ECE 3510 Discrete-time Signals and Systems

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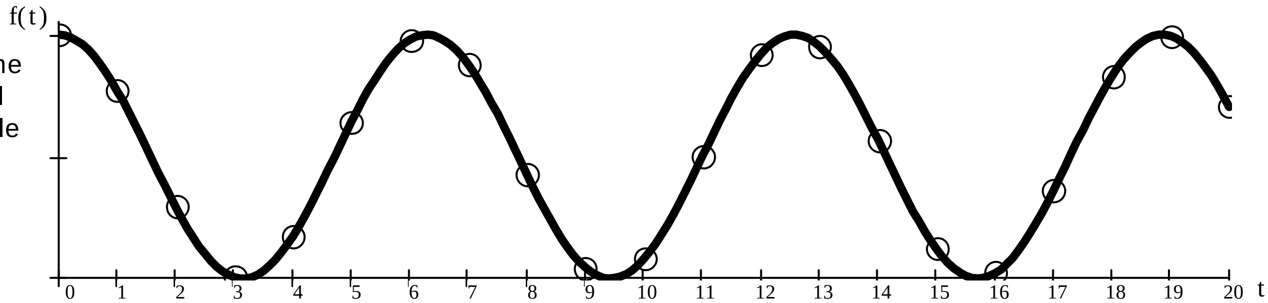
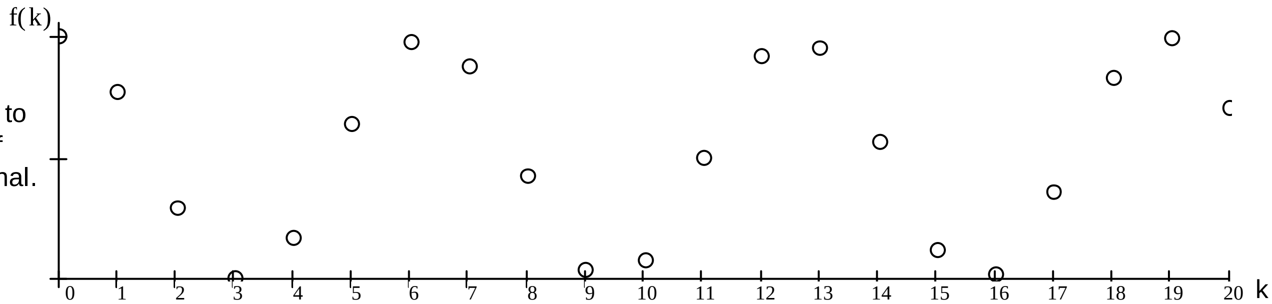
4/16/20

Continuous-time (analog) signal showing sample times



Discrete-time (digital) signal corresponding to the samples of the analog signal.

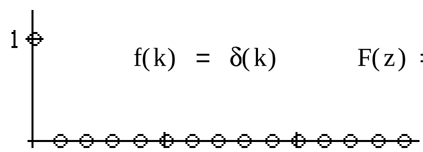


The z - transform

The z - transform will help us deal with discrete-time (digital) signals just like the Laplace transform helped us with continuous-time signals. So let's start making a table.

z - transform: $$F(z) = \sum_{k=0}^{\infty} f(k)\cdot z^{-k}$$

Impulse

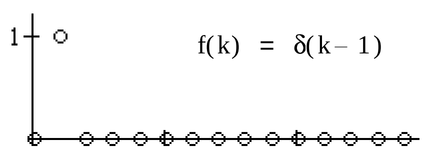


$$f(k) = \delta(k) \qquad F(z) = \sum_{k=0}^{\infty} \delta(k)\cdot z^{-k} = 1 + 0 + 0 + 0 + \ . \ . \ . \qquad F(z) = 1$$

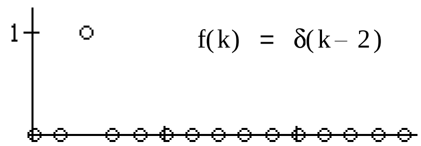
no pole

Just like Laplace:
$f(t) = \delta(t)$ & $\mathbf{F}(s) = 1$

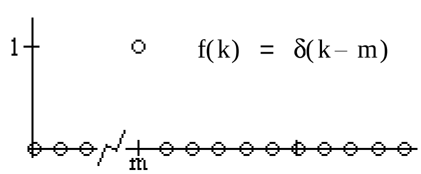
Delayed Impulses



$$f(k) = \delta(k-1) \qquad F(z) = \sum_{k=0}^{\infty} \delta(k-1)\cdot z^{-k} = 0 + \frac{1}{z} + 0 + 0 + \ . \ . \ . \qquad F(z) = \frac{1}{z}$$



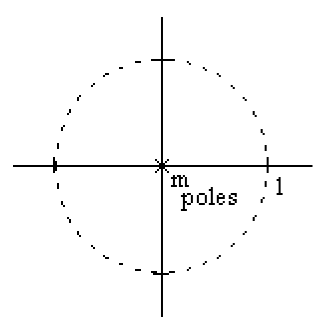
$$f(k) = \delta(k-2) \qquad F(z) = \sum_{k=0}^{\infty} \delta(k-2)\cdot z^{-k} = 0 + 0 + \frac{1}{z^2} + 0 + \ . \ . \ . \qquad F(z) = \frac{1}{z^2}$$



$$f(k) = \delta(k-m) \qquad F(z) = \sum_{k=0}^{\infty} \delta(k-m)\cdot z^{-k}$$

$$= 0 + \ . \ . \ . \ 0 + \frac{1}{z^m} + 0 + 0 + \ . \ . \ . = \frac{1}{z^m}$$

$$F(z) = \frac{1}{z^m}$$



Any finite-length signal can be made of delayed impulses, so all its poles are at the origin.

Power Series (Needed to solve the next few cases)

$$\text{SUM} = \sum_{k=0}^{n} \alpha^k = 1+\alpha+\alpha^2+\alpha^3+\alpha^4+\ldots\ \alpha^n$$

$$\text{SUM}\cdot(1-\alpha) = (1-\alpha)\left(1+\alpha+\alpha^2+\alpha^3+\alpha^4+\ldots\ \alpha^n\right)$$

$$= 1\left(1+\alpha+\alpha^2+\alpha^3+\alpha^4+\ldots\ \alpha^n\right) - \alpha\left(1+\alpha+\alpha^2+\alpha^3+\alpha^4+\ldots\ \alpha^n\right)$$

$$= 1\left(1+\alpha+\alpha^2+\alpha^3+\alpha^4+\ldots\ \alpha^n\right)$$
$$-\left(\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\ldots\ .\alpha^n+\alpha^{n+1}\right)$$
$$1 \qquad\qquad -\alpha^{n+1}$$

$$\text{SUM}\cdot(1-\alpha) = 1-\alpha^{n+1}$$

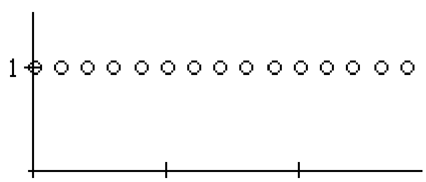
$$\text{SUM} = \frac{1-\alpha^{n+1}}{(1-\alpha)} \qquad \text{If } n=\infty \qquad \text{SUM} = \sum_{k=0}^{\infty}\alpha^{\infty} = \frac{1-\alpha^{\infty+1}}{(1-\alpha)} \overset{\text{if }(\alpha<1)}{=} \frac{1}{(1-\alpha)}$$

in region of convergence $(\alpha < 1)$

Unit Step

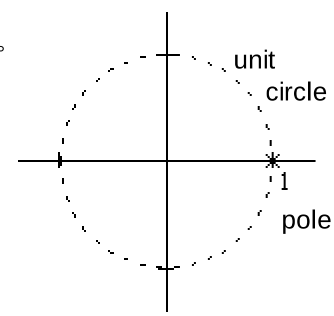
$f(k) = u(k)$

$$F(z) = \sum_{k=0}^{\infty} u(k)\cdot z^{-k} = 1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\ldots\ \left(\frac{1}{z}\right)^{\infty}$$



$$= \frac{1}{\left(1-\frac{1}{z}\right)} = F(z) = \frac{z}{(z-1)}$$

in region of convergence



Like Laplace:
$f(t) = u(t)$ & $\mathbf{F}(s) = \frac{1}{s}$

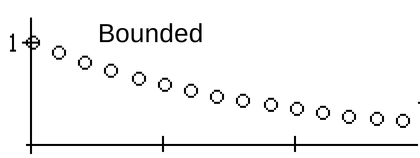
Geometric Progression

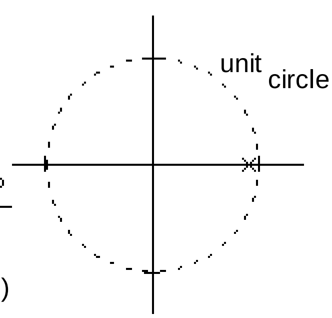
$f(k) = a^k\cdot u(k)$

$$F(z) = \sum_{k=0}^{\infty} a^k\cdot z^{-k} = 1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\left(\frac{a}{z}\right)^3+\ldots\ \left(\frac{a}{z}\right)^{\infty} = \frac{1}{\left(1-\frac{a}{z}\right)} = F(z) = \frac{z}{(z-a)}$$

If $(a<1)$

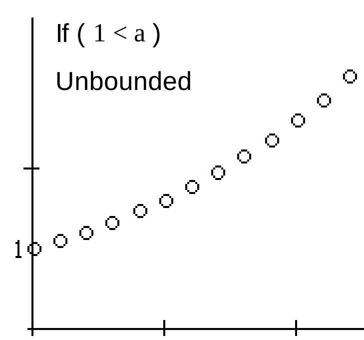
Bounded

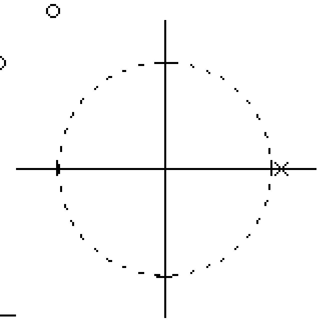




If $(1<a)$

Unbounded



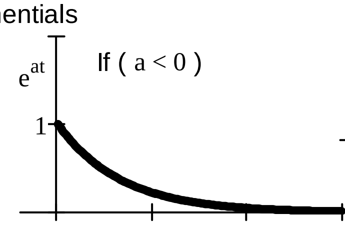


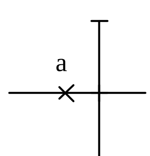
Unit step is a special case: $(a=1)$

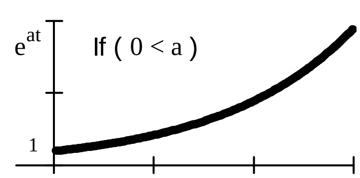
Like Laplace exponentials

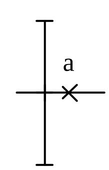
$f(t) = e^{a\cdot t}\cdot u(t)$

$\mathbf{F}(s) = \frac{1}{s-a}$









$f(k) = \left(C_1 \cdot a_1^{\,k} + C_2 \cdot a_2^{\,k}\right) \cdot u(k)$ $\qquad F(z) = C_1 \cdot \frac{z}{z - a_1} + C_2 \cdot \frac{z}{z - a_2}$ Linearity

<u>Sinusoidals</u>

If $C_2 = \overline{C_1}$ (complex conjugate) and $a_2 = \overline{a_1}$ and we'll now call $C_1 = C$ and $a_1 = p$

Then $f(k) = \left[C \cdot p^k + \overline{C} \cdot \left(\overline{p}\right)^k\right] \cdot u(t)$ and $F(z) = C \cdot \frac{z}{z - p} + \overline{C} \cdot \frac{z}{z - \overline{p}}$

$$
\begin{aligned}
f(k) &= \left[C \cdot p^k + \overline{C} \cdot \left(\overline{p}\right)^k\right] \cdot u(k) \\
&= \left[|C| \cdot e^{j \cdot \theta_C} \cdot (|p|)^k \cdot e^{j \cdot \theta_p \cdot k} + |C| \cdot e^{-j \cdot \theta_C} \cdot (|p|)^k \cdot e^{-j \cdot \theta_p \cdot k}\right] \cdot u(k) \\
&= |C| \cdot (|p|)^k \cdot \left(e^{j \cdot \theta_C} \cdot e^{j \cdot \theta_p} + e^{-j \cdot \theta_C} \cdot e^{-j \cdot \theta_p \cdot k}\right) \cdot u(k) \\
&= |C| \cdot (|p|)^k \cdot \left[e^{j \cdot \left(\theta_C + \theta_p \cdot k\right)} + e^{-j \cdot \left(\theta_C + \theta_p \cdot k\right)}\right] \cdot u(k) \\
&= 2 \cdot |C| \cdot (|p|)^k \cdot \left[\frac{e^{j \cdot \left(\theta_p \cdot k + \theta_C\right)} + e^{-j \cdot \left(\theta_p \cdot k + \theta_C\right)}}{2}\right] \cdot u(k) \\
&= 2 \cdot |C| \cdot (|p|)^k \cdot \cos\left(\theta_p \cdot k + \theta_C\right) \cdot u(k)
\end{aligned}
$$

$$= \frac{C \cdot z \cdot \left(z - \overline{p}\right) + \overline{C} \cdot z \cdot (z - p)}{(z - p) \cdot \left(z - \overline{p}\right)}$$

2 complex-conjugate poles at p and $\overline{p}$

Recall Euler's eq.: $\cos(\theta \cdot t) = \frac{e^{j \cdot \theta \cdot t} + e^{-j \cdot \theta \cdot t}}{2}$

If C is real ($\theta_C = 0$)

