Nyquist Plot Notes

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4/9/09,
4/3/21

A Nyquist plot is essentially a polar Bode plot. Like a Bode plot, it is plotted for the Open-Loop (OL) Transfer function and will give information about the stability of the Closed-Loop (CL) system.

Open-Loop (OL) Transfer function: $\mathbf{G}(s) = \dfrac{N_G(s)}{D_G(s)}$ m = number of zeros, n = number of poles

Basic Nyquist Rules

1. "Clean up" any "-s" terms in $\mathbf{G}(s)$ by multiplying by -1 as needed.
   If an overall "-" remains in $\mathbf{G}(s)$, then add 180° to all the angles below. (rare)

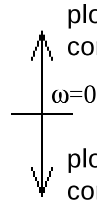
2. Start at $\mathbf{G}(0)$, the DC gain, a point on the real axis.

   If $\mathbf{G}(s)$ has a zero at the origin: $\mathbf{G}(0) = 0$

   If $\mathbf{G}(s)$ has a pole at the origin: $\mathbf{G}(0) = \pm\infty$

   If $\mathbf{G}(s)$ has no poles or zeros in the right-half plane then:

   plot heads upward if first corner frequency is a zero



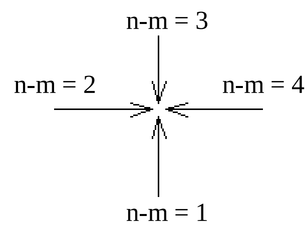
   plot heads downward if first corner frequency is a pole

3. End at $\mathbf{G}(\infty)$.

   n < m (rare) Plot ---> ∞ , almost always +∞

   n = m Plot ---> $\mathbf{G}(\infty)$, a point on the real axis

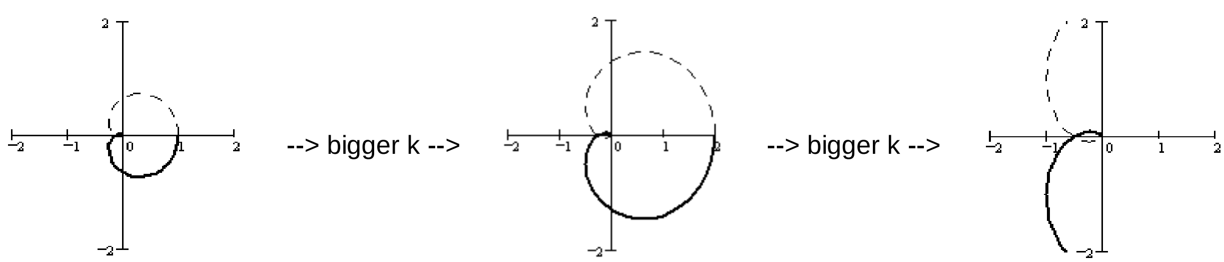
   n > m (most common) Plot ---> 0 Angle of approach to origin = $(n - m)\cdot(-90\cdot deg)$



4. Plot the rest of the frequency response of $\mathbf{G}(j\omega)$. Use Bode plot to gude you.

5. Add the $\omega < 0$ curve (dashed line). It is simply the mirror image of the $\omega > 0$ curve about the real axis.
   This part of the curve is usually not necessary, it doesn't provide any more information.

6. Gain, k, makes entire plot grow in all directions (or shrink if k<1).



7. $Z = N + P$

   P = OL ($\mathbf{G}(s)$) poles in RHP (0 if open-loop stable). P cannot be -

   N = CW encirclements of -1, CCW encirclements are counted as negative and may make up for P.

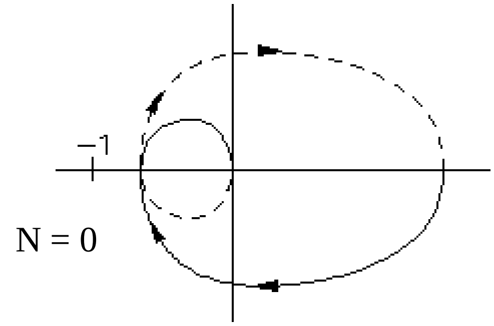
   Z = CL poles in RHP (must be zero if closed-loop stable). Z cannot be -

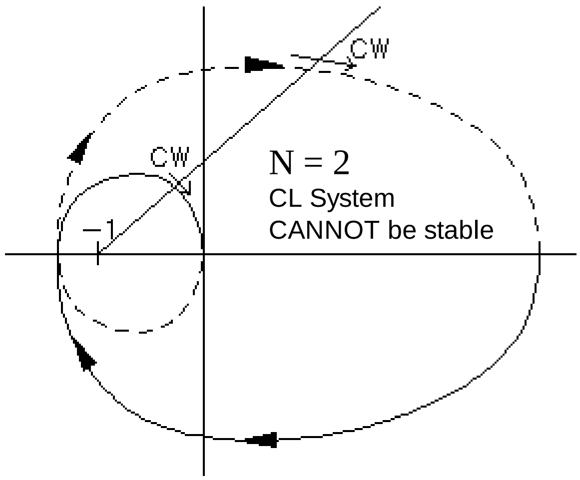
8. ANY CW encirclements means Closed-Loop system is UNSTABLE

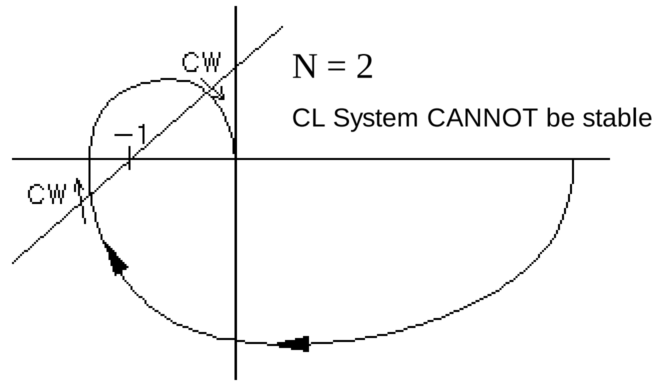
   $\mathbf{N} > 0$ --> CL unstable (P cannot be -)

Counting Clockwise Encirclements

N = CW encirclements of -1,
CCW encirclements are counted as negative and may make up for P.

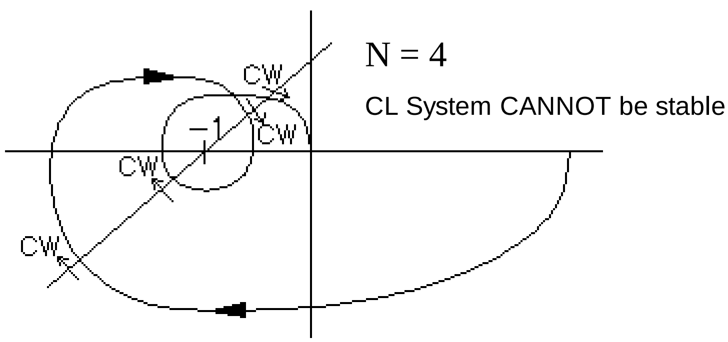


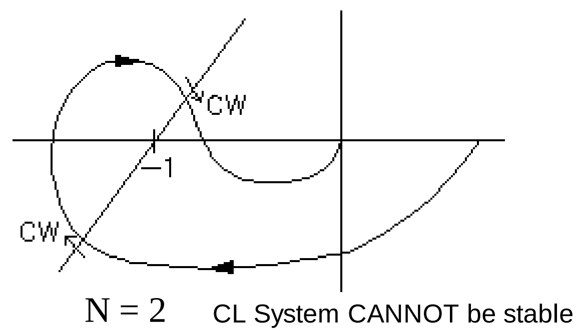


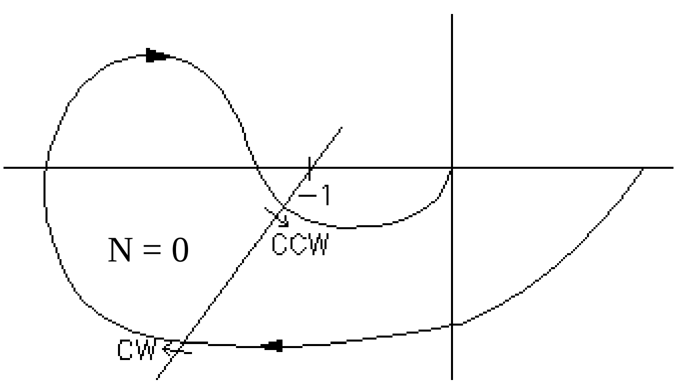


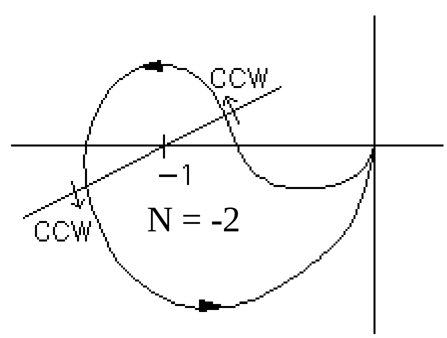
If you have the $\omega < 0$ curve (dashed line), then you can use any single-ended line that starts at -1 to help you count encirclements.

