e^{\alpha\cdot t}\cdot\sin(\omega\cdot t)$



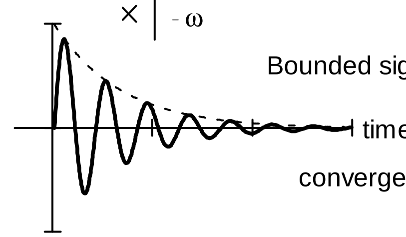
$$\frac{A\cdot s + B}{(s - \alpha)^2 + \omega^2}$$



Unbounded signal



Doesn't converge



Bounded signal

converges

Ex. 5 Multiply by time property

$$f(t) = u(t)\cdot t\cdot e^{a\cdot t} \qquad \mathbf{F}(s) = \int_0^\infty t\cdot e^{a\cdot t}\cdot e^{-s\cdot t}\,dt = \int_0^\infty t\cdot e^{(a-s)\cdot t}\,dt$$

Remember integration by parts:

$$\int h(t)\cdot\frac{d}{dt}g(t)\,dt = h(t)\cdot g(t) - \int g(t)\cdot\frac{d}{dt}h(t)\,dt$$

choose: $h(t) = t$ from which: $\frac{d}{dt}h(t) = 1$

and: $\frac{d}{dt}g(t) = e^{(a-s)\cdot t}$ from which: $g(t) = \int e^{(a-s)\cdot t}\,dt = \frac{e^{(a-s)\cdot t}}{(a-s)}$

$$h(t)\cdot g(t) \quad - \int g(t)\cdot\frac{d}{dt}h(t)\,dt$$

$$\mathbf{F}(s) = \int_0^\infty t\cdot e^{(a-s)\cdot t}\,dt = t\cdot\frac{e^{(a-s)\cdot t}}{(a-s)}\Bigg|_0^\infty - \int_0^\infty \frac{e^{(a-s)\cdot t}}{(a-s)}\cdot(1)\,dt = t\cdot\frac{e^{(a-s)\cdot t}}{(a-s)}\Bigg|_0^\infty - \frac{e^{(a-s)\cdot t}}{(a-s)^2}\Bigg|_0^\infty$$

$$= 0 - 0 - \left[0 - \frac{1}{(a-s)^2}\right]$$

$$= \frac{1}{(a-s)^2} = \frac{1}{(s-a)^2}$$

The easy way:

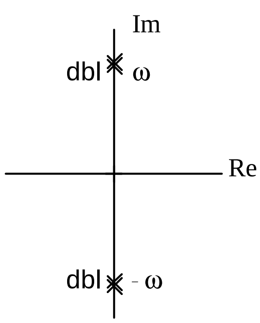
Use the "multiplication by time" property # 5 on p.8 of the Bodson textbook

$$t\cdot x(t) \quad \Leftrightarrow \quad -\frac{d}{ds}X(s)$$

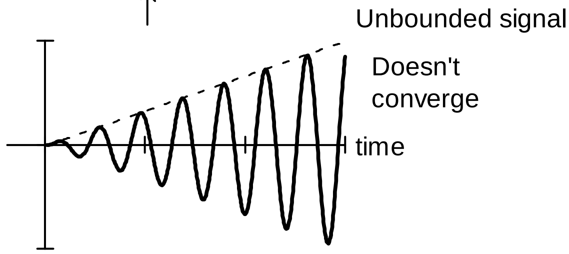
$$t\cdot e^{a\cdot t} \quad \Leftrightarrow \quad -\frac{d}{ds}\left(\frac{1}{s-a}\right) = -\frac{d}{ds}\left[(s-a)^{-1}\right] = -\frac{1}{-1}\cdot\frac{1}{(s-a)^2}\cdot\left[\frac{d}{ds}(s-a)\right] = \frac{1}{(s-a)^2}\cdot 1 = \frac{1}{(s-a)^2}$$

Anything that works for exponentials also works for sines and cosines...