$f(k) = 2 \cdot C \cdot (|p|)^k \cdot \cos\left(\theta_p \cdot k\right) \cdot u(k)$

$$F(z) = \frac{C \cdot z \cdot \left[\left(z - \overline{p}\right) + (z - p)\right]}{(z - p) \cdot \left(z - \overline{p}\right)} = \frac{C \cdot z \cdot \left(2 \cdot z - \overline{p} - p\right)}{z^2 - z\left(\overline{p} + p\right) + p \cdot \overline{p}}$$

$$
\begin{aligned}
\overline{p} + p &= |p| \cdot \cos\left(\theta_p\right) - j \cdot |p| \cdot \sin\left(\theta_p\right) + |p| \cdot \cos\left(\theta_p\right) + j \cdot |p| \cdot \sin\left(\theta_p\right) \\
&= |p| \cdot \cos\left(\theta_p\right) + |p| \cdot \cos\left(\theta_p\right) \qquad = 2 \cdot |p| \cdot \cos\left(\theta_p\right)
\end{aligned}
$$

$$F(z) = \frac{2 \cdot C \cdot z \cdot \left(z - |p| \cdot \cos\left(\theta_p\right)\right)}{z^2 - z \cdot \left(2 \cdot |p| \cdot \cos\left(\theta_p\right)\right) + p \cdot \overline{p}}$$

This leads directly to (let C=1/2):

If $f(k) = (|p|)^k \cdot \cos\left(\theta_p \cdot k\right) \cdot u(k)$ $\qquad F(z) = \frac{z \cdot \left(z - |p| \cdot \cos\left(\theta_p\right)\right)}{z^2 - 2 \cdot |p| \cdot \cos\left(\theta_p\right) \cdot z + (|p|)^2}$

And If $|p| = 1$ (poles are right in the unit circle)

$f(k) = \cos\left(\theta_p \cdot k\right) \cdot u(k)$ $\qquad F(z) = \frac{z \cdot \left(z - \cos\left(\theta_p\right)\right)}{z^2 - 2 \cdot \cos\left(\theta_p\right) \cdot z + 1}$

Then sometimes θ_p is replaced by Ω_o

If C is -j |C|, imaginary (θ_C = -90º) ($\overline{C} = j \cdot |C|$)

$f(k) = 2 \cdot (|p|)^k \cdot \sin\left(\theta_p \cdot k\right) \cdot u(k)$

$$
\begin{aligned}
F(z) &= \frac{-j \cdot |C| \cdot z \cdot \left(z - \overline{p}\right) + j \cdot |C| \cdot z \cdot (z - p)}{(z - p) \cdot \left(z - \overline{p}\right)} \\
&= \frac{|C| \cdot z \cdot \left(-j \cdot z + j \cdot z + j \cdot \overline{p} - j \cdot p\right)}{z^2 - 2 \cdot |p| \cdot \cos\left(\theta_p\right) \cdot z + (|p|)^2}
\end{aligned}
$$

$$
\begin{aligned}
j \cdot \overline{p} - j \cdot p &= \left(j \cdot |p| \cdot \cos\left(\theta_p\right) - j \cdot j \cdot |p| \cdot \sin\left(\theta_p\right)\right) - j \cdot |p| \cdot \cos\left(\theta_p\right) - j \cdot j \cdot |p| \cdot \sin\left(\theta_p\right) \\
&= j \cdot |p| \cdot \cos\left(\theta_p\right) + |p| \cdot \sin\left(\theta_p\right) - j \cdot |p| \cdot \cos\left(\theta_p\right) + |p| \cdot \sin\left(\theta_p\right)
\end{aligned}
$$

$$F(z) = \frac{z \cdot \left(2 \cdot |p| \cdot \sin\left(\theta_p\right)\right)}{z^2 - 2 \cdot |p| \cdot \cos\left(\theta_p\right) \cdot z + (|p|)^2}$$

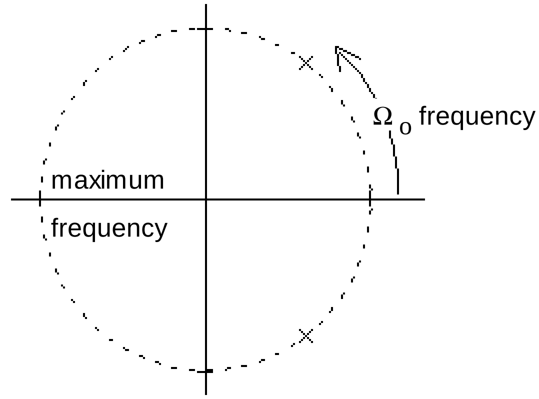
which gets us to the sine terms in the table

<u>Sinusoidals</u>

$$f(k) = \cos(\Omega_o \cdot k)\cdot u(k) \qquad F(z) = \frac{z(z - \cos(\Omega_o))}{z^2 - 2\cdot\cos(\Omega_o)\cdot z + 1}$$

AND

$$f(k) = \sin(\Omega_o \cdot k)\cdot u(k) \qquad F(z) = \frac{z\cdot\sin(\Omega_o)}{z^2 - 2\cdot\cos(\Omega_o)\cdot z + 1}$$

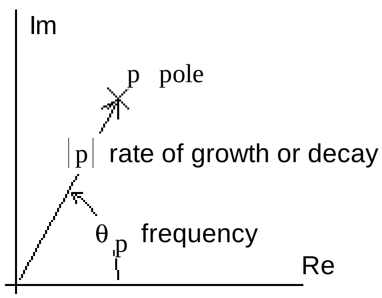


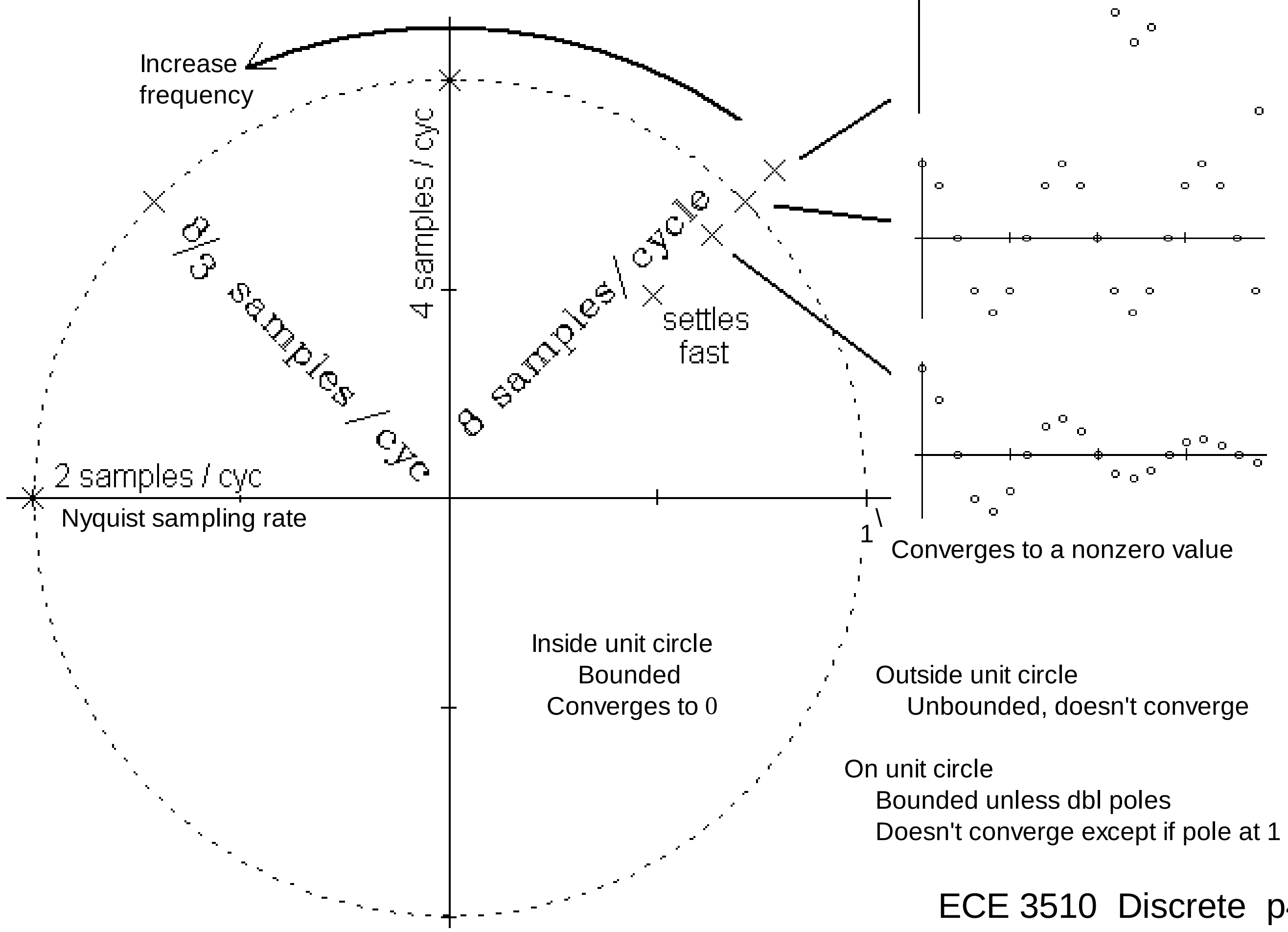
<u>Sinusoidals with growth or decay</u>

$$f(k) = \left|p^k\right|\cdot\cos(\theta_p \cdot k)\cdot u(k) \qquad F(z) = \frac{z\cdot(z - |p|\cdot\cos(\theta_p))}{z^2 - 2\cdot|p|\cdot\cos(\theta_p)\cdot z + (|p|)^2}$$

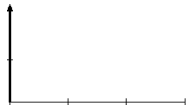
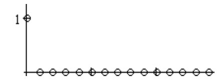
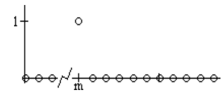
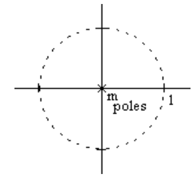
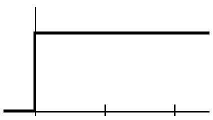
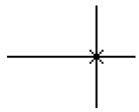
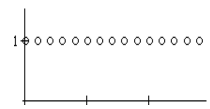
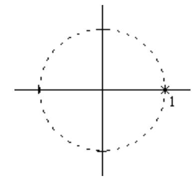
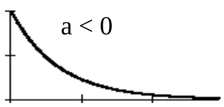
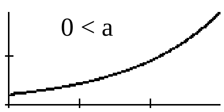
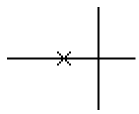
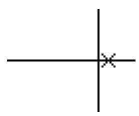
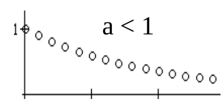
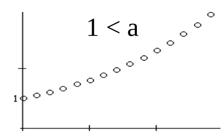
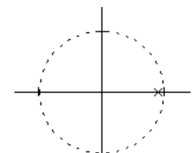
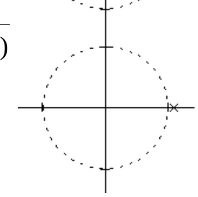
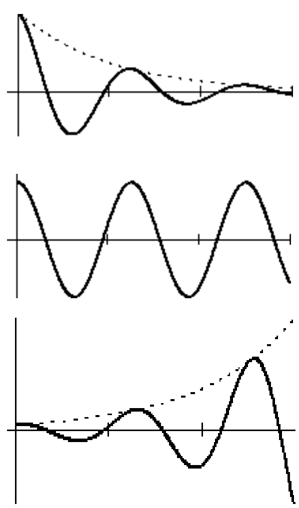
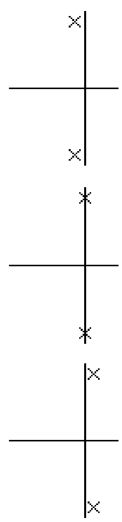
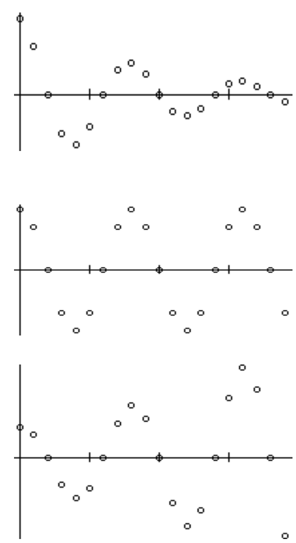
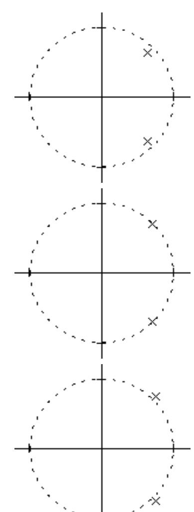
AND

$$f(k) = (|p|)^k\cdot\sin(\theta_p \cdot k)\cdot u(k) \qquad F(z) = \frac{z\cdot(|p|\cdot\sin(\theta_p))}{z^2 - 2\cdot|p|\cdot\cos(\theta_p)\cdot z + (|p|)^2}$$





Laplace Transform (Unilateral) / z - transforms

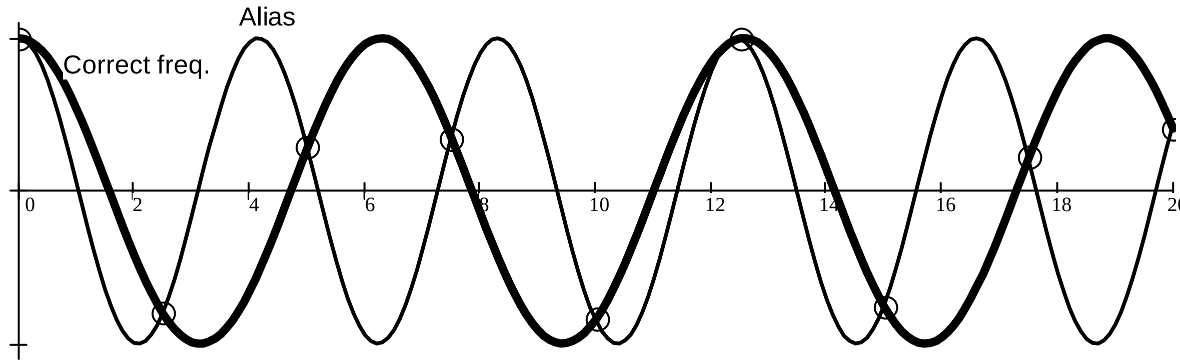
	Laplace $f(t)$		$F(s)$	pole	z $f(k)$		$F(z)$	pole
impulse	$\delta(t)$		1	none	$\delta(k)$		1	none
delayed impulse					$\delta(k-m)$		$\frac{1}{z^m}$	
unit step	$u(t)$		$\frac{1}{s}$		$u(k)$		$\frac{z}{z-1}$	
Exponential or Geometric Progression	$e^{a \cdot t} \cdot u(t)$	$a < 0$  $0 < a$ 	$\frac{1}{s-a}$	 	$a^k \cdot u(k)$	$a < 1$  $1 < a$ 	$\frac{z}{(z-a)}$	 
Sinusoidals Only cosine shown here	$e^{a \cdot t} \cdot \cos(b \cdot t) \cdot u(t)$	$a < 0$, $a = 0$, $0 < a$ 	$\frac{s-a}{(s-a)^2 + b^2}$		$\lvert p \rvert^k \cdot \cos(\theta_p \cdot k) \cdot u(k)$	$\lvert p \rvert < 1$, $\lvert p \rvert = 1$, $\lvert p \rvert > 1$ 	$\frac{z \cdot \left(z - \lvert p \rvert \cdot \cos(\theta_p)\right)}{z^2 - 2 \cdot \lvert p \rvert \cdot \cos(\theta_p) \cdot z + (\lvert p \rvert)^2}$	