If you don't have the $\omega < 0$ curve (dashed line), then make your line extend both directions from -1.









CCW encirclements are counted as negative.

CL System CAN be stable, if $P \leq 2$

$-N$ can make up for $+P$. and stabilize an OL unstable system

$$Z = N + P$$

P = OL poles in RHP (0 if open-loop stable)

N = CW encirclements of -1. CL System CANNOT be stable if $\mathbf{N > 0}$

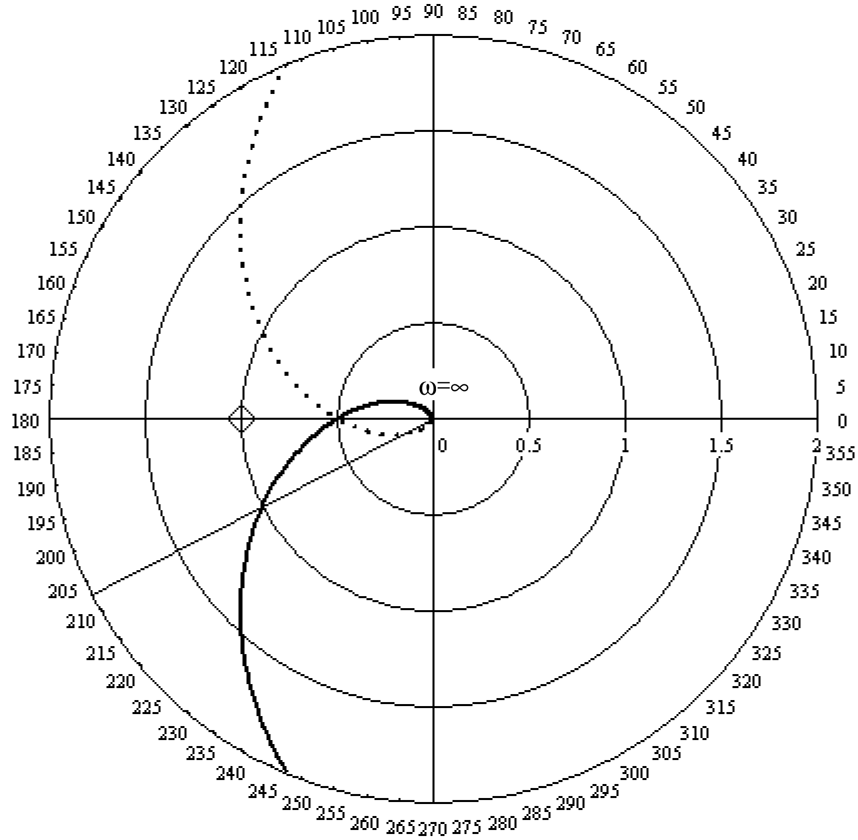
Z = CL poles in RHP (must be zero if closed-loop stable)

To find the Phase Margin (PM):

1. Find where the Nyquist plot crosses the unit circle. These crossings separate the unit circle into regions.
2. Decide which of these regions have unacceptable CW encirclements.
3. Determine what phase change would cause the -1 point to be an unacceptable region, usually 180^o - / crossing

$$GM = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

$$PM = 207 - 180 = 27 \text{ deg}$$



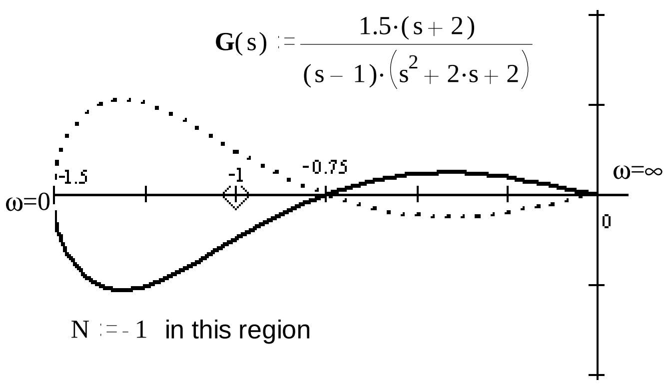
To find the Gain Margin (GM):

1. Find where the Nyquist plot crosses the negative real axis. These crossings separate the negative real axis into regions.
2. Decide which of these regions have unacceptable CW encirclements.
3. Determine what gain would cause the -1 point to be an unacceptable region, usually $\frac{1}{-\text{crossing}}$ into the unacceptable region.
4. Usually there is just one upper limit of gain-- in that case report that as the Gain Margin.
5. If there is a lower limit of gain, report the Gain Margin as: $GM = \left[\text{Lower limit , upper limit}\right]$

If there is no upper limit, then report it as ∞

$$\mathbf{G}(s) := \frac{1.5\cdot(s+2)}{(s-1)\cdot\left(s^2+2\cdot s+2\right)}$$

$P := 1$ For CL stability, $N := -1$ or more

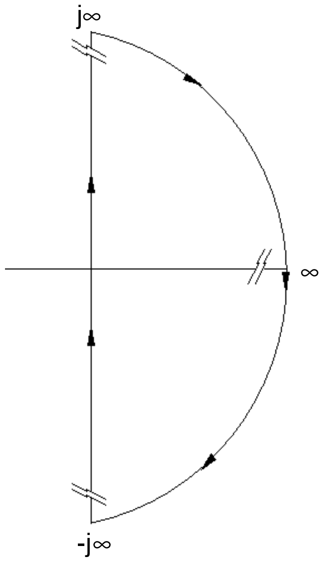


$N := -1$ in this region

$$GM = \left[\frac{1}{1.5}\ ,\ \frac{1}{0.75}\right] = \left[0.667\ ,\ 1.333\right]$$

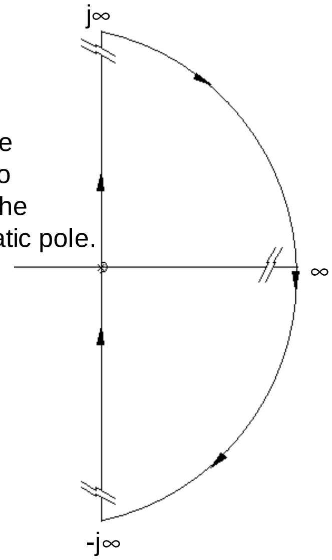
Poles on the imaginary (jω) axis

The normal contour



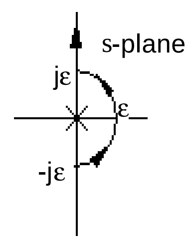
A pole on the imaginary axis causes a problem.
Is it inside or outside of the contour?

Modify the contour to exclude the problematic pole.