$$\frac{A\cdot s + B}{\left(s^2+\omega^2\right)^2}$$

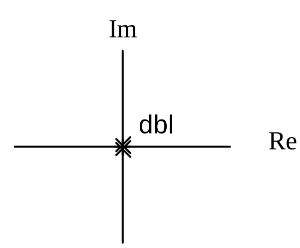


$t\cdot\sin(\omega\cdot t)\cdot u(t)$

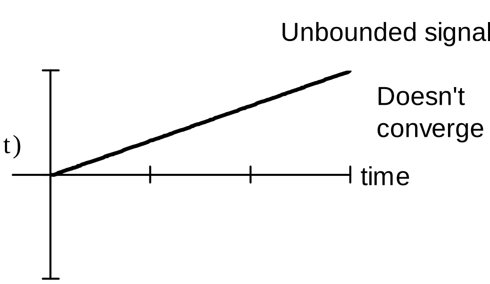


And "DC" too...

$$\frac{A}{s^2}$$



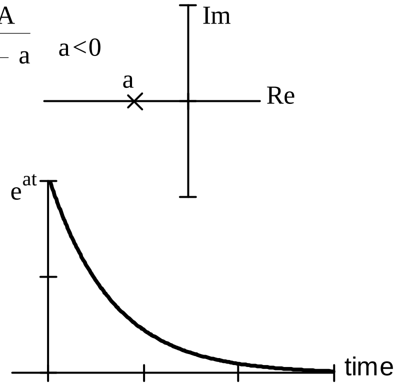
$t\cdot u(t)$

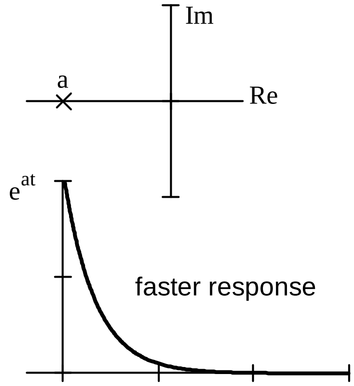


Signal Type, Boundedness, and Convergence can be predicted from the poles

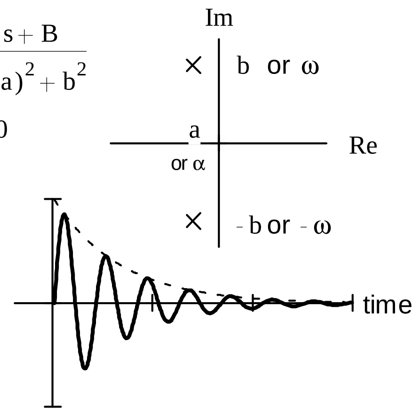
Poles in the Open-Left-Half-Plane (OLHP) Real part of pole is negative $\mathrm{Re}(s_p) < 0$