Time constant $\tau = -\frac{1}{\ln(\lvert p \rvert)}$ Settling time $T_s = 4 \cdot \tau$

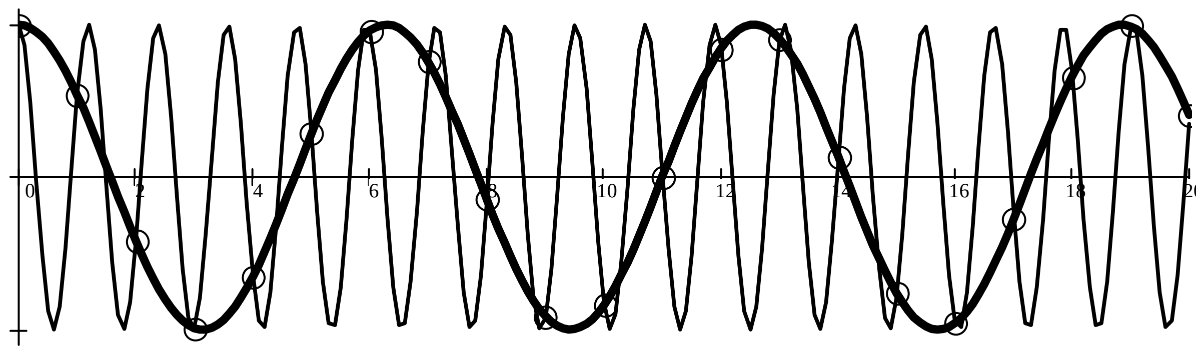
Damping factor $\zeta = \frac{-\ln(\lvert p \rvert)}{\sqrt{\ln(\lvert p \rvert)^2 - \theta_p^{\,2}}}$

Aliasing

Close to the maximum frequency (2 samples per cycle)



Far from the maximum frequency.



z - transform Properties

All the following are multiplied by $u(k)$ unless specified otherwise

Operation	f(k)	F(z)
Linearity	$c\cdot f(k) + d\cdot g(k)$	$c\cdot F(z) + d\cdot G(z)$
Right shift (Delay) $m \ge 0$	$f(k-m)\cdot u(k-m)$	$\frac{1}{z^m}\cdot F(z) = z^{-m}\cdot F(z)$
	$f(k-1)$	$z^{-1}\cdot F(z) + f(-1)$
	$f(k-2)$	$z^{-2}\cdot F(z) + z^{-1}\cdot f(-1) + f(-2)$
	$f(k-m)$	$\frac{1}{z^m}\cdot F(z) + \frac{1}{z^m}\cdot \sum_{k=1}^{m} f(-k)\cdot z^k$
Left shift $m \ge 0$	$f(k+m)$	$z^m\cdot F(z) - z^m\cdot \sum_{k=0}^{m-1} f(k)\cdot z^{-k}$
not common	$f(k+1)$	$z\cdot F(z) - z\cdot f(0)$
	$f(k+2)$	$z^2\cdot F(z) - z^2\cdot f(0) - z\cdot f(1)$
Initial value	$f(0)$	$\lim_{z \to \infty} F(z)$
Final value	$f(\infty)$	$\lim_{z \to 1} (z-1)\cdot F(z)$ (all poles of $(z-1)F(z)$ inside unit circle)

A. Stolp
4/15/09,
4/18/20

f(k)		F(z)
$f(k) = \frac{1}{2\cdot\pi\cdot j}\cdot\oint F(z)\cdot z^{k-1}\,dz$ integral around a closed path in the complex plane		$F(z) = \sum_{k=0}^{\infty} f(k)\cdot z^{-k}$
$\delta(k)$	impulse	1
$\delta(k-m)$	shifted impulse	$\frac{1}{z^m}$
$u(k)$	unit step	$\frac{z}{z-1}$

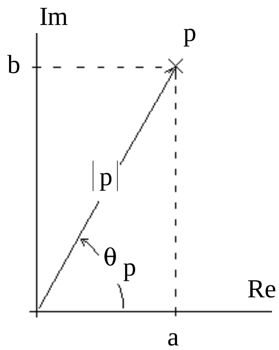
All the following are multiplied by $u(k)$

f(k)	F(z)
k	$\frac{z}{(z-1)^2}$
k^2	$\frac{z\cdot(z+1)}{(z-1)^3}$
k^3	$\frac{z\cdot\left(z^2+4\cdot z+1\right)}{(z-1)^4}$

Geometric Progression or Power Series

f(k)	F(z)
a^k	$\frac{z}{z-a}$
$k\cdot a^k$	$\frac{a\cdot z}{(z-a)^2}$
$k^2\cdot a^k$	$\frac{a\cdot z\cdot(z+a)}{(z-a)^3}$
$k^3\cdot a^k$	$\frac{a\cdot z\cdot\left(z^2+4\cdot a\cdot z+a^2\right)}{(z-a)^4}$

Sinusoids



f(k)	F(z)
$\cos\left(\Omega_o\cdot k\right)$	$\frac{z(z-a)}{z^2-2\cdot a\cdot z+1} = \frac{z\left(z-\cos\left(\Omega_o\right)\right)}{z^2-2\cdot\cos\left(\Omega_o\right)\cdot z+1}$
$\sin\left(\Omega_o\cdot k\right)$	$\frac{z\cdot b}{z^2-2\cdot a\cdot z+1} = \frac{z\cdot\sin\left(\Omega_o\right)}{z^2-2\cdot\cos\left(\Omega_o\right)\cdot z+1}$
$(\lvert p\rvert)^k\cdot\cos\left(\theta_p\cdot k\right)$	$\frac{z\cdot(z-a)}{z^2-2\cdot a\cdot z+\left(a^2+b^2\right)} = \frac{z\cdot\left(z-\lvert p\rvert\cdot\cos\left(\theta_p\right)\right)}{z^2-2\cdot\lvert p\rvert\cdot\cos\left(\theta_p\right)\cdot z+(\lvert p\rvert)^2}$
$(\lvert p\rvert)^k\cdot\sin\left(\theta_p\cdot k\right)$	$\frac{z\cdot b}{z^2-2\cdot a\cdot z+\left(a^2+b^2\right)} = \frac{z\cdot\left(\lvert p\rvert\cdot\sin\left(\theta_p\right)\right)}{z^2-2\cdot\lvert p\rvert\cdot\cos\left(\theta_p\right)\cdot z+(\lvert p\rvert)^2}$

z - transforms

Good for inverse z-transforms

A. Stolp
4/16/06
4/15/09

F(z)	f(k)
	All the following are multiplied by u(k) unless specified otherwise
Poles at zero	
$\frac{A\cdot z}{z} = A$	$A\cdot\delta(k)$
$\frac{B\cdot z}{z^2} = \frac{B}{z}$	$B\cdot\delta(k-1)$
$\frac{C\cdot z}{z^3} = \frac{C}{z^2}$	$C\cdot\delta(k-2)$
$\frac{D\cdot z}{z^4} = \frac{D}{z^3}$	$D\cdot\delta(k-3)$
Poles on real axis (not at zero)	
$\frac{B\cdot z}{(z-p)}$	$B\cdot p^k$
$\frac{C\cdot z}{(z-p)^2}$	$C\cdot k\cdot p^{k-1}$
$\frac{D\cdot z}{(z-p)^3}$	$D\cdot\frac{1}{2}\cdot k\cdot(k-1)\cdot p^{k-2}$
$\frac{E\cdot z}{(z-p)^4}$	$E\cdot\frac{1}{6}\cdot k\cdot(k-1)\cdot(k-2)\cdot p^{k-3}$
Complex poles	
$\frac{B\cdot z}{(z-p)} + \frac{\overline{B}\cdot z}{\left(z-\overline{p}\right)}$	$2\cdot\lvert B\rvert\cdot(\lvert p\rvert)^k\cdot\cos\left(\theta_p\cdot k+\theta_B\right)$
$\frac{B\cdot z}{(z-p)^2} + \frac{\overline{B}\cdot z}{\left(z-\overline{p}\right)^2}$	$2\cdot\lvert B\rvert\cdot k\cdot(\lvert p\rvert)^{k-1}\cdot\cos\left[\theta_p\cdot(k-1)+\theta_B\right]$
$\frac{B\cdot z}{(z-p)^3} + \frac{\overline{B}\cdot z}{\left(z-\overline{p}\right)^3}$	$\lvert B\rvert\cdot k\cdot(k-1)\cdot(\lvert p\rvert)^{k-2}\cdot\cos\left[\theta_p\cdot(k-2)+\theta_B\right]$
$\frac{B\cdot z}{(z-p)^4} + \frac{\overline{B}\cdot z}{\left(z-\overline{p}\right)^4}$	$\frac{1}{3}\cdot\lvert B\rvert\cdot k\cdot(k-1)\cdot(k-2)\cdot(\lvert p\rvert)^{k-3}\cdot\cos\left[\theta_p\cdot(k-3)+\theta_B\right]$

where $B = \lvert B\rvert\cdot e^{j\cdot\theta_B}$ and $p = \lvert p\rvert\cdot e^{j\cdot\theta_p}$

if $B = C + D\cdot j$ and $p = q + r\cdot j$

then $\lvert B\rvert = \sqrt{C^2+D^2}$ and $\lvert p\rvert = \sqrt{q^2+r^2}$

$\theta_B = \operatorname{atan}\left(\frac{D}{C}\right)$ $\theta_p = \operatorname{atan}\left(\frac{r}{q}\right)$

z - transform Properties

A. Stolp
4/15/09

All the following are multiplied by u(k) unless specified otherwise

Operation	f(k)	F(z)	
Addition	$f(k)+g(k)$	$F(z)+G(z)$	
Scalar multiplication	$c\cdot f(k)$	$c\cdot F(z)$	
Linearity	$c\cdot f(k)+d\cdot g(k)$	$c\cdot F(z)+d\cdot G(z)$	
Right shift $m\ge 0$	$f(k-m)\cdot u(k-m)$	$\frac{1}{z^m}\cdot F(z) = z^{-m}\cdot F(z)$	
	$f(k-m)$	$\frac{1}{z^m}\cdot F(z)+\frac{1}{z^m}\cdot\sum_{k=1}^{m} f(-k)\cdot z^k$	
	$f(k-1)$	$z^{-1}\cdot F(z)+f(-1)$	
	$f(k-2)$	$z^{-2}\cdot F(z)+z^{-1}\cdot f(-1)+f(-2)$	
	$f(k-3)$	$z^{-3}\cdot F(z)+z^{-2}\cdot f(-1)+z^{-1}\cdot f(-2)+f(-3)$	
Left shift $m\ge 0$	$f(k+m)$	$z^m\cdot F(z)-z^m\cdot\sum_{k=0}^{m-1} f(k)\cdot z^{-k}$	
	$f(k+1)$	$z\cdot F(z)-z\cdot f(0)$	
	$f(k+2)$	$z^2\cdot F(z)-z^2\cdot f(0)-z\cdot f(1)$	
	$f(k+3)$	$z^3\cdot F(z)-z^3\cdot f(0)-z^2\cdot f(1)-z\cdot f(2)$	
Multiplication by p^k	$p^k\cdot f(k)$	$F\left(\frac{z}{p}\right)$	Frequency scaling
Multiplication by k	$k\cdot f(k)$	$-z\cdot\frac{d}{dz}F(z)$	Frequency differentiation
Time convolution	$f(k)*g(k)$	$F(z)\cdot G(z)$	
Initial value	$f(0)$	$\lim_{z\to\infty} F(z)$	
Final value	$f(\infty)$	$\lim_{z\to 1} (z-1)\cdot F(z)$ (all poles of $(z-1)F(z)$ inside unit circle)	