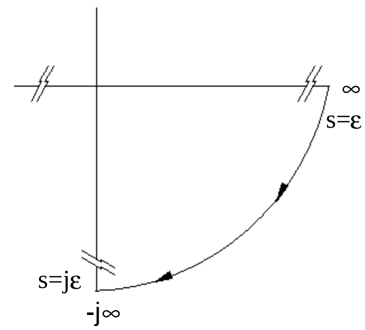


A **single pole** at the origin

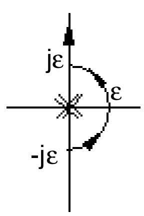
A closer look



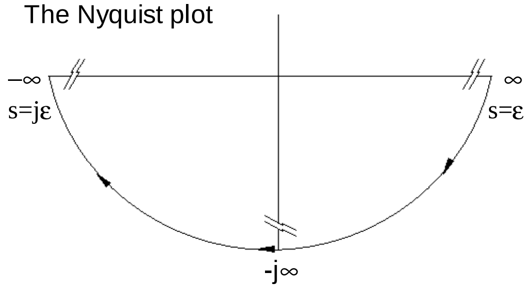
The Nyquist plot of this



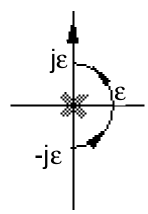
A **double pole** at the origin



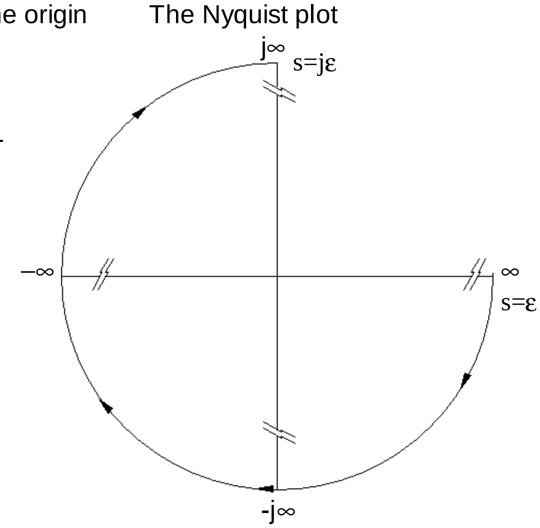
The Nyquist plot



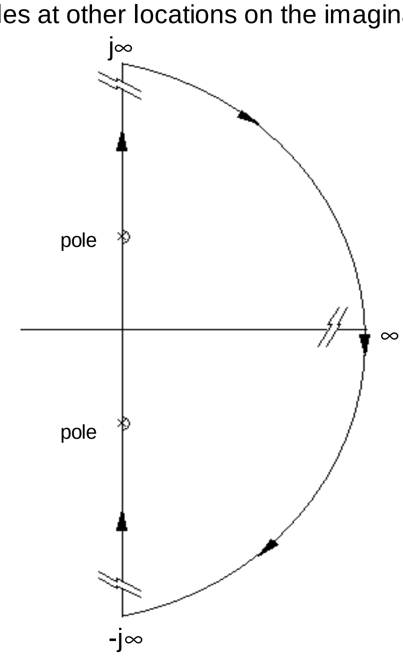
A **triple pole** at the origin



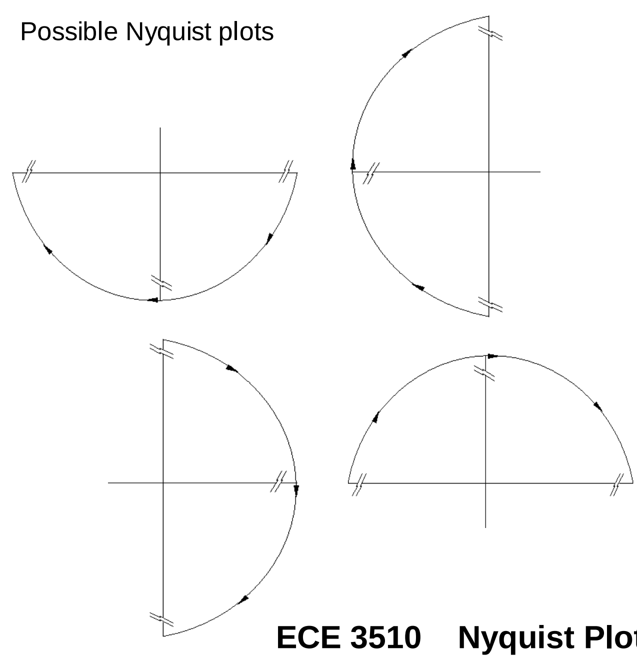
The Nyquist plot



Poles at other locations on the imaginary axis

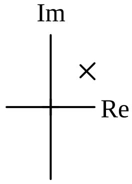


Possible Nyquist plots

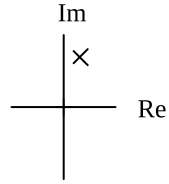


ECE 3510 How Nyquist Plots Work see pages 615 and 616 in Nise

For any point on the s-plane

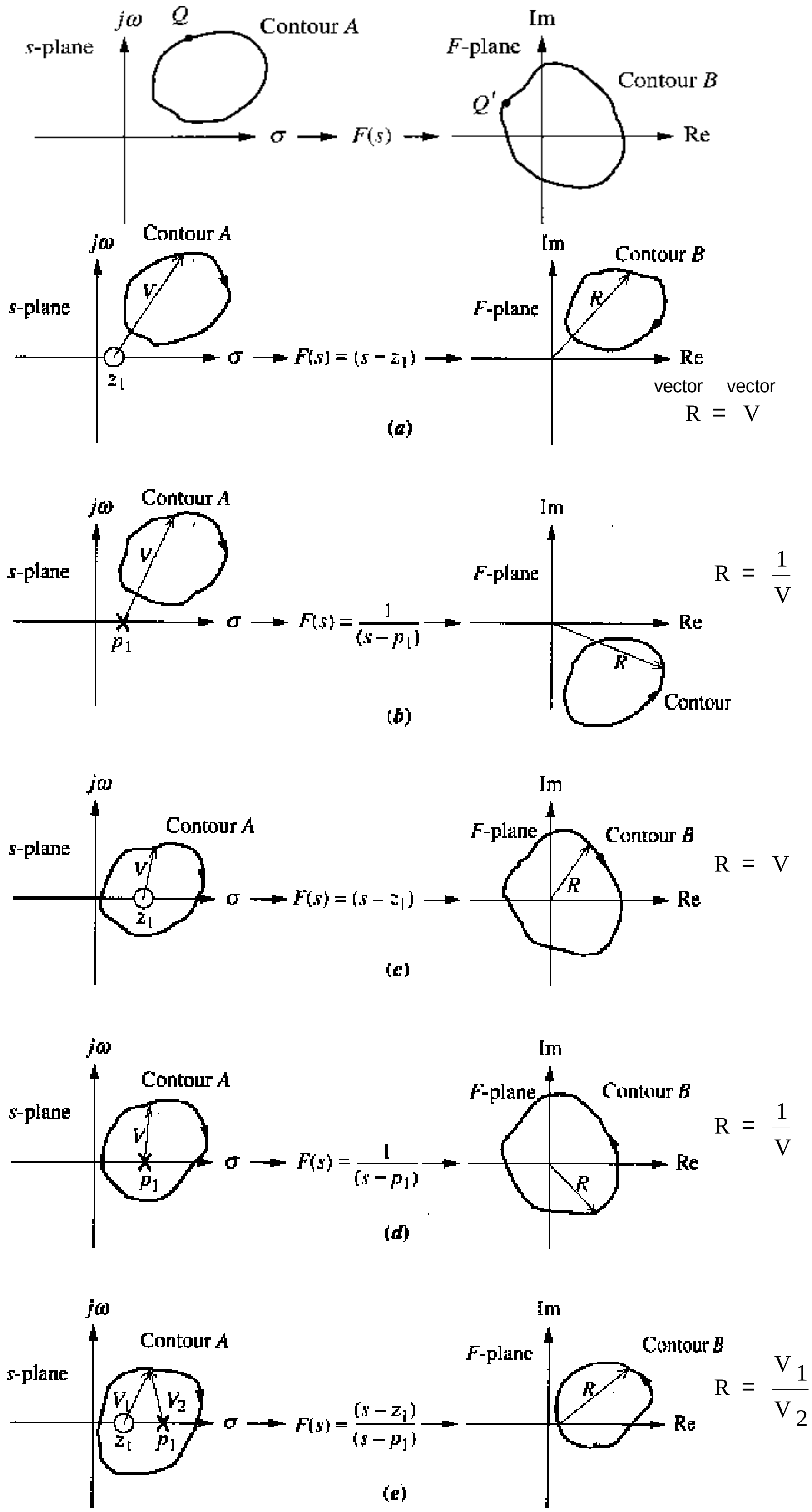


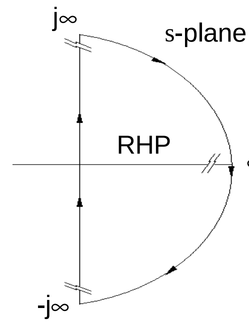
You can find a corresponding point on the **F**(s) plane



For any contour on the s-plane

There is a corresponding contour on the **F**(s) plane





When we make a Nyquist plot for frequencies from o to ∞, and then from -∞ to 0 again, we are essentially making a contour that circles the entire RHP

This gives information about poles and zeros in the RHP. This information reveals itself in the encirclements of 0, OR if we want the same information about 1 + **G**(s), in the encirclements of -1.