$\dfrac{A}{s-a}$ $a<0$





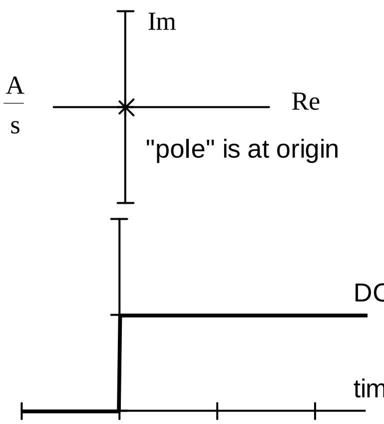
$\dfrac{A\cdot s+B}{(s-a)^2+b^2}$ $a<0$



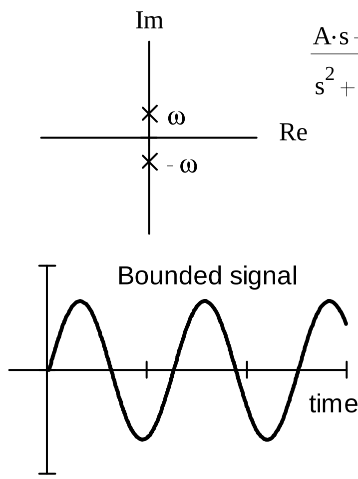
Bounded signals, Converge to zero

Single Poles on Imaginary Axis Real part of pole is zero $\mathrm{Re}(s_p) = 0$

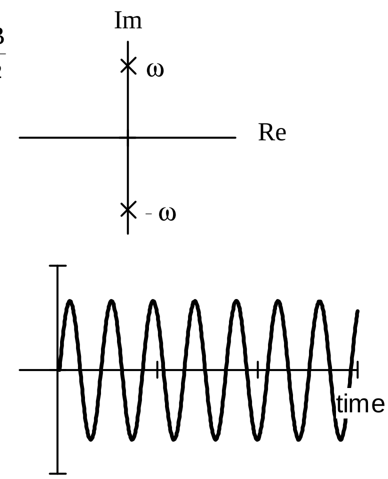
$\dfrac{A}{s}$



Bounded signal,
Converges to DC value



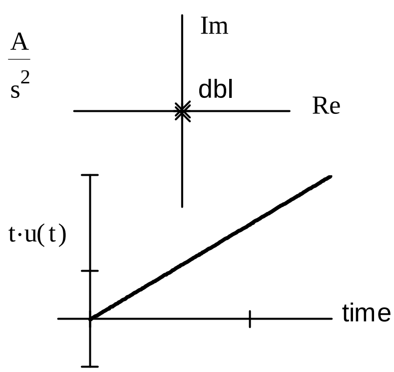
$\dfrac{A\cdot s+B}{s^2+\omega^2}$



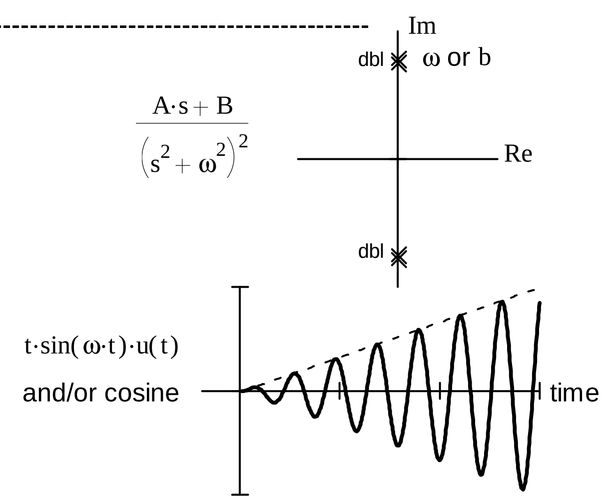
Bounded signals, Don't Converge

Double Poles on Imaginary Axis or

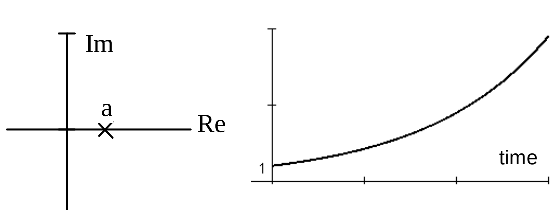
$\dfrac{A}{s^2}$

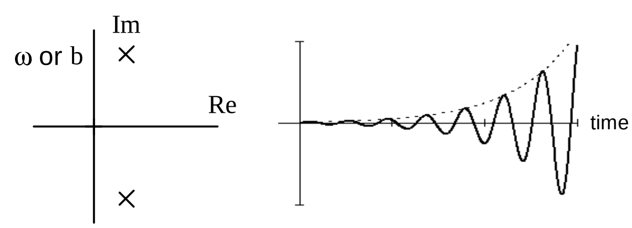


$\dfrac{A\cdot s+B}{\left(s^2+\omega^2\right)^2}$



In the Open-Right-Half-Plane (ORHP)





Unbounded signals, Don't Converge