ECE 3510 Lines of Constant Damping in the z-plane

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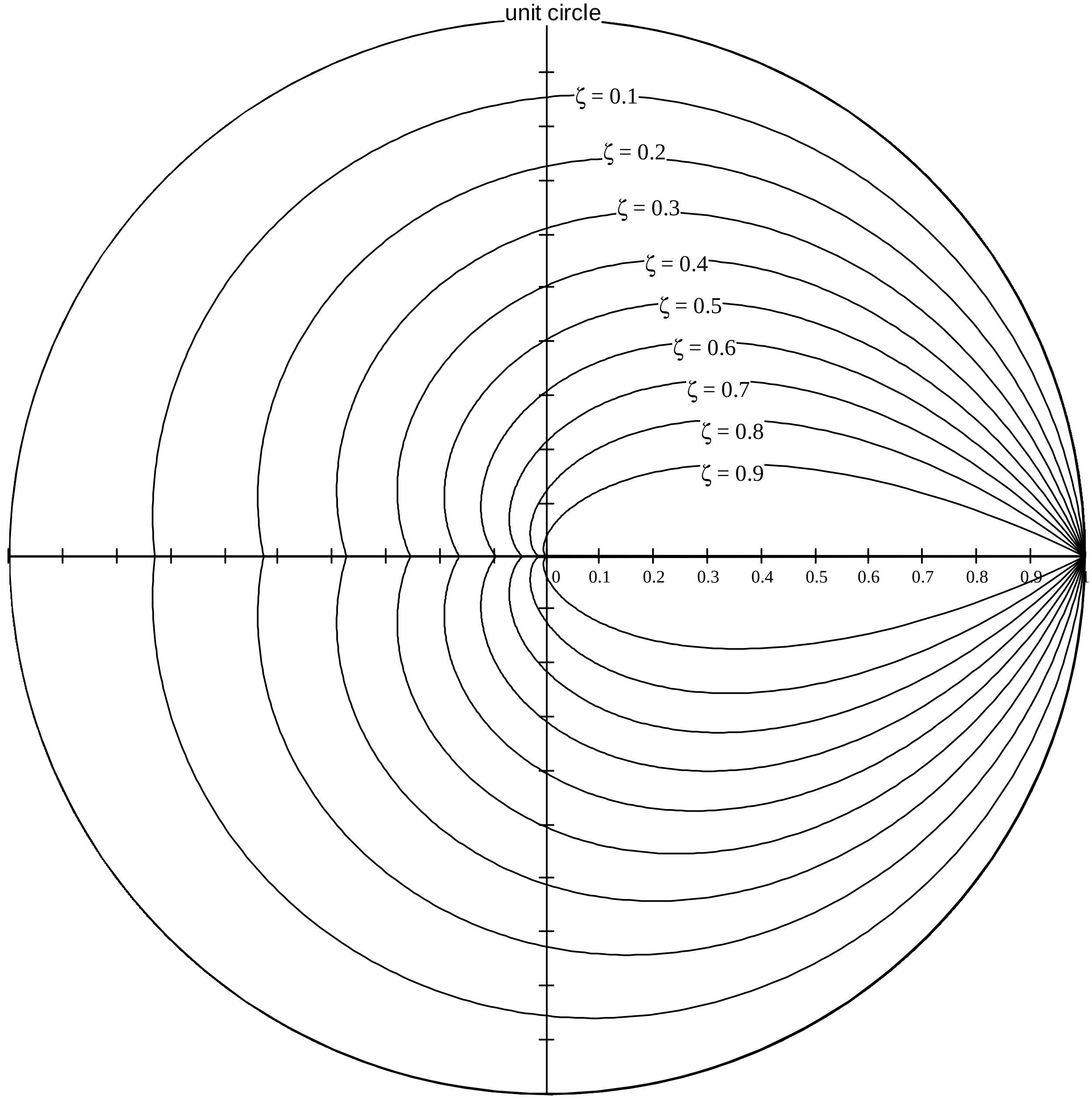
4/4/07



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4/17/09,
4/17/20

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Partial Fraction Expansion

Ex.1 $F(z) = \frac{1}{(z-1)\cdot(z+1)}$ Example 1 from Bodson, page 197

Divide by z first, because all the table entries have a z in the numerator, you can remultiply by z at the end.

$$\frac{F(z)}{z} = \frac{1}{z\cdot(z-1)\cdot(z+1)} = \frac{A}{z} + \frac{B}{(z-1)} + \frac{C}{z+1}$$

Multiply both sides by: $z\cdot(z-1)\cdot(z+1)$

$$1 = A\cdot(z-1)\cdot(z+1) + B\cdot z\cdot(z+1) + C\cdot z\cdot(z-1)$$

Set $z := 0$

$$1 = \underbrace{A\cdot(0-1)\cdot(0+1)}_{-1} + 0 + 0 \qquad A := \frac{1}{-1} \qquad A = -1$$

Set $z := 1$

$$1 = 0 + \underbrace{B\cdot 1\cdot(1+1)}_{2} + 0 \qquad B := \frac{1}{2} \qquad B = 0.5$$

Set $z := -1$

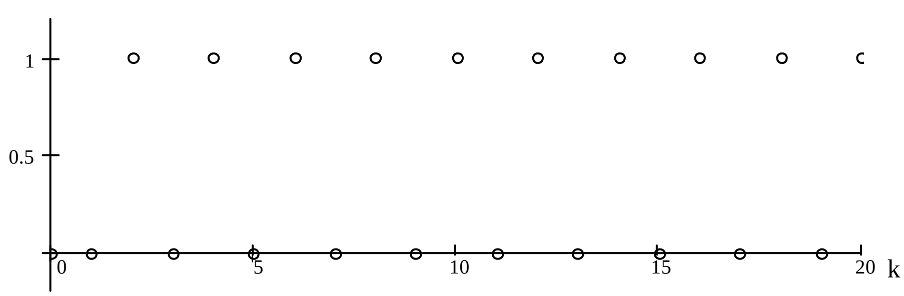
$$1 = 0 + 0 + \underbrace{C\cdot(-1)\cdot(-1-1)}_{2} \qquad C := \frac{1}{2} \qquad C = 0.5$$

$$\frac{F(z)}{z} = \frac{1}{z\cdot(z-1)\cdot(z+1)} = \frac{-1}{z} + \frac{1}{2}\cdot\frac{1}{(z-1)} + \frac{1}{2}\cdot\frac{1}{z+1}$$

Now multiply back through by z to get partial fractions that can actually be found in the table.

$$F(z) = \frac{1}{(z-1)\cdot(z+1)} = \frac{-1\cdot z}{z} + \frac{1}{2}\cdot\frac{z}{(z-1)} + \frac{1}{2}\cdot\frac{z}{z+1}$$

$$f(k) := \left[-1\cdot\delta(k) + \frac{1}{2} + \frac{1}{2}\cdot(-1)^k\right]\cdot u(k)$$



By long division, as shown in section 6.3.2 in Bodson text.

$$(z-1)\cdot(z+1) = \left(z^2-1\right)$$

$$\begin{array}{r|l} & z^{-2} + z^{-4} + z^{-6} + \;.\;.\;. \\ \left(z^2-1\right) & 1 \\ & 1 - z^{-2} \\ \hline & z^{-2} \\ & z^{-2} - z^{-4} \\ \hline & z^{-4} \\ & z^{-4} - z^{-6} \\ \hline & z^{-6} \quad \text{etc} \end{array}$$

never ends

Ex.2 $F(z) = \dfrac{z}{(z-0.9)^2\cdot(z+0.8)}$

$$\frac{F(z)}{z} = \frac{1}{(z-0.9)^2\cdot(z+0.8)} = \frac{A}{z-0.9} + \frac{0.9\cdot B}{(z-0.9)^2} + \frac{C}{z+0.8}$$

Multiply both sides by: $(z-0.9)^2\cdot(z+0.8)$

$$1 = A\cdot(z-0.9)\cdot(z+0.8) + 0.9\cdot B\cdot(z+0.8) + C\cdot(z-0.9)^2$$

Set $z := 0.9$

$$1 = 0 + 0.9\cdot B\cdot\underbrace{(0.9+0.8)}_{1.7} + 0 \qquad B := \frac{1}{0.9\cdot 1.7} \qquad B = 0.654$$

Set $z := -0.8$

$$1 = 0 + 0 + C\cdot\underbrace{(-0.8-0.9)^2}_{(-1.7)^2} \qquad C := \frac{1}{(-1.7)^2} \qquad C = 0.346$$

Back to equation above

$$1 = A\cdot(z+0.9)\cdot(z+0.8) + 0.9\cdot B\cdot(z+0.8) + C\cdot(z+0.9)^2$$

$$1 = A\cdot z^2 + 1.7\cdot A\cdot z + .72\cdot A + 0.9\cdot B\cdot z + 0.72\cdot B + C\cdot z^2 + 1.8\cdot C\cdot z + .81\cdot C$$

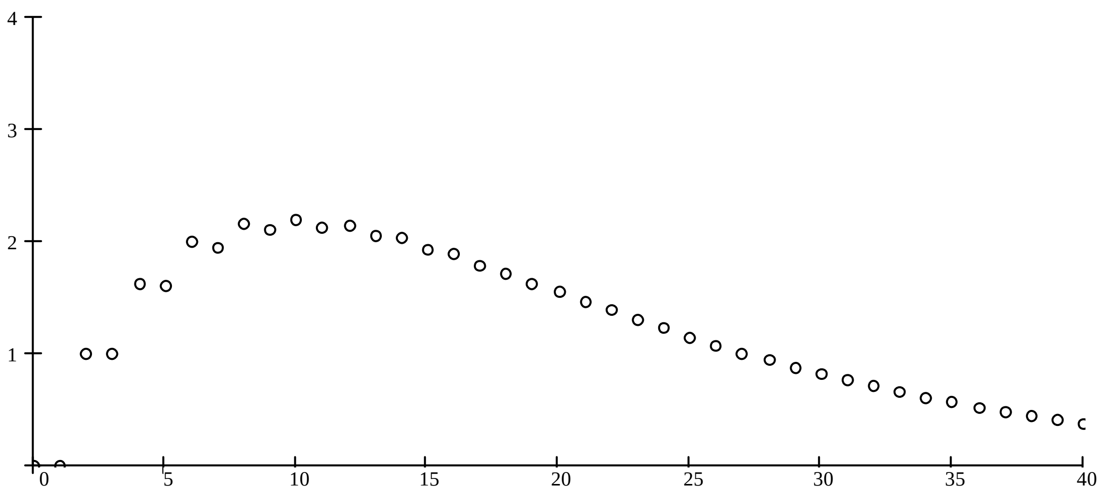
$$0\cdot z^2 = A\cdot z^2 + 0 + C\cdot z^2 \qquad A := -C \qquad A = -0.346$$

no z^2 term on the left

$$F(z) = \frac{z}{(z-0.9)^2\cdot(z+0.8)} = \frac{-0.346\cdot z}{z-0.9} + \frac{0.654\cdot 0.9\cdot z}{(z-0.9)^2} + \frac{0.346\cdot z}{z+0.8}$$

$$f(k) = -0.346\cdot(0.9)^k + 0.654\cdot k\cdot(0.9)^k + 0.346\cdot(-0.8)^k$$

$$f(k) := \left[-0.346\cdot(0.9)^k + 0.654\cdot k\cdot(0.9)^k + 0.346\cdot(-0.8)^k\right]\cdot u(k) \qquad k := 0, 1 .. 40$$



Ex.3 $F(z) = \dfrac{z}{z^2 - 2\cdot z + 2}$

The complex coefficient way (not recommended)

$$\frac{F(z)}{z} = \frac{1}{(z-(1+j))\cdot(z-(1-j))} = \frac{A}{(z-(1+j))} + \frac{B}{(z-(1-j))}$$

$$\left.\frac{1}{(z-(1-j))}\right|_{z\ =\ (1+j)} = A = \frac{1}{((1+j)-(1-j))} = -0.5j$$

$$\left.\frac{1}{(z-(1+j))}\right|_{z\ =\ 1-j} = B = \frac{1}{((1-j)-(1+j))} = 0.5j$$

$p := (1+j) \qquad |p| = \sqrt{2} \qquad \underline{/p} = \theta_p = \dfrac{\pi}{4}$

$c = -\dfrac{1}{2}\cdot j \qquad |c| = \dfrac{1}{2} \qquad \underline{/c} = \theta_c = -\dfrac{\pi}{2}$

Use this Table entry $\dfrac{C\cdot z}{(z-p)} + \dfrac{\overline{C}\cdot z}{\left(z-\overline{p}\right)}$ <--> $2\cdot|C|\cdot(|p|)^k\cdot\cos\left(\theta_p\cdot k + \theta_C\right)$ Note: table shows B, where I've changed to C for clarity here

$$f(k) = 2\cdot|c|\cdot(|p|)^k\cdot\cos\left(\theta_p\cdot k + \theta_c\right) = 2\cdot\frac{1}{2}\cdot\left(\sqrt{2}\right)^k\cdot\cos\left(\frac{\pi}{4}\cdot k - \frac{\pi}{2}\right)\cdot u(k) = \left(\sqrt{2}\right)^k\cdot\sin\left(\frac{\pi}{4}\cdot k\right)\cdot u(k)$$

The easy way

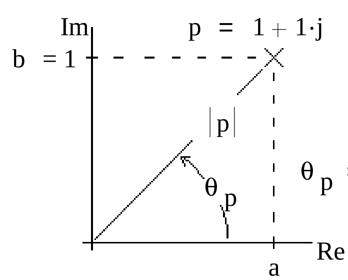
$$(|p|)^k\cdot\cos\left(\theta_p\cdot k\right) \quad \text{<-->} \quad \frac{z\cdot(z-a)}{z^2 - 2\cdot a\cdot z + \left(a^2+b^2\right)} = \frac{z\cdot\left(z - |p|\cdot\cos\left(\theta_p\right)\right)}{z^2 - 2\cdot|p|\cdot\cos\left(\theta_p\right)\cdot z + (|p|)^2}$$

$$(|p|)^k\cdot\sin\left(\theta_p\cdot k\right) \quad \text{<-->} \quad \frac{z\cdot b}{z^2 - 2\cdot a\cdot z + \left(a^2+b^2\right)} = \frac{z\cdot\left(|p|\cdot\sin\left(\theta_p\right)\right)}{z^2 - 2\cdot|p|\cdot\cos\left(\theta_p\right)\cdot z + (|p|)^2}$$

Fit to our denominator: $z^2 - 2\cdot z + 2$ $\qquad |p| = \sqrt{2}$

$a := 1 \qquad b := \sqrt{2 - a^2} \qquad b = 1$

|p| and θ_p are found here, but only needed in the very last step, to find f(k).