Closed-loop transfer function

(**G**(s) includes gain, k)

$$\frac{\mathbf{G}(s)}{1+\mathbf{G}(s)} = \frac{\mathbf{G}(s)}{\left(\frac{D_G + N_G}{D_G}\right)}$$

poles of CL = zeros of $1 + \mathbf{G}(s)$

$= Z$ (RHP)

poles of $1 + \mathbf{G}(s)$

= poles of $\mathbf{G}(s)$

$= P$ (RHP)

We plot **G**(s), but count CW encirclements of -1, rather than 0 to get the CW encirclements that 1 + **G**(s) would have of 0, Thus.

N = CW encirclements of -1

$= Z - P$

rearrange

$$Z = N + P$$

P = OL poles in RHP

(0 if open-loop stable)

(**P** cannot be negative)

N = CW encirclements of -1

CL System CANNOT be stable if **N > 0**

Z = CL poles in RHP

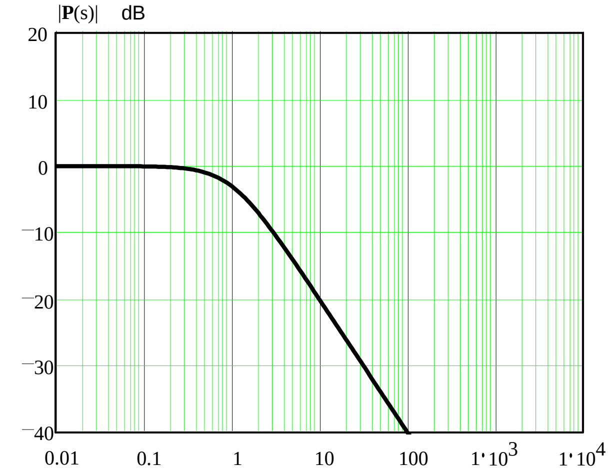
(must be zero (or ≤ 0) if closed-loop stable)

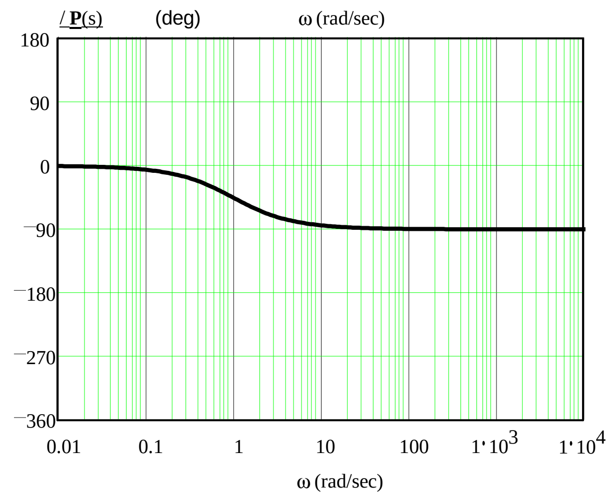
ECE 3510 Nyquist Examples

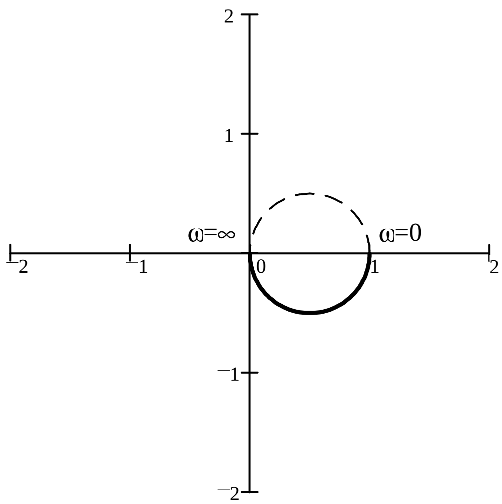
Example 1, Bodson, section 5.2 $k := 1$

$$\mathbf{P}(s) := \frac{1}{(s+1)}$$

$$\frac{1}{j\cdot\omega + 1} = \frac{1}{\sqrt{\omega^2 + 1^2}} \quad -\operatorname{atan}\left(\frac{\omega}{1}\right)$$

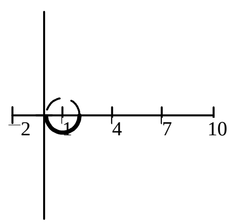




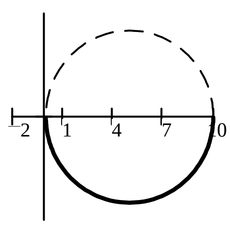


$$\mathbf{G}(s) = k\cdot\mathbf{P}(s) = k\cdot\frac{1}{(s+1)}$$

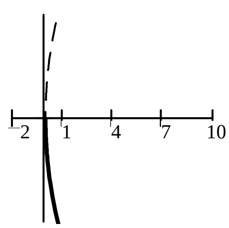
$k := 2$



$k := 10$



$k := 50$



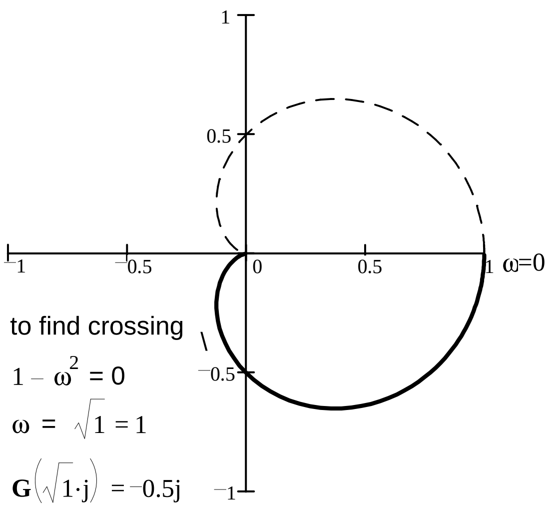
This plot never circles -1, no matter what the gain, so $N = 0$, no matter what k is.

ECE 3510 Nyquist Examples p.2

$k := 1$

$$\mathbf{G}(s) := \frac{k}{(s+1)^2} = \frac{1}{s^2 + 2\cdot s + 1} = \frac{1}{(j\cdot\omega)^2 + 2\cdot(j\cdot\omega) + 1} = \frac{1}{\left(1-\omega^2\right) + 2\cdot j\cdot\omega}\cdot\left[\frac{\left(1-\omega^2\right) - 2\cdot j\cdot\omega}{\left(1-\omega^2\right) - 2\cdot j\cdot\omega}\right] = \frac{\left(1-\omega^2\right) - 2\cdot j\cdot\omega}{\left(1-\omega^2\right)^2 + (2\cdot j\cdot\omega)^2}$$

$$\frac{1}{(j\cdot\omega + 1)^2} = \frac{1}{\omega^2 + 1^2} \qquad -2\cdot\operatorname{atan}\left(\frac{\omega}{1}\right)$$

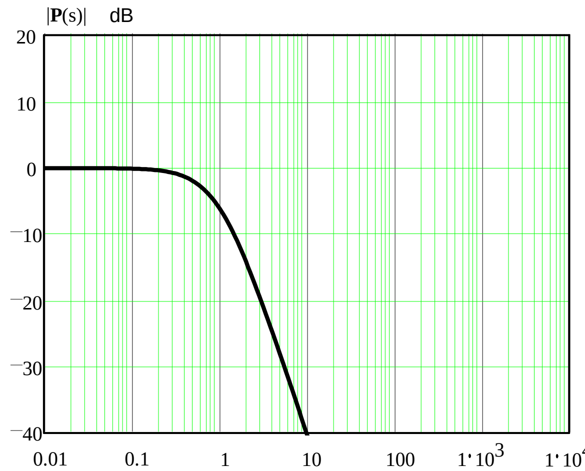


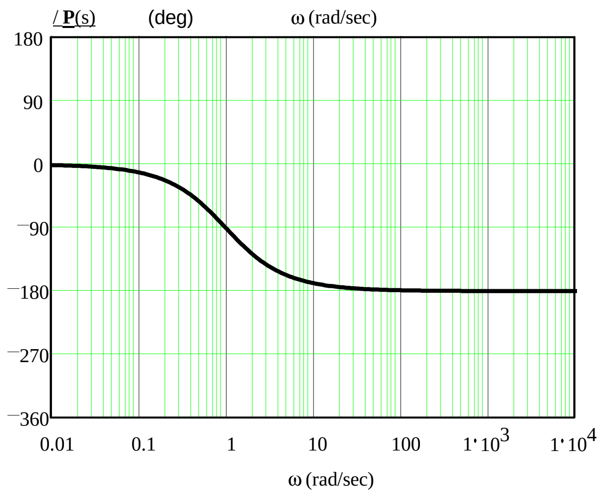
to find crossing

$1 - \omega^2 = 0$

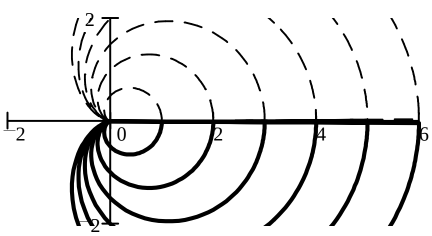
$\omega = \sqrt{1} = 1$

$\mathbf{G}\left(\sqrt{1}\cdot j\right) = -0.5j$



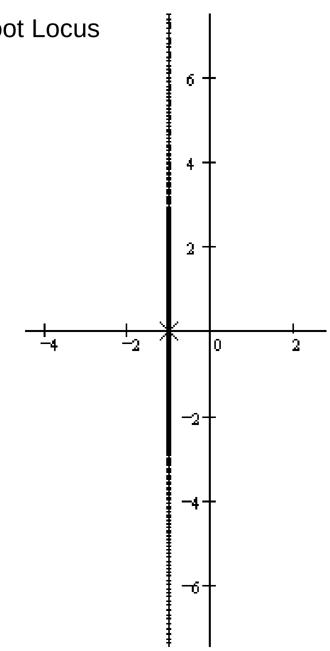


$$\mathbf{G}(s) = k\cdot\mathbf{P}(s) = k\cdot\frac{1}{(s+1)^2}$$

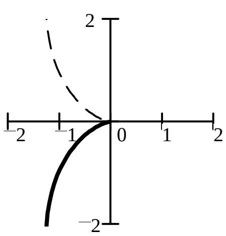


bigger gain -->

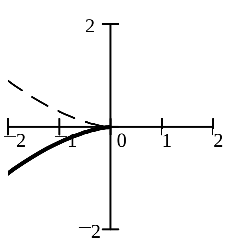
Root Locus



$k := 10$



$k := 50$



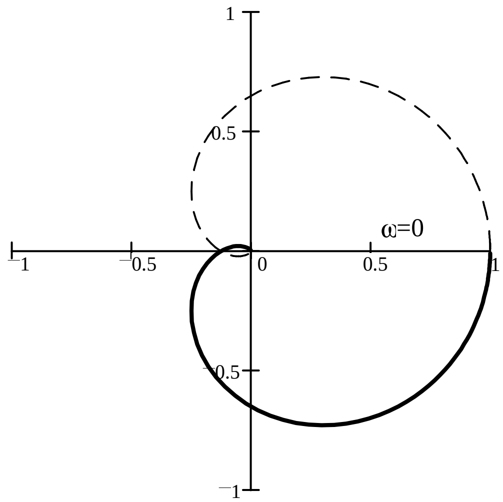
N = 0, no matter what k is

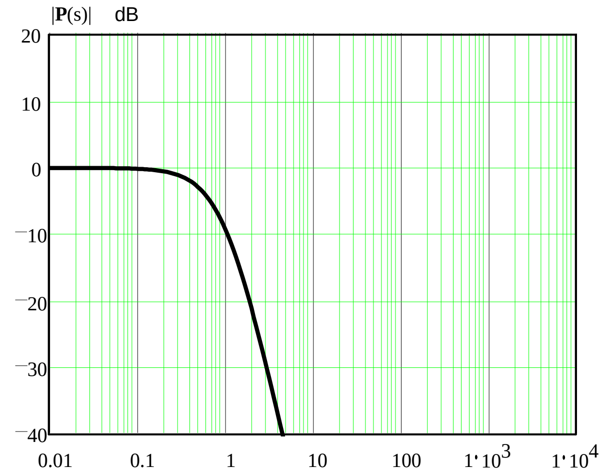
But ringing gets worse with k

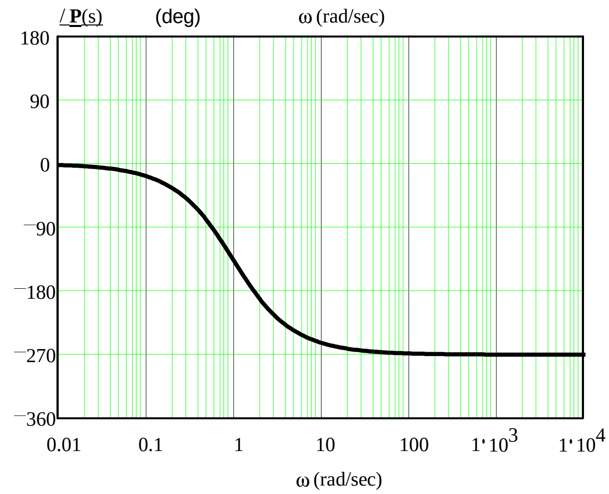
Example 2, Bodson, section 5.2.

$$\mathbf{P}(s) := \frac{1}{(s+1)^3}$$

$$\frac{1}{(j\cdot\omega+1)^3} = \frac{1}{\left(\sqrt{\omega^2+1^2}\right)^3} \qquad -3\cdot\operatorname{atan}\left(\frac{\omega}{1}\right)$$







Find real-axis crossing (in left half plane)

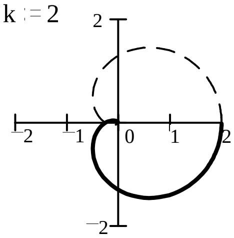
$$\frac{180}{3} = 60 \ \deg \qquad \tan(60\cdot\deg) = \sqrt{3} = \frac{\omega}{1} = \omega$$

$$\frac{1}{\left(\sqrt{\omega^2+1^2}\right)^3} = \frac{1}{\left[\sqrt{\left(\sqrt{3}\right)^2+1^2}\right]^3} = \frac{1}{8}$$

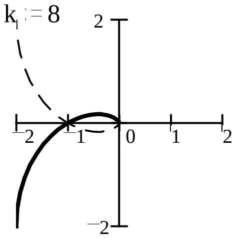
$$\mathbf{G}(s) = k\cdot\mathbf{P}(s)$$

$$= k\cdot\frac{1}{(s+1)^3}$$

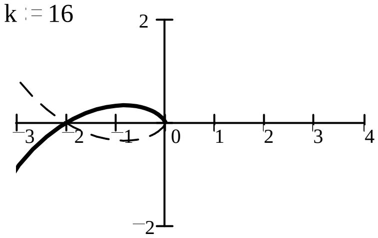
$k := 2$



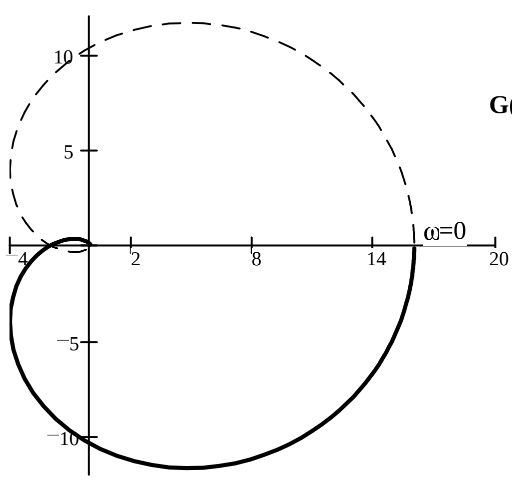
$k_1 := 8$



$k := 16$



$k := 16$



$$\mathbf{G}(s) = k\cdot\frac{1}{(s+1)^3}$$

$P := 0$

$N := 2$

$Z := N + P \qquad Z = 2$ 2 closed-loop poles in RHP

Closed-loop UNSTABLE

Gain Margins

For: $\mathbf{G}(s) = \dfrac{1}{(s+1)^3}$ Real-axis crossing (in left half plane) is at $1/8$