$\theta_p = \operatorname{acos}\left(\dfrac{1}{\sqrt{2}}\right) = 0.785 = \dfrac{\pi}{4}$

$a = 1$

Continue partial fraction expansion

$$\frac{F(z)}{z} = \frac{1}{z^2 - 2\cdot z + 2} = \frac{A(z-1)}{z^2 - 2\cdot z + 2} + \frac{B(1)}{z^2 - 2\cdot z + 2}$$

Let: $z = 1 \qquad B := 1$

$1 = A(z-1) + B$

$0\cdot z = A\cdot z \qquad A := 0$

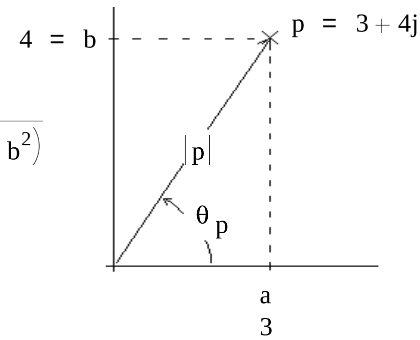
|p| and θ_p are needed here, to find f(k).

$$f(k) = A\cdot\left[(|p|)^k\cdot\cos\left(\theta_p\cdot k\right)\right]\cdot u(k) + B\cdot\left[(|p|)^k\cdot\sin\left(\theta_p\cdot k\right)\right]\cdot u(k)$$

$$f(k) = \left(\sqrt{2}\right)^k\cdot\sin\left(\frac{\pi}{4}\cdot k\right)\cdot u(k)$$

Ex.4

$$F(z) = \frac{2\cdot z\cdot(3\cdot z+17)}{(z-1)\cdot\left(z^2-6\cdot z+25\right)}$$



$$\frac{F(z)}{z} = \frac{2\cdot(3\cdot z+17)}{(z-1)\cdot\left(z^2-6\cdot z+25\right)} = \frac{A}{z-1} + \frac{B\cdot(z-a)}{z^2-2\cdot a\cdot z+\left(a^2+b^2\right)} + \frac{C\cdot b}{z^2-2\cdot a\cdot z+\left(a^2+b^2\right)}$$

$$= \frac{A}{z-1} + \frac{B(z-a)}{z^2-6\cdot z+25} + \frac{C\cdot b}{z^2-6\cdot z+25}$$

$z^2-6\cdot z+25$

$z^2-2\cdot a\cdot z+\left(a^2+b^2\right)$

$a := 3$ $\quad b := \sqrt{25-a^2}$ $\quad b = 4$

|p| and θ_p are found here, but only needed in the very last step, to find f(k). $\quad |p| = \sqrt{25} = 5$

$\theta_p = \operatorname{asin}\left(\frac{4}{5}\right) = 0.927 = \operatorname{acos}\left(\frac{3}{5}\right) = 0.927 = \operatorname{atan}\left(\frac{4}{3}\right) = 0.927$ (in radians)

several ways to find θ_p (in radians)

$(z-a)$

$a = 3 \qquad b = 4$

$$\frac{F(z)}{z} = \frac{2\cdot(3\cdot z+17)}{(z-1)\cdot\left(z^2-6\cdot z+25\right)} = \frac{A}{z-1} + \frac{B(z-3)}{z^2-6\cdot z+25} + \frac{C\cdot 4}{z^2-6\cdot z+25}$$

$$\left.\frac{2\cdot(3\cdot z+17)}{\left(z^2-6\cdot z+25\right)}\right|_{z\,=\,1} = A = \frac{2\cdot(3\cdot 1+17)}{\left(1^2-6\cdot 1+25\right)} = 2$$

$$2\cdot(3\cdot z+17) = A\cdot\left(z^2-6\cdot z+25\right) + B\cdot(z-3)\cdot(z-1) + C\cdot 4\cdot(z-1)$$

$$6\cdot z+34 = 2\cdot\left(z^2-6\cdot z+25\right) + B\cdot\left(z^2-4\cdot z+3\right) + C\cdot 4\cdot(z-1)$$

$$6\cdot z+34 = 2\cdot z^2-12\cdot z+50 + B\cdot z^2-4\cdot B\cdot z+3\cdot B + C\cdot 4\cdot z-C\cdot 4$$

$B := -2$

$$6\cdot z = -12\cdot z + 4\cdot 2\cdot z + C\cdot 4\cdot z$$

$$C = \frac{6+12-8}{4} = 2.5$$

OR $\quad \frac{34-50+6}{-4} = 2.5$

$$F(z) = \frac{2\cdot z\cdot(3\cdot z+17)}{(z-1)\cdot\left(z^2-6\cdot z+25\right)} = \frac{2\cdot z}{z-1} + \frac{-2\cdot z(z-3)}{z^2-6\cdot z+25} + \frac{2.5\cdot 4}{z^2-6\cdot z+25}$$

|p| and θ_p are needed here, to find f(k). $\quad u(k) \quad (|p|)^k\cdot\cos(\theta_p\cdot k) \quad (|p|)^k\cdot\sin(\theta_p\cdot k)$

$$f(k) = 2\cdot u(k) - 2\cdot 5^k\cdot\cos(0.927\cdot k) + 2.5\cdot 5^k\cdot\sin(0.927\cdot k)$$

$$f(k) = \left(2\cdot u(k) - 2\cdot 5^k\cdot\cos(0.927\cdot k) + 2.5\cdot 5^k\cdot\sin(0.927\cdot k)\right)\ u(k)$$

ECE 3510 Discrete-time Systems & Transfer Functions

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4/20/20

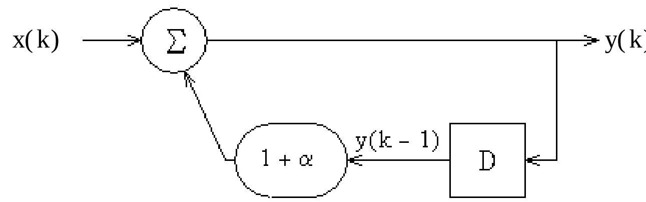
Section 6.4 in Bodson text (p.200) Follow along in the Textbook

Ex.1 ($ I got in bank) = ($ I had) + interest + ($ I add)

Define: $y(k)$ = bank account balance at end of day k

$x(k)$ = money deposited on day k

α = interest earned in one day



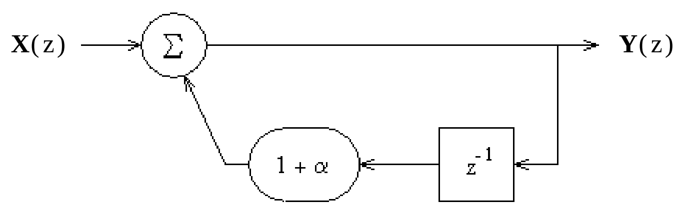
$$y(k) = y(k-1) + \alpha \cdot y(k-1) + x(k)$$

$$\mathbf{Y}(z) = z^{-1} \cdot \mathbf{Y}(z) + \alpha \cdot z^{-1} \cdot \mathbf{Y}(z) + \mathbf{X}(z)$$

$$= z^{-1} \cdot \mathbf{Y}(z) \cdot (1+\alpha) + \mathbf{X}(z)$$

$$\mathbf{Y}(z) - z^{-1} \cdot \mathbf{Y}(z) \cdot (1+\alpha) = \mathbf{X}(z)$$

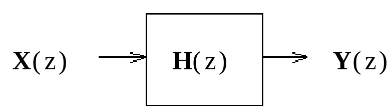
$$\mathbf{Y}(z) \cdot \left[1 - z^{-1} \cdot (1+\alpha)\right] = \mathbf{X}(z)$$



$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = \frac{1}{\left[1 - z^{-1} \cdot (1+\alpha)\right]} \cdot \frac{z}{z}$$

$$\mathbf{H}(z) = \frac{z}{z - (1+\alpha)}$$

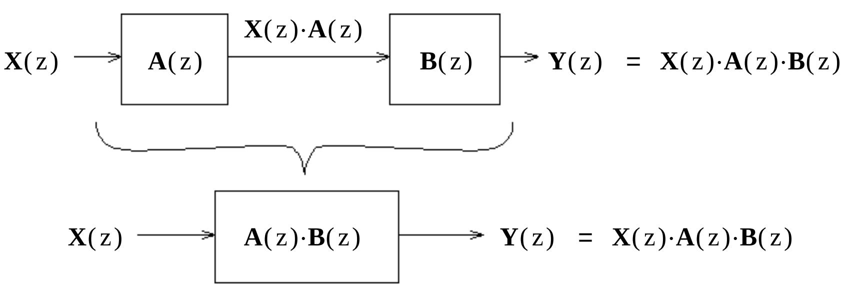
In general: $\mathbf{H}(z) = \frac{\text{output}}{\text{input}} = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)}$



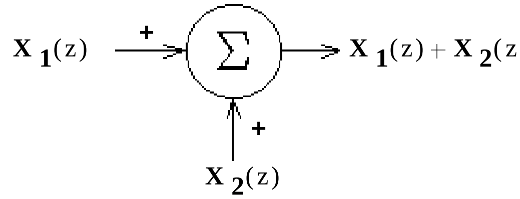
$\mathbf{Y}(z) = \mathbf{X}(z) \cdot \mathbf{H}(z)$

All Transfer - Function and Block - Diagrams we already know from Laplace work with z-transforms

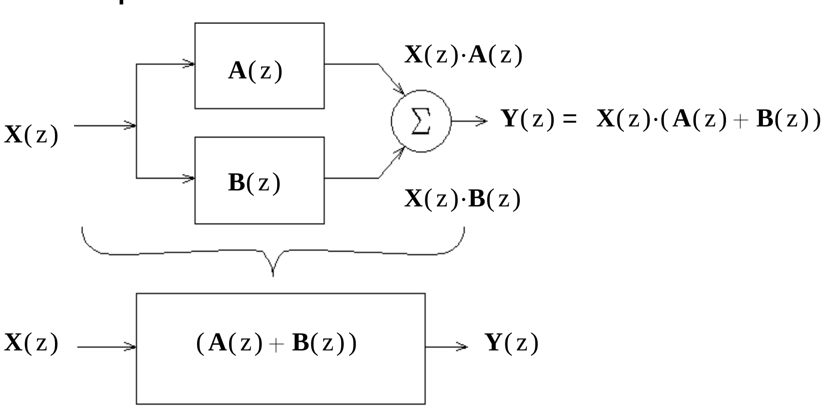
Serial - path systems



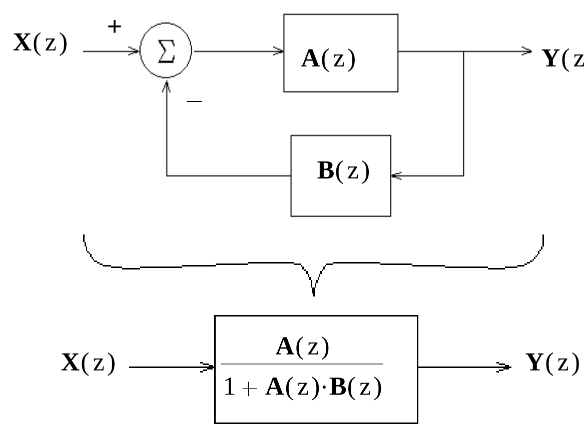
Summers



Parallel - paths



Feedback loop



Impulse Response

$\mathbf{X}(z) = 1 \longrightarrow$ [$\mathbf{H}(z)$] $\longrightarrow \mathbf{Y}(z) = \mathbf{X}(z)\cdot\mathbf{H}(z) = 1\cdot\mathbf{H}(z) = \mathbf{H}(z)$ = Impulse response = z - transform of $h(k)$

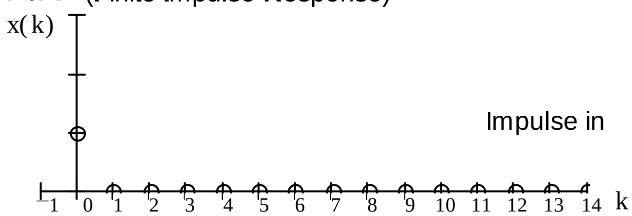
Input is an impulse $\qquad y(k) = h(k)$

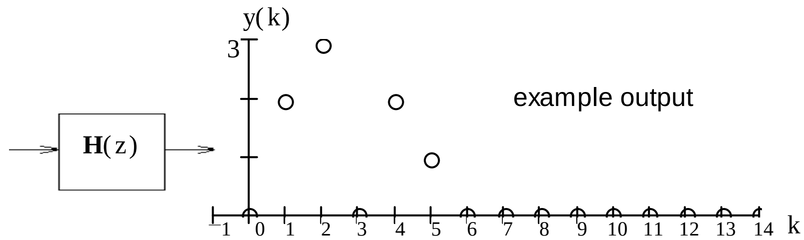
Sometimes the term "impulse response" is used in place of the term "transfer function"

FIR **F**inite **I**mpulse **R**esponse (FIR) means that output goes to and stays at absolute 0 within a finite number of steps.

IIR **I**nfinite **I**mpulse **R**esponse (IIR) means output never completely goes away. (It may approach 0 like exponential decay)

FIR (**F**inite **I**mpulse **R**esponse)





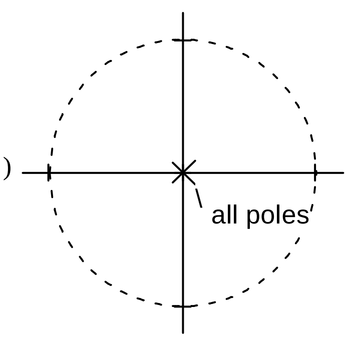
$$y(k) = 2\cdot x(k-1) + 3\cdot x(k-2) + 2\cdot x(k-4) + x(k-5)$$

$$\mathbf{Y}(z) = 2\cdot z^{-1}\cdot\mathbf{X}(z) + 3\cdot z^{-2}\cdot\mathbf{X}(z) + 2\cdot z^{-4}\cdot\mathbf{X}(z) + z^{-5}\cdot\mathbf{X}(z)$$

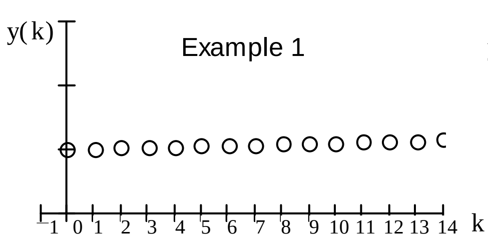
$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = \left(2\cdot z^{-1} + 3\cdot z^{-2} + 2\cdot z^{-4} + z^{-5}\right)\cdot\frac{z^5}{z^5}$$

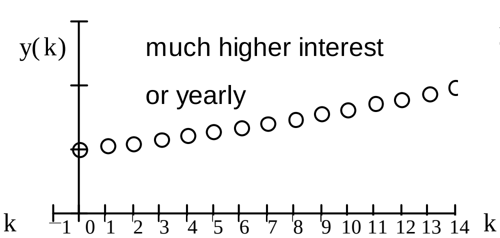
$$\mathbf{H}(z) = \frac{2\cdot z^4 + 3\cdot z^3 + 2\cdot z + 1}{z^5}$$