Gain can be 8 times bigger before $N = 2$ $GM := 8$

For: $\mathbf{G}(s) = \dfrac{2}{(s+1)^3}$ $GM := 4$

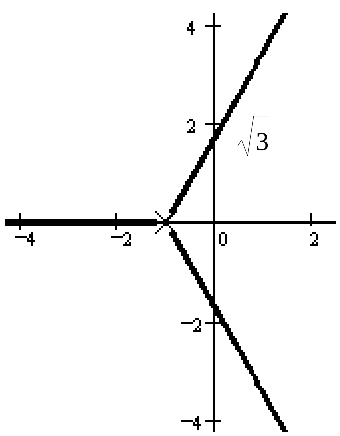
For: $\mathbf{G}(s) = \dfrac{4}{(s+1)^3}$ $GM := 2$

For: $\mathbf{G}(s) = \dfrac{20}{(s+1)^3}$ $GM := \dfrac{8}{20} = \dfrac{2}{5}$

Other ways to find the same maximum gain

$$G(s) = \frac{1}{(s+1)^3}$$

Root Locus



By Routh-Hurwitz:

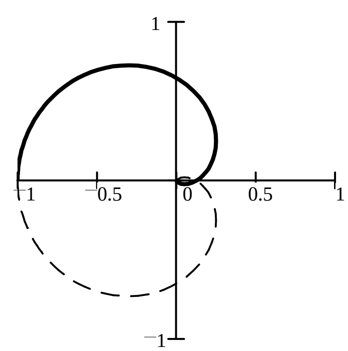
Closed-loop denominator

$$(s+1)^3 + k = s^3 + 3\cdot s^2 + 3\cdot s + 1 + k$$

s^3	1	3	
s^2	3	$1+k$	
s^1	$\dfrac{9 - 1\cdot(1+k)}{3}$	$k < 8$	same result as previous page
s^0	$1+k$	$k > -1$	

Nyquist plot for negative k

(plot rotates 180^o around the origin)



$N = 2$ for any k more negative than -1

Example 10.4, Nise 3 p.620.

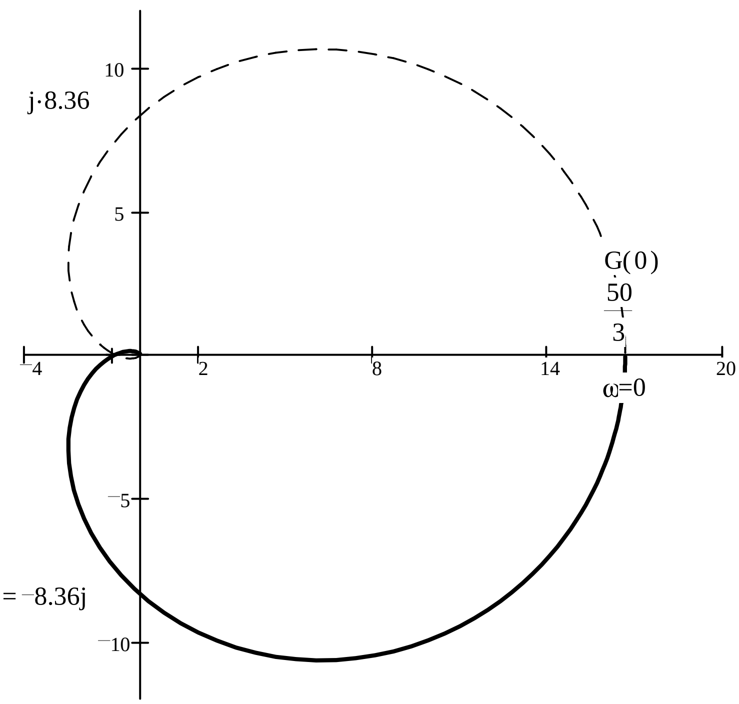
$$\mathbf{G}(s) := \frac{100}{(s+10)} \cdot \frac{1}{(s+3)} \cdot \frac{5}{(s+1)} = \frac{500}{(s+10)\cdot(s+3)\cdot(s+1)}$$

$$\mathbf{G}(0) = \frac{500}{10 \cdot 3 \cdot 1} = \frac{50}{3}$$

$$\mathbf{G}(\omega) = \frac{500}{(j\cdot\omega+10)\cdot(j\cdot\omega+3)\cdot(j\cdot\omega+1)} = \frac{500}{\left[(-1)\cdot\omega^2 + 13\cdot j\cdot\omega + 30\right]\cdot(j\cdot\omega+1)} = \frac{500}{-\omega^3\cdot j - \omega^2 + 13\cdot(-1)\cdot\omega^2 + 43\cdot j\cdot\omega + 30}$$

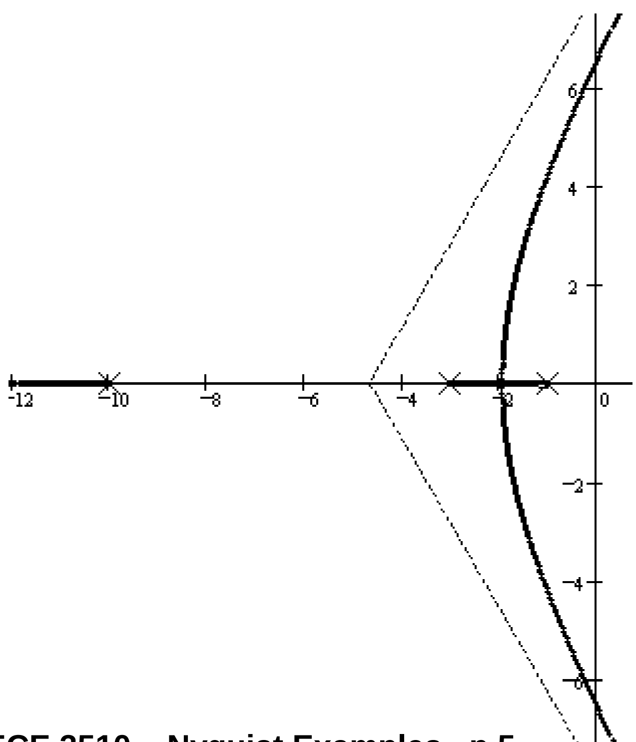
$$= \frac{500}{\left(-14\cdot\omega^2+30\right) + j\cdot\left(43\cdot\omega-\omega^3\right)} \cdot \frac{\left(-14\cdot\omega^2+30\right) - j\cdot\left(43\cdot\omega-\omega^3\right)}{\left(-14\cdot\omega^2+30\right) - j\cdot\left(43\cdot\omega-\omega^3\right)}$$

$$= \frac{\left(-14\cdot\omega^2+30\right) - j\cdot\left(43\cdot\omega-\omega^3\right)}{\left(-14\cdot\omega^2+30\right)^2 + \left(43\cdot\omega-\omega^3\right)^2}$$



Real part goes to zero

$-14\cdot\omega^2 + 30 = 0$ $\qquad \boldsymbol{\omega} := \sqrt{\frac{30}{14}}$ $\qquad \mathbf{G}(j\cdot\boldsymbol{\omega}) = -8.36j$