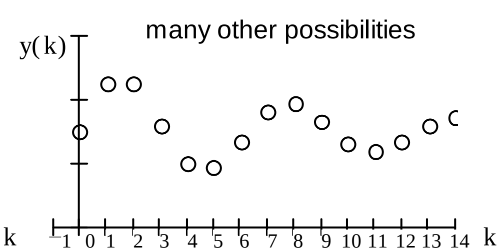
Poles of $\mathbf{H}(z)$ all at origin

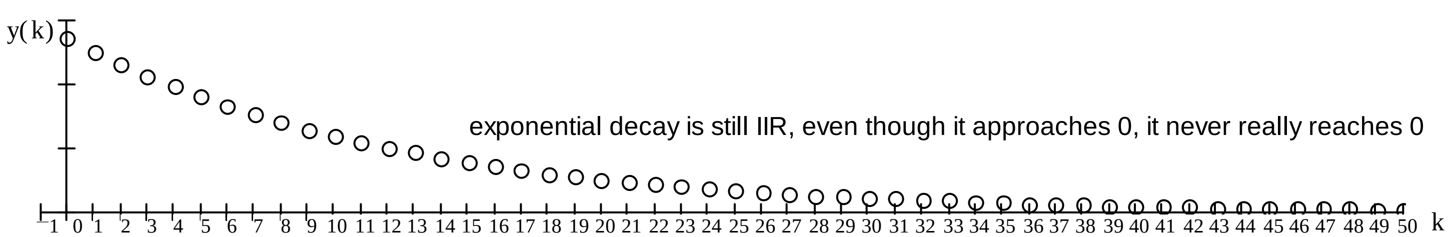


IIR (**I**nfinite **I**mpulse **R**esponse)









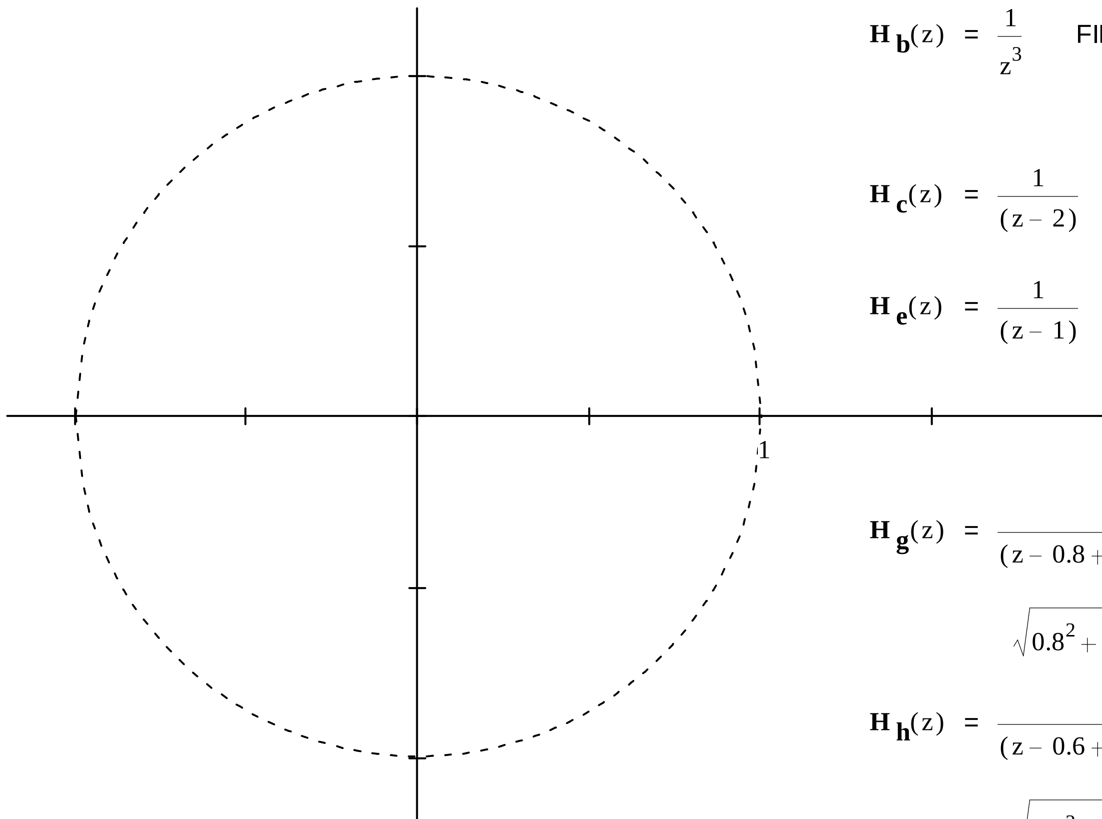
Bounded-Input, Bounded-Output (BIBO) Stable

A system is considered BIBO stable if the output is bounded for any bounded input.

A bounded input could have single poles on the unit circle at any location.
A bounded output may not have double poles on the unit circle or any poles outside the unit circle.
The output will have all the poles of the input plus all the poles of the system. (except in rare pole-zero cancellations.)

Therefore: A BIBO system may not have any poles on the unit circle or outside the unit circle.

Draw the poles on this unit circle



$$\mathbf{H}_{\mathbf{a}}(z) = \frac{1}{z\cdot(z-0.5)}$$

$$\mathbf{H}_{\mathbf{b}}(z) = \frac{1}{z^3}$$ FIR

$$\mathbf{H}_{\mathbf{c}}(z) = \frac{1}{(z-2)} \qquad \mathbf{H}_{\mathbf{d}}(z) = \frac{1}{(z+2)}$$

$$\mathbf{H}_{\mathbf{e}}(z) = \frac{1}{(z-1)} \qquad \mathbf{H}_{\mathbf{f}}(z) = \frac{1}{(z+1)}$$

$$\mathbf{H}_{\mathbf{g}}(z) = \frac{1}{(z-0.8+0.8\cdot j)\cdot(z-0.8-0.8\cdot j)}$$

$$\sqrt{0.8^2+0.8^2} = 1.131 = |p|$$

$$\mathbf{H}_{\mathbf{h}}(z) = \frac{1}{(z-0.6+0.8\cdot j)\cdot(z-0.6-0.8\cdot j)}$$

$$\sqrt{0.6^2+0.8^2} = 1 = |p|$$

$$\mathbf{H}_{\mathbf{i}}(z) = \frac{1}{(z-0.6+0.6\cdot j)\cdot(z-0.6-0.6\cdot j)}$$

$$\sqrt{0.6^2+0.6^2} = 0.849 = |p|$$

a,b, YES poles all inside unit circle

c,d, NO pole outide

e,f, NO right on unit circle

g, NO outside

h, NO right on unit circle

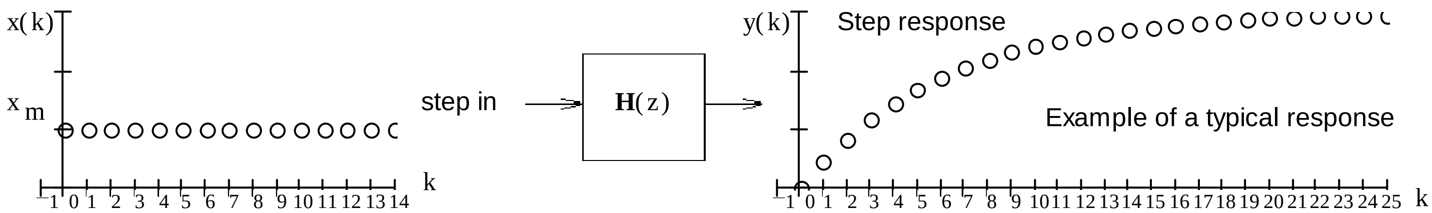
i, YES inside unit circle

Step Response

Remember: Continuous-time (Laplace) $\mathbf{Y}_{ss}(s) = \dfrac{X_m \cdot \mathbf{H}(0)}{z}$ $\quad y_{ss}(t) = X_m \cdot \mathbf{H}(0) \cdot u(t)$ $\quad H(0)$ = DC Gain

Sooo.. yesterday

Today . . .



$\mathbf{X}(z) = X_m \cdot u(k)$

Steady-State Response & DC Gain

For BIBO Systems

$$\mathbf{Y}(z) = \mathbf{X}(z) \cdot \mathbf{H}(z)$$

Complete step response = steady-state response + transient response

partial fraction expansion: $\mathbf{Y}(z) = X_m \cdot \dfrac{z}{z-1} \cdot \mathbf{H}(z) = A + \left[\dfrac{B}{(\blacksquare)} + \dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)}\right]$

divide both sides by z: $\dfrac{\mathbf{Y}(z)}{z} = X_m \cdot \dfrac{1}{z-1} \cdot \mathbf{H}(z) = \dfrac{A}{z-1} + \left[\dfrac{B}{(\blacksquare)} + \dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)}\right] \cdot \dfrac{1}{z}$

multiply both sides by $(z-1)$: $\mathbf{Y}(z) \cdot \dfrac{z-1}{z} = X_m \cdot \mathbf{H}(z) = A + \left[\dfrac{B}{(\blacksquare)} + \dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)}\right] \cdot \dfrac{z-1}{z}$

set $z := 1$: $X_m \cdot \mathbf{H}(1) = A$

$$\mathbf{Y}(z) = X_m \cdot \frac{z}{z-1} \cdot \mathbf{H}(z) = X_m \cdot \mathbf{H}(1) \cdot \frac{z}{z-1} + \left[\frac{B}{(\blacksquare)} + \frac{C}{(\blacksquare)} + \frac{D}{(\blacksquare)}\right]$$

steady-state response

transient response (all other poles are inside unit circle (BIBO))

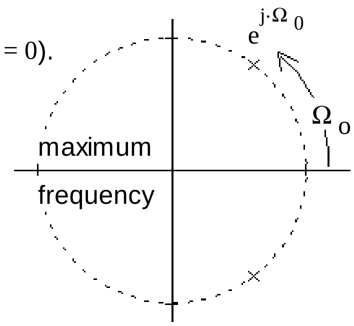
$$y_{ss}(k) = X_m \cdot \mathbf{H}(1) \cdot u(k)$$

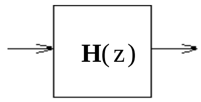
$H(1)$ = DC Gain

The **transient part** would be found by finishing the partial-fraction expansion.

Sinusoidal Response   For BIBO Systems

The sinusoidal response of a system is the output when the input is a sinusoidal (starting at $t = 0$).





For continuous time, we found $\mathbf{H}(j\omega) = |\mathbf{H}(j\omega)| \angle \mathbf{H}(j\omega)$ all $j\omega$ are on the Imaginary axis

For discrete time, we find $\mathbf{H}(p) = |\mathbf{H}(p)| \angle \mathbf{H}(p)$ where all p are on the unit circle

That means that $p = 1 \angle \Omega_0 = 1\cdot e^{j\cdot\Omega_0} = e^{j\cdot\Omega_0}$

$$\mathbf{H}\left(e^{j\cdot\Omega_0}\right) = \left|\mathbf{H}\left(e^{j\cdot\Omega_0}\right)\right| \angle \mathbf{H}(e^{j\Omega_0})$$

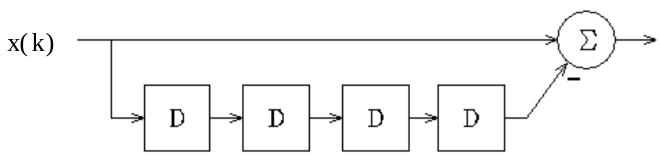
Use in the same way.
Either:

Modify the magnitude and phase of the input to get the steady-state output, $y_{ss}(k)$ (multiply magnitudes & add phases)

OR $\mathbf{Y}(z) = \mathbf{X}(z)\cdot\mathbf{H}\left(e^{j\cdot\Omega_0}\right)$ Which gives both steady-state and transient outputs.

to get a frequency response plot, allow to vary from 0 (or near 0) to the maximum frequency.

Example from text:



$y(k) = x(k) - x(k-4)$ FIR system

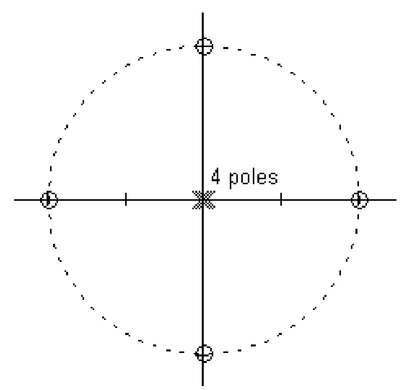
$\mathbf{Y}(z) = \mathbf{X}(z) - z^{-4}\cdot\mathbf{X}(z) = \mathbf{X}(z)\cdot\left(1 - z^{-4}\right)$

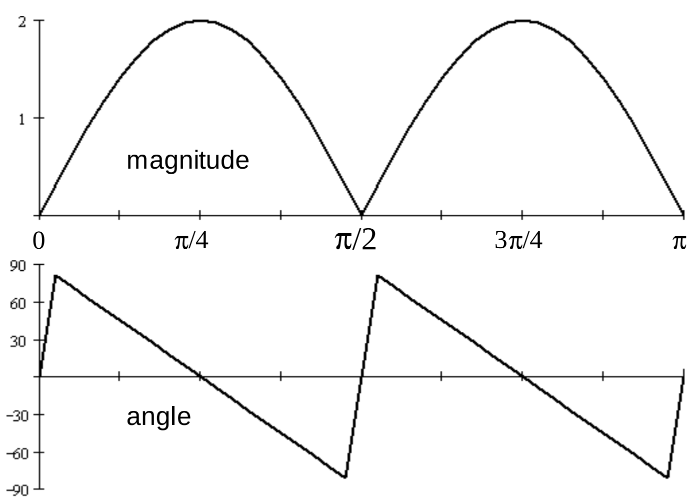
$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = 1 - z^{-4} = 1 - \frac{1}{z^4}$$

$$= \frac{z^4}{z^4} - \frac{1}{z^4} = \frac{z^4 - 1}{z^4} = \frac{\left(z^2+1\right)\cdot\left(z^2-1\right)}{z^4}$$

$$\mathbf{H}(z) = \frac{z^4 - 1}{z^4}$$

$$\mathbf{H}\left(e^{j\cdot\Omega_0}\right) = \frac{\left(e^{j\cdot\Omega_0}\right)^4 - 1}{\left(e^{j\cdot\Omega_0}\right)^4} = \frac{e^{j\cdot\Omega_0\cdot 4} - 1}{e^{j\cdot\Omega_0\cdot 4}}$$





These strange, repeating frequency-response curves are common in digital signal processing. Take a class in DSP to learn more. Here, this is about as deep as we're going.