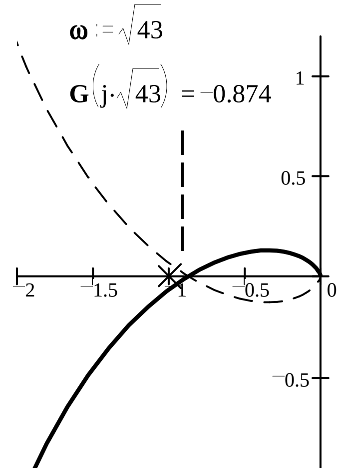


Imaginary part goes to zero

$43\cdot\boldsymbol{\omega} - \boldsymbol{\omega}^3 = 0$

$\boldsymbol{\omega} := \sqrt{43}$

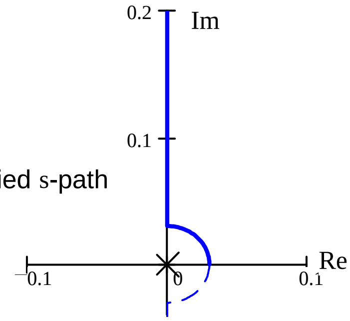
$\mathbf{G}\left(j\cdot\sqrt{43}\right) = -0.874$



$\text{GM} = \frac{1}{0.874} = 1.144$

Examples of Poles on the imaginary (jω) axis

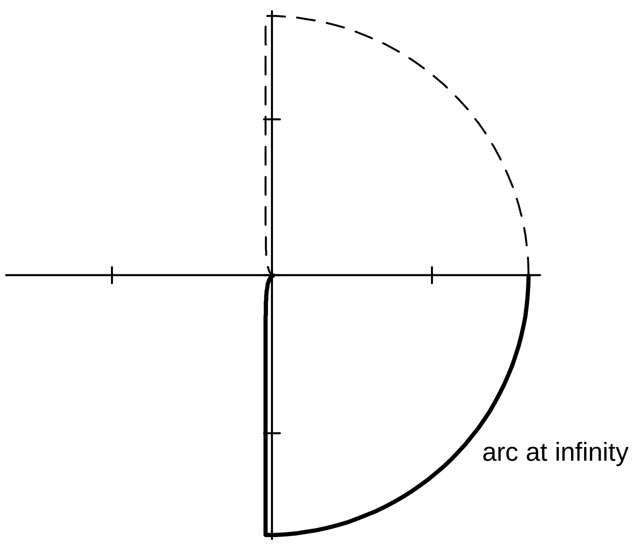
Modified s-path

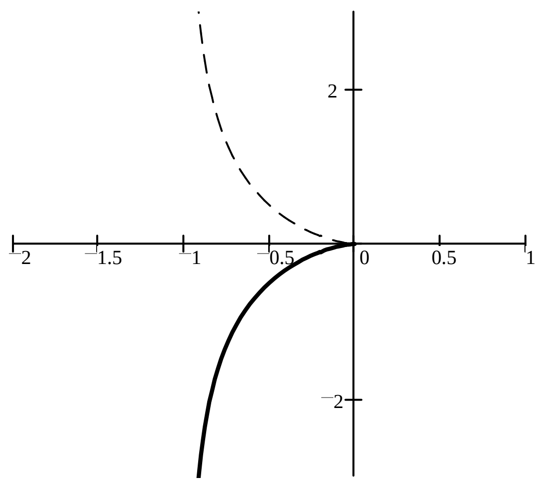


Single Pole at the origin

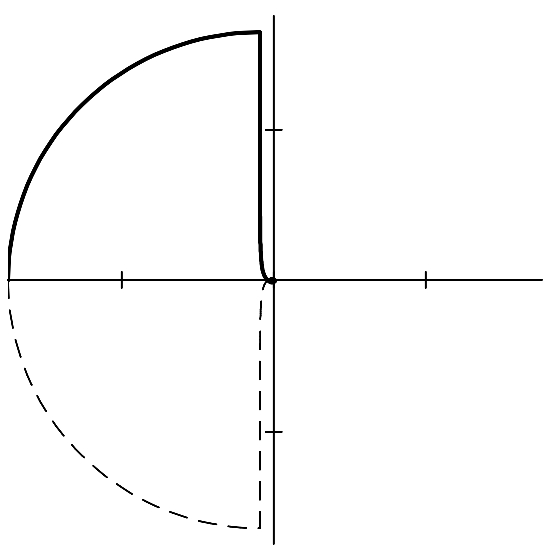
$$\mathbf{G}(s) := \frac{1}{s\cdot(s+1)}$$

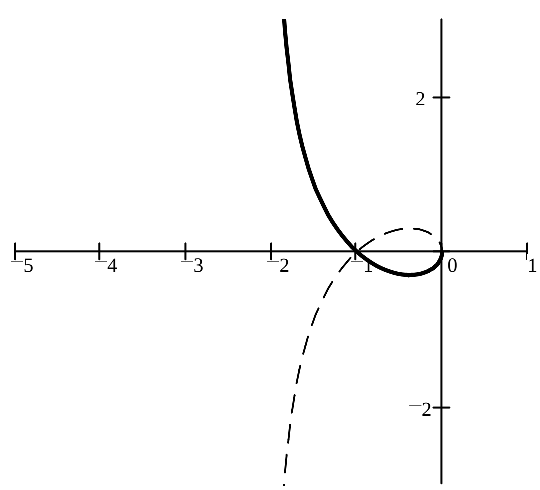
$siz := \mathbf{G}(sz) + 1.5$
(for plotting)