The **transient part** would be found by partial-fraction expansion.

Initial Conditions

Initial Conditions are handled here much like they are in continuous time, with similar results.
In a BIBO system their effects dissappear quickly and are very similar to the impulse response.

Integration

$y(k) = y(k-1) + x(k)$ Accumulation

old sum + new

$\mathbf{Y}(z) = z^{-1}\cdot\mathbf{Y}(z) + \mathbf{X}(z)$

$\mathbf{Y}(z) - z^{-1}\cdot\mathbf{Y}(z) = \mathbf{X}(z)$

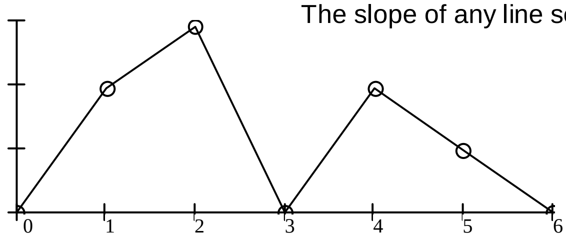
$\mathbf{Y}(z)\cdot\left(1 - z^{-1}\right) = \mathbf{X}(z)$

$\mathbf{H}(z) = \dfrac{\mathbf{Y}(z)}{\mathbf{X}(z)} = \dfrac{1}{1 - z^{-1}} = \dfrac{z}{z-1}$ Compare to Laplace, where the transfer function for integration is $\dfrac{1}{s}$

In both cases this is also the transform of the unit step function.

That's because convolution of a signal with the unit step function is the same as integration.

Differentiation



The slope of any line segment is $y(k) = \dfrac{\text{rise}}{\text{run}} = \dfrac{x(k) - x(k-1)}{1}$

$\mathbf{Y}(z) = \mathbf{X}(z) - z^{-1}\cdot\mathbf{X}(z)$

$\mathbf{Y}(z) = \mathbf{X}(z)\cdot\left(1 - z^{-1}\right)$

$\mathbf{H}(z) = \dfrac{\mathbf{Y}(z)}{\mathbf{X}(z)} = 1 - z^{-1} = \dfrac{z-1}{z}$

Compare to Laplace, where the transfer function for integration is s.

In both cases this is the inverse of transform of integration.

In continuous time, diffential equations play a very important role in describing the world.
In the digital, they become difference equations.

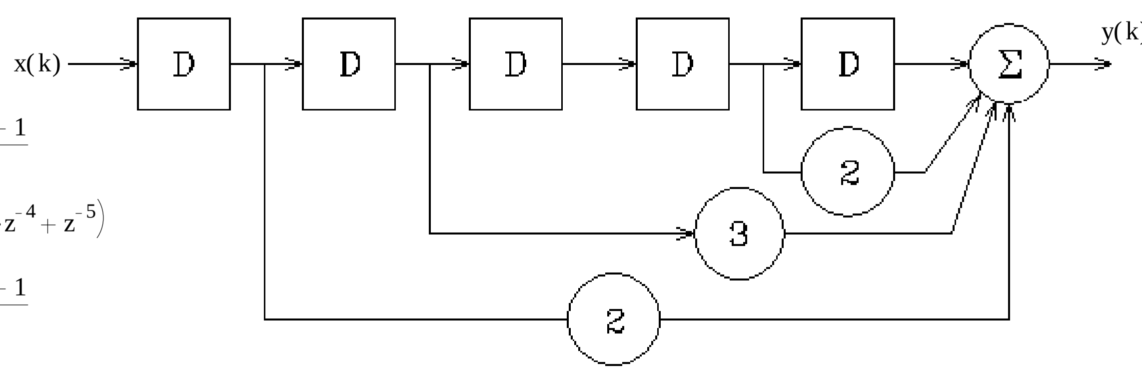
Implementation

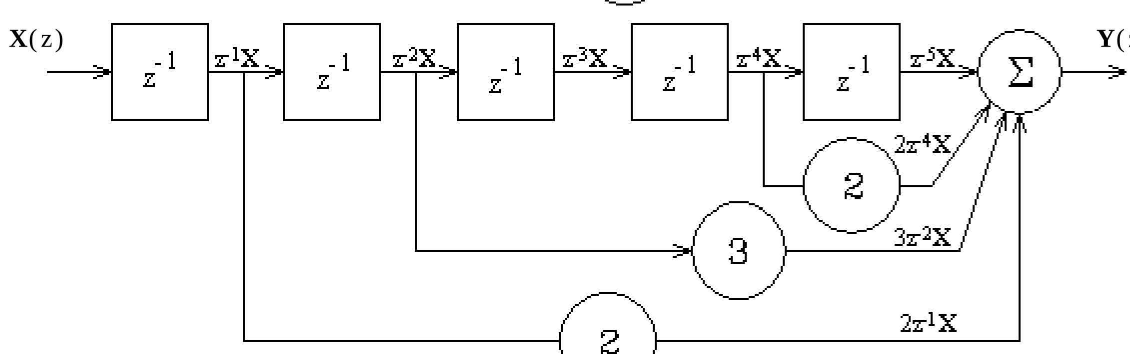
FIR Example:

$\mathbf{H}(z) = \dfrac{2\cdot z^4 + 3\cdot z^3 + 2\cdot z + 1}{z^5}$

$= \left(2\cdot z^{-1} + 3\cdot z^{-2} + 2\cdot z^{-4} + z^{-5}\right)$

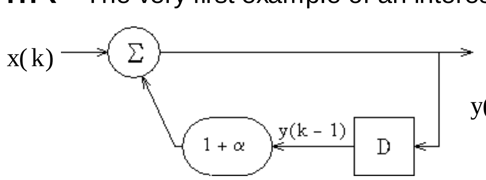
$= \dfrac{2\cdot z^4 + 3\cdot z^3 + 2\cdot z + 1}{z^5}$



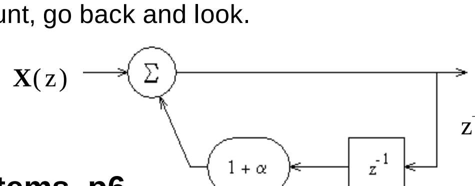


IIR

The very first example of an interest bearing bank account, go back and look.



$y(k) = y(k-1)\cdot(1+\alpha) + x(k)$



$\mathbf{Y}(z) = z^{-1}\cdot\mathbf{Y}(z)\cdot(1+\alpha) + \mathbf{X}(z)$

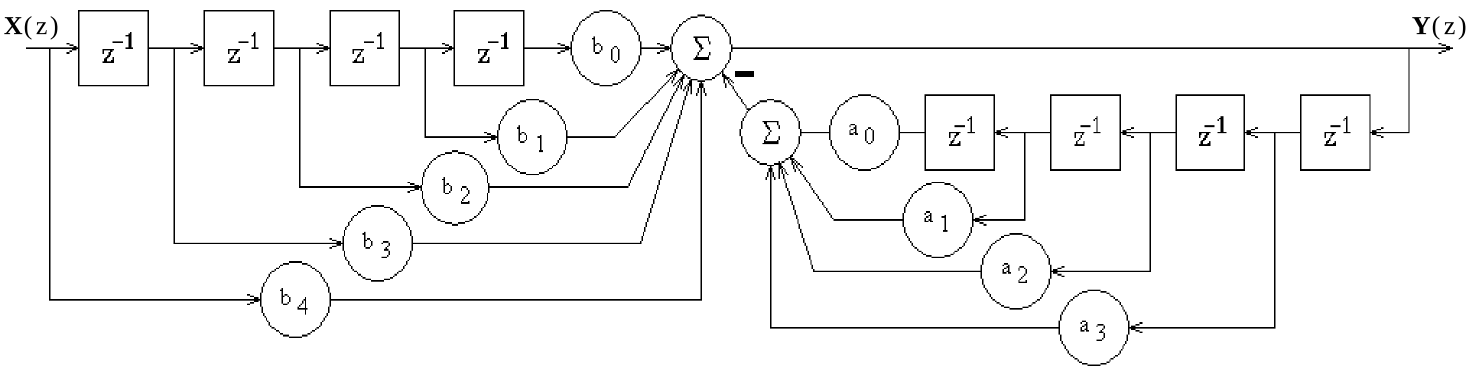
IIR General Example

$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = \frac{b_4 \cdot z^4 + b_3 \cdot z^3 + b_2 \cdot z^2 + b_1 \cdot z + b_0}{z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0} = \frac{b_4 + b_3 \cdot z^{-1} + b_2 \cdot z^{-2} + b_1 \cdot z^{-3} + b_0 \cdot z^{-4}}{1 + a_3 \cdot z^{-1} + a_2 \cdot z^{-2} + a_1 \cdot z^{-3} + a_0 \cdot z^{-4}}$$

$$\mathbf{Y}(z) \cdot \left(1 + a_3 \cdot z^{-1} + a_2 \cdot z^{-2} + a_1 \cdot z^{-3} + a_0 \cdot z^{-4}\right) = \mathbf{X}(z) \cdot \left(b_4 + b_3 \cdot z^{-1} + b_2 \cdot z^{-2} + b_1 \cdot z^{-3} + b_0 \cdot z^{-4}\right)$$

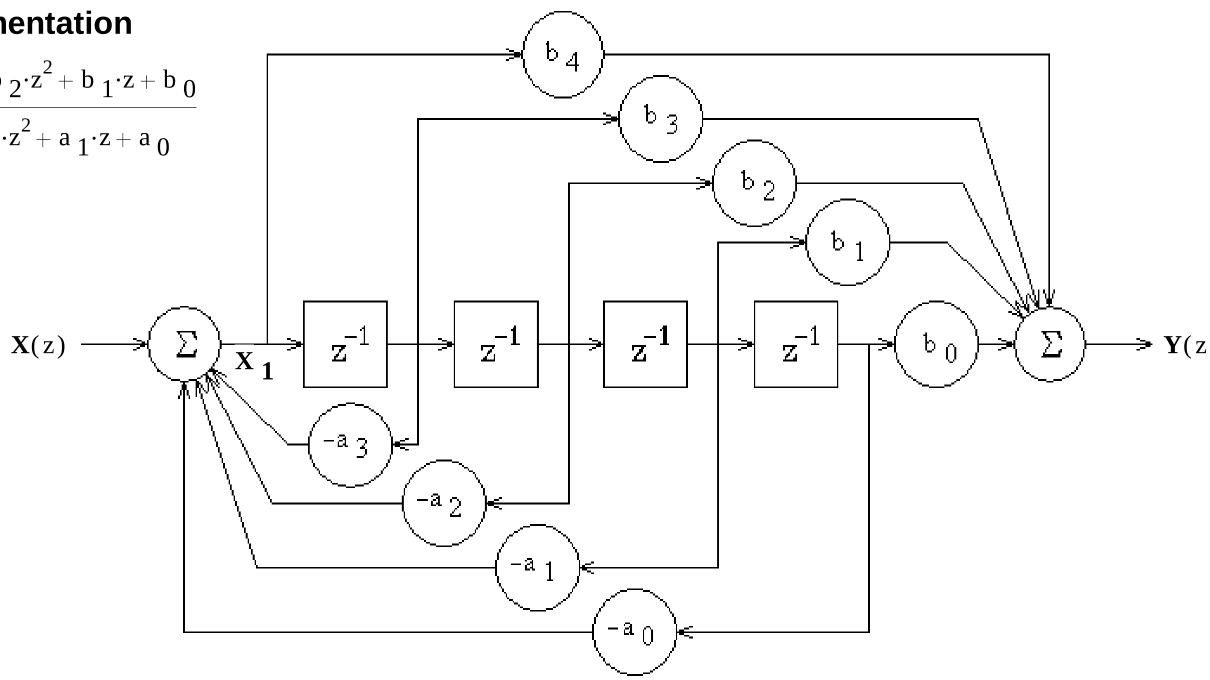
$$\mathbf{Y}(z) = \mathbf{X}(z) \cdot \left(b_4 + b_3 \cdot z^{-1} + b_2 \cdot z^{-2} + b_1 \cdot z^{-3} + b_0 \cdot z^{-4}\right) - \mathbf{Y}(z) \cdot \left(a_3 \cdot z^{-1} + a_2 \cdot z^{-2} + a_1 \cdot z^{-3} + a_0 \cdot z^{-4}\right)$$

Direct Implementation



Minimal Implementation

$$\frac{b_4 \cdot z^4 + b_3 \cdot z^3 + b_2 \cdot z^2 + b_1 \cdot z + b_0}{z^4 + a_3 \cdot z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0}$$



$$\mathbf{X_1} = \mathbf{X}(z) - a_3 \cdot z^{-1} \cdot \mathbf{X_1} - a_2 \cdot z^{-2} \cdot \mathbf{X_1} - a_1 \cdot z^{-3} \cdot \mathbf{X_1} - a_0 \cdot z^{-4} \cdot \mathbf{X_1}$$

$$\mathbf{X_1} \cdot \left(1 + a_3 \cdot z^{-1} + a_2 \cdot z^{-2} + a_1 \cdot z^{-3} + a_0 \cdot z^{-4}\right) = \mathbf{X}(z)$$

$$\mathbf{X_1} = \frac{\mathbf{X}}{1 + a_3 \cdot z^{-1} + a_2 \cdot z^{-2} + a_1 \cdot z^{-3} + a_0 \cdot z^{-4}}$$