$$\mathbf{G}(s) := \frac{s+1}{s\cdot(s-1)} \qquad siz := |\mathbf{G}(sz)|$$

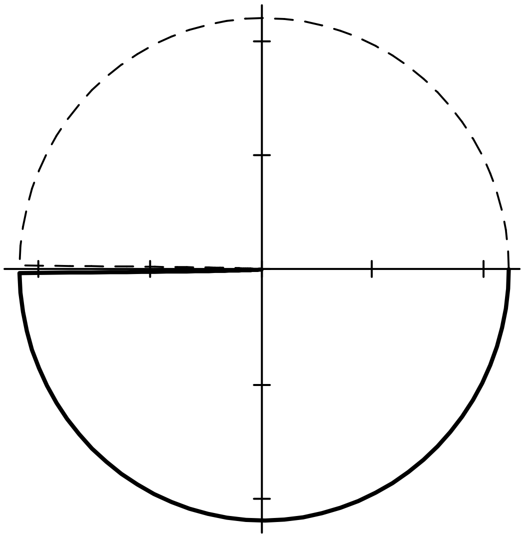


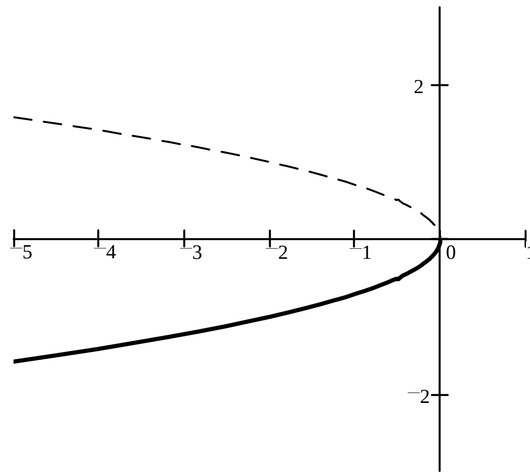


Double Pole at the origin

$$G(s) := \frac{(s+2)}{s^2}$$

$$siz := G(sz) + 100$$

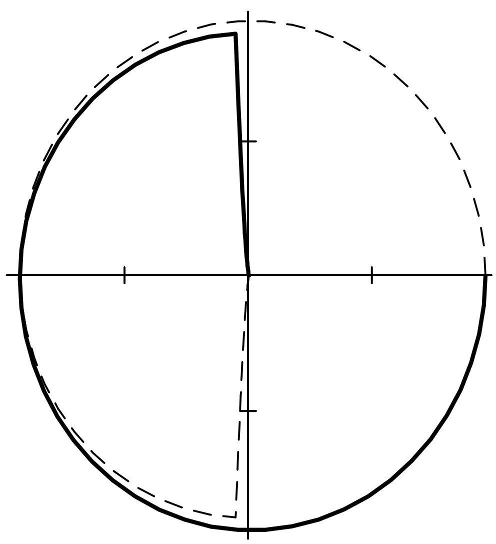


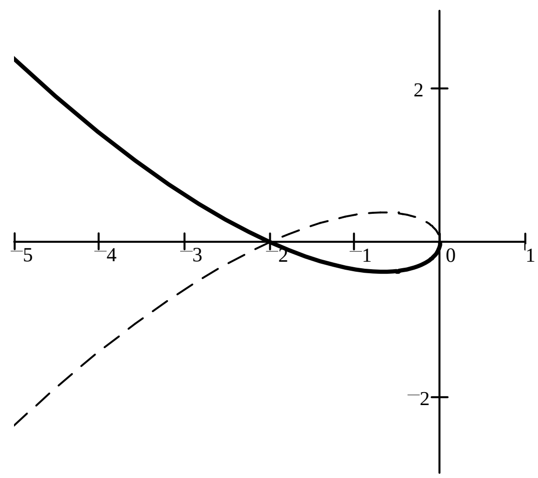


Triple Pole at the origin

$$G(s) := \frac{(s+1)^2}{s^3}$$

$$siz := G(sz) + 1000$$

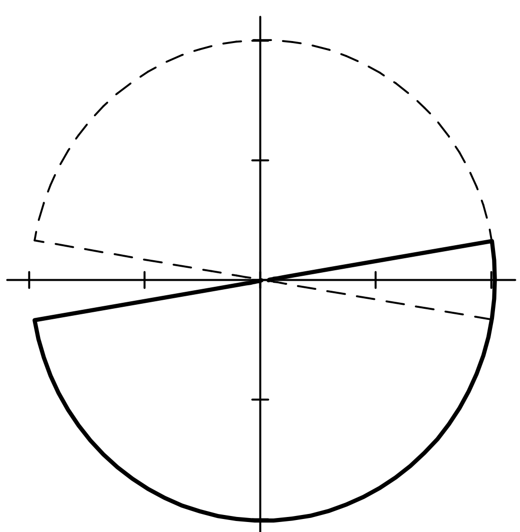


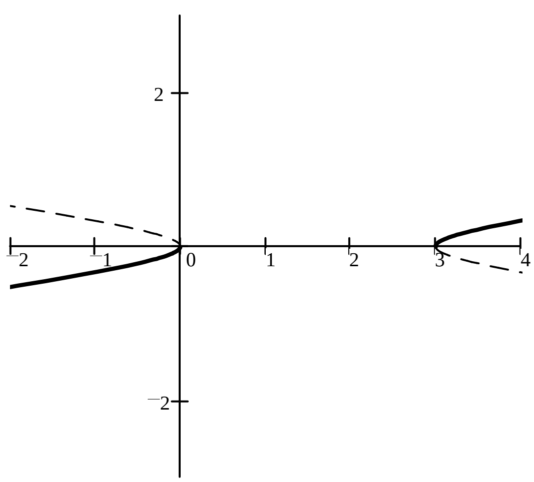


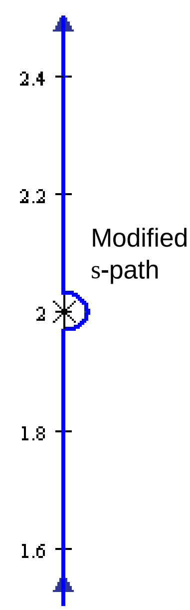
Poles at the ± 2j

$$G(s) := \frac{s+12}{\left(s^2+4\right)}$$

$$siz := 110$$

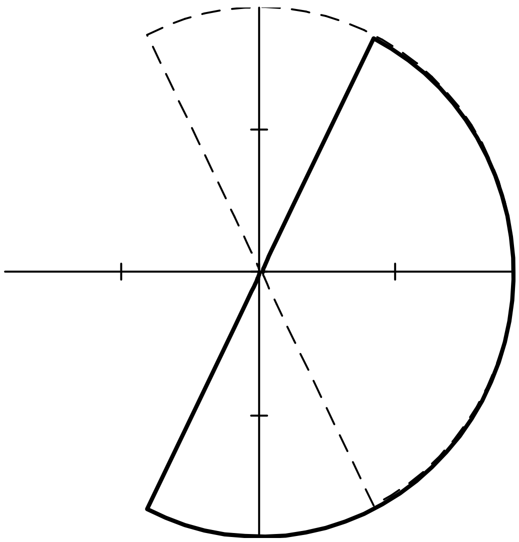


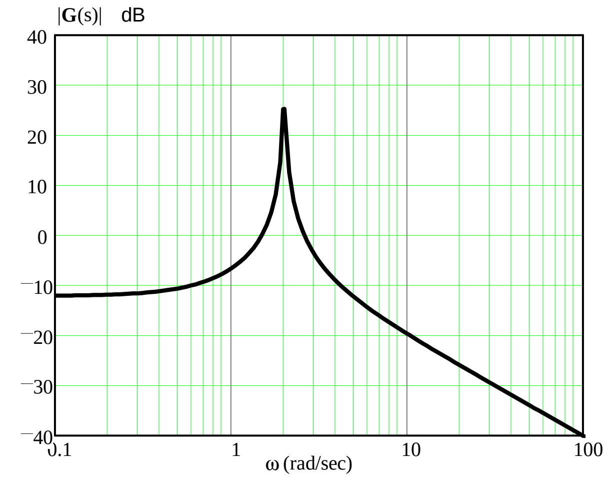


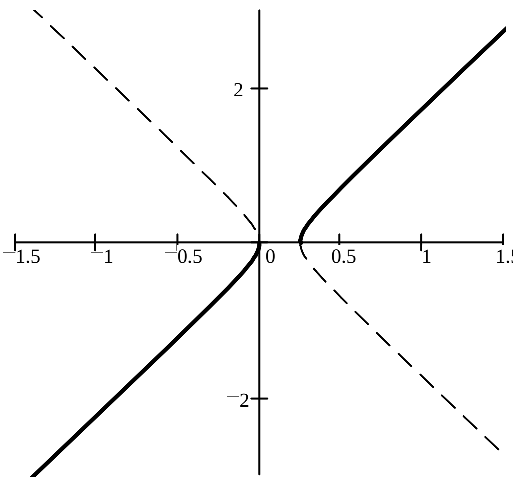


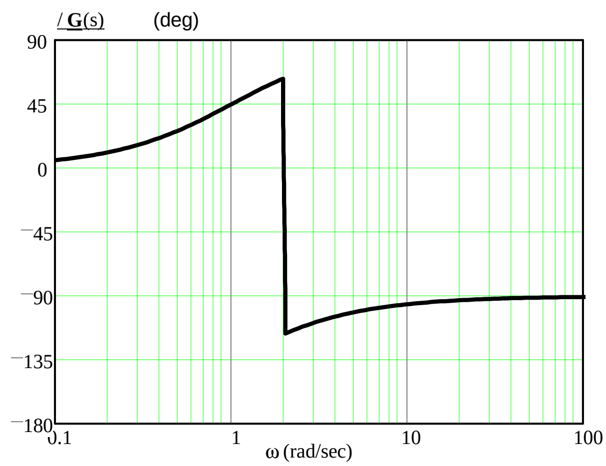
$\mathbf{G}(s) := \dfrac{s+1}{\left(s^2+4\right)}$

$siz := 18.5$



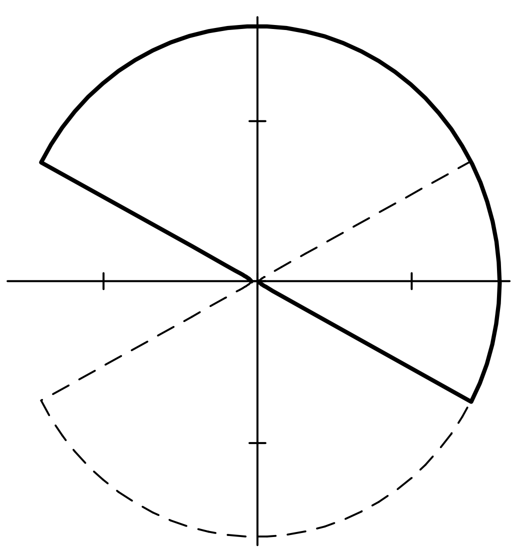