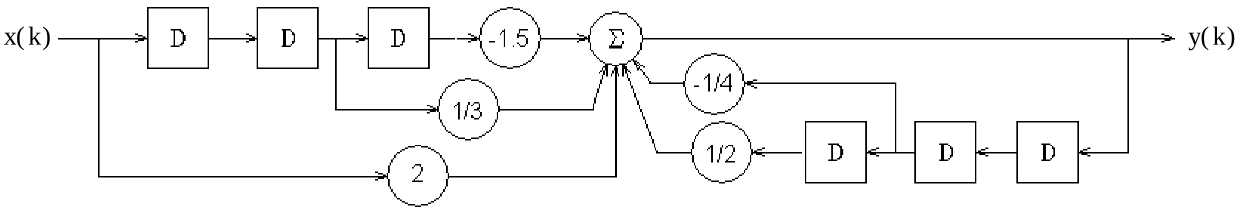
$$\mathbf{Y}(z) = \mathbf{X_1} \cdot \left(b_4 + b_3 \cdot z^{-1} + b_2 \cdot z^{-2} + b_1 \cdot z^{-3} + b_0 \cdot z^{-4}\right)$$

$$\mathbf{Y}(z) = \frac{\mathbf{X}}{1 + a_3 \cdot z^{-1} + a_2 \cdot z^{-2} + a_1 \cdot z^{-3} + a_0 \cdot z^{-4}} \cdot \left(b_4 + b_3 \cdot z^{-1} + b_2 \cdot z^{-2} + b_1 \cdot z^{-3} + b_0 \cdot z^{-4}\right)$$

$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = \frac{b_4 + b_3 \cdot z^{-1} + b_2 \cdot z^{-2} + b_1 \cdot z^{-3} + b_0 \cdot z^{-4}}{1 + a_3 \cdot z^{-1} + a_2 \cdot z^{-2} + a_1 \cdot z^{-3} + a_0 \cdot z^{-4}}$$ Check, it works

Example From Spring 2011 Final   a) Draw the block diagram of a simple direct implementation of the difference equation.

$$y(k) = 2\cdot x(k) + \frac{x(k-2)}{3} - 1.5\cdot x(k-3) - \frac{1}{4}\cdot y(k-2) + \frac{1}{2}\cdot y(k-3)$$



b) Find the $\mathbf{H}(z)$ corresponding to the difference equation above. Show your work.

$$\mathbf{Y}(z) = 2\cdot\mathbf{X}(z) + \frac{1}{3}\cdot\mathbf{X}(z)\cdot z^{-2} - 1.5\cdot\mathbf{X}(z)\cdot z^{-3} - \frac{1}{4}\cdot\mathbf{Y}(z)\cdot z^{-1} + \frac{1}{2}\cdot\mathbf{Y}(z)\cdot z^{-2}$$

$$\mathbf{Y}(z) + \frac{1}{4}\cdot\mathbf{Y}(z)\cdot z^{-2} - \frac{1}{2}\cdot\mathbf{Y}(z)\cdot z^{-3} = 2\cdot\mathbf{X}(z) + \frac{1}{3}\cdot\mathbf{X}(z)\cdot z^{-2} - 1.5\cdot\mathbf{X}(z)\cdot z^{-3}$$

$$\mathbf{Y}(z)\cdot\left(1 + \frac{1}{4}\cdot z^{-2} - \frac{1}{2}\cdot z^{-3}\right) = \mathbf{X}(z)\cdot\left(2 + \frac{1}{3}\cdot z^{-2} - 1.5\cdot z^{-3}\right)$$

$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = \frac{2 + \frac{1}{3}\cdot z^{-2} - 1.5\cdot z^{-3}}{1 + \frac{1}{4}\cdot z^{-2} - \frac{1}{2}\cdot z^{-3}}\cdot\left(\frac{z^3}{z^3}\right) = \frac{2\cdot z^3 + \frac{1}{3}\cdot z - 1.5}{z^3 + \frac{1}{4}\cdot z - \frac{1}{2}}$$

c) List the poles of $\mathbf{H}(z)$. Indicate multiple poles if there are any.

$$0 = z^3 + \frac{1}{4}\cdot z - \frac{1}{2} \quad \text{solves to} \quad \begin{pmatrix} 0.689 \\ -0.345 + 0.779\cdot j \\ -0.345 - 0.779\cdot j \end{pmatrix}$$

Poles at: 0.689

$-0.345 + 0.779\cdot j$

$-0.345 - 0.779\cdot j$

d) Is this system BIBO stable? Justify your answer.

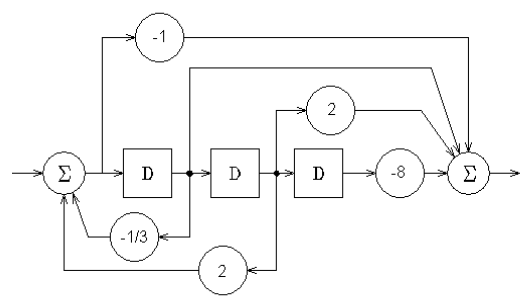
Yes, all poles are inside the unit circle

$0.689 < 1 \qquad \sqrt{0.345^2 + 0.779^2} = 0.852 < 1$

Another Example from the same Final

Draw a minimal implementation of a system with the following transfer function

$$\mathbf{H}(z) = \frac{-z^3 + (z-2)\cdot(z+4)}{z\cdot\left(z^2 + \frac{z}{3} - 2\right)} \quad \text{find} \quad = \frac{-z^3 + z^2 + 2\cdot z - 8}{z^3 + \frac{1}{3}\cdot z^2 - 2\cdot z}$$

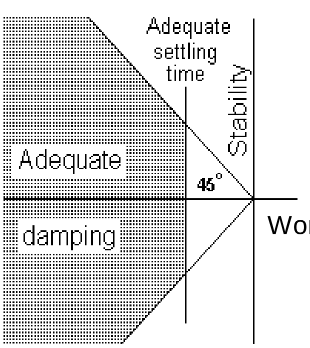


Continuous Time

Differential Equations

Laplace Transform

- Left-half plane / Right-half plane
- Origin
- Frequency increases as pole goes up, vertically



Discrete Time

Difference Equations

z transform

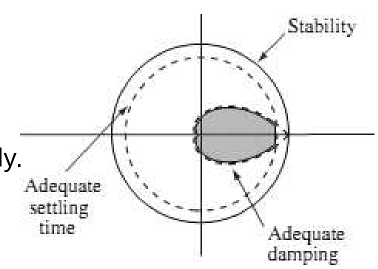
- Inside unit circle / outside unit circle
- Point at (1,0), the right-most point on unit circle
- Frequency increases as pole goes around unit circle
- Extra z in numerator of most terms
  - Divide by z before partial-fraction expansion

Transfer functions and Block diagrams

Same   Lots of z^{-1} blocks

Root Locus

Works exactly the same way, but results are interpreted very differently.



ECE 3510 homework # Z1 Due: Fri, 4/23/21

1. Problem 6.1 (p.215) in the Bodson text. Find $x(0)$ if the z-transform of $x(k)$ is

   a) $X(z) = \dfrac{a\cdot z - 1}{z - 1}$ b) $X(z) = \dfrac{z}{z^2 - a\cdot z + a^2}$

2. Problem 6.3 in the text. Use partial fraction expansions to find the $x(k)$ whose z-transform is

   a) $X(z) = \dfrac{1}{(z-1)\cdot(z-2)}$ b) $X(z) = \dfrac{z}{z^2 - 2\cdot z + 2}$

3. Problem 6.4 in the text. Sketch the time function $x(k)$ that you would associate with the following poles. Only a sketch is required, but be as precise as possible.

   a) $p_1 = 0.9\cdot j$, $= -0.9\cdot j$ b) $p_1 = 1$, $p_2 = -1$ c) $p_1 = 0.3$, $p_2 = 0.9$ d) $p_1 = e^{j\cdot\frac{\pi}{6}}$, $p_2 = e^{-j\cdot\frac{\pi}{6}}$

4. Problem 6.6 (p.217) in the Bodson text.
5. Problem 6.7 in the text.

homework # Z2 Due: Tue, 4/27/21

1. Problem 6.8 in the text
2. Problem 6.9 in the text
3. Problem 6.10 in the text b) hint: find y(k) by partial fraction expansion, then $\dfrac{y(k)}{y(k-1)}$ and then let k -> ∞ .
4. Problem 6.11 (p.219) in the Bodson text.
5. Problem 6.12 in the text.
   Hints: $r(k) = r\cdot u(k)$ Find $\dfrac{e(z)}{R(z)}$ and make sure its poles are inside unit circle

homework # Z3 Due Tue, 4/27/21 May be handed in with final for full credit

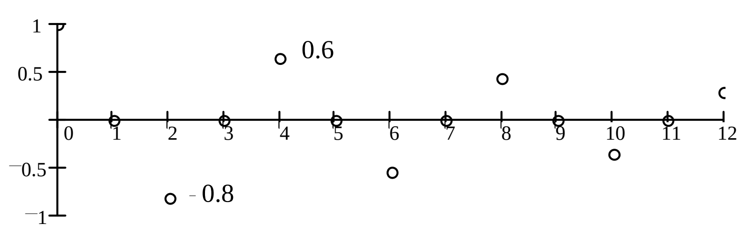
1. Problem 7.1 (p.253) in the Bodson text.
2. Problem 7.2 in the text

<u>Z1 Answers</u>

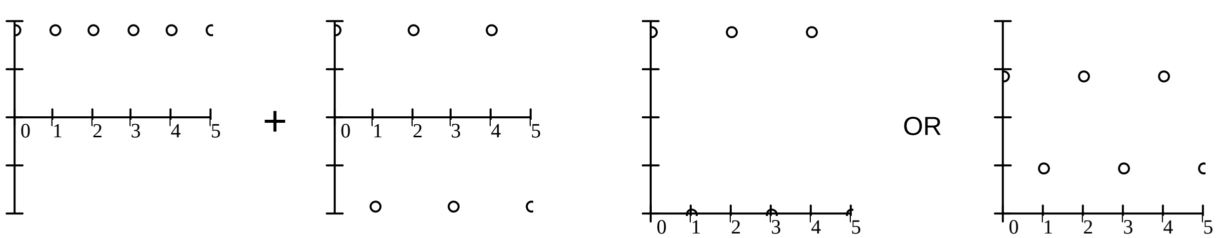
1. a) a b) 0 2. a) $\frac{1}{2}\cdot\delta(k) - 1 + \frac{1}{2}\cdot 2^k$ b) $\left(\sqrt{2}\right)^k\cdot\sin\left(\frac{\pi}{4}\cdot k\right)$

3. Actual signals may have different magnitudes and/or phase angles. You can't tell those things from the pole locations.

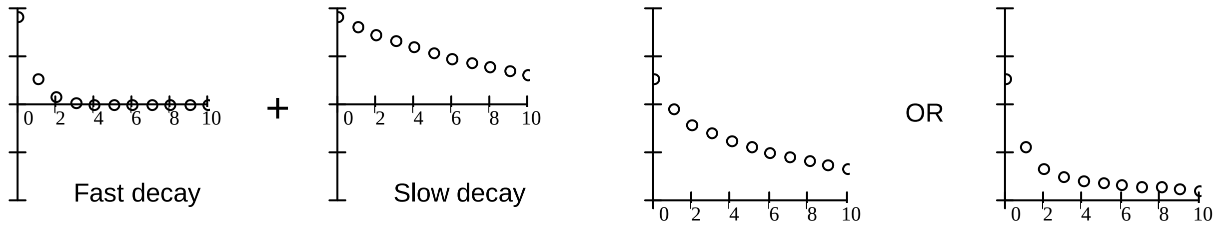
   a) $x(k) = 0.9^k\cdot\cos\left(\frac{\pi}{2}\cdot k\right)$



   b)

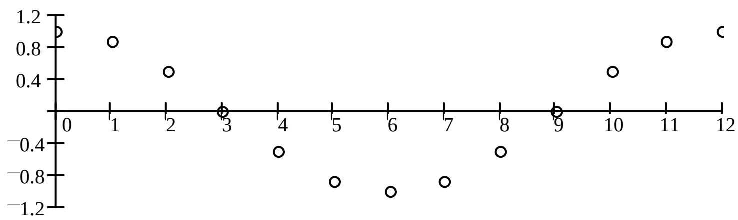


Or many others, depending on relative magnitudes c)



Or many others, depending on relative magnitudes

d) $x(k) = \cos\left(\frac{\pi}{6} \cdot k\right)$



4. (6.6) a) $x(k) := -4 \cdot \delta(k) + 2 + 2 \cdot \sqrt{2} \cdot \cos\left(\frac{\pi}{2} \cdot k + \frac{\pi}{4}\right)$

$x(0) = 0$ $\quad x(1) = 0$ $\quad x(2) = 0$ $\quad x(3) = 4$ $\quad x(4) = 4$ $\quad x(5) = 0$ $\quad x(6) = 0$ $\quad x(7) = 4$ $\quad x(8) = 4$

5. (6.7)	Bounded	Converges	$x(\infty)$	
a)	yes	yes	0	
b)	yes	yes	0	vanishes in a finite time (all poles are at zero)
c)	yes	no		
d)	yes	yes	8/9	
e)	yes	yes	2	
f)	no			
g)	yes	no		
h)	yes	yes	1	

Z2 Answers

1. (6.8) a) yes
   b) yes
   c) no
   d) yes
   e) no
   f) yes

2. (6.9) a) $H(z) = \dfrac{z^2}{z^2 - a \cdot z + a^2}$ stable if: $|a| < 1$

   b) $H(z) = \dfrac{12 \cdot z^2 + 48 \cdot z - 3}{z \cdot (2 \cdot z - 1)}$ stable

3. (6.10) a) $H(z) = \dfrac{z^2}{z^2 - z - 1}$ unstable b) $\dfrac{1 + \sqrt{5}}{2} = 1.618$

4. (6.11) a) gain = $-\dfrac{2}{3}$ $y_{ss} = -2$ b) $2 \cdot e^{j \cdot \frac{\pi}{2}}$ (frequency response) $-2 \cdot \sin\left(\frac{\pi}{2} \cdot k\right)$

5. (6.12) $a = 1$ $g < 1$

Z3 Answers

1. (7.1) a) $H_d(z) = \dfrac{z \cdot \left(T - 1 + e^{-T}\right) + \left(1 - e^{-T} - T \cdot e^{-T}\right)}{(z - 1) \cdot \left(z - e^{-T}\right)}$ b) $H_d(z) = \dfrac{(1 - \cos(T)) \cdot (z + 1)}{z^2 - 2 \cdot \cos(T) \cdot z + 1} = 0$ @ $T = 2 \cdot \pi$

2. (7.2) $60 \cdot Hz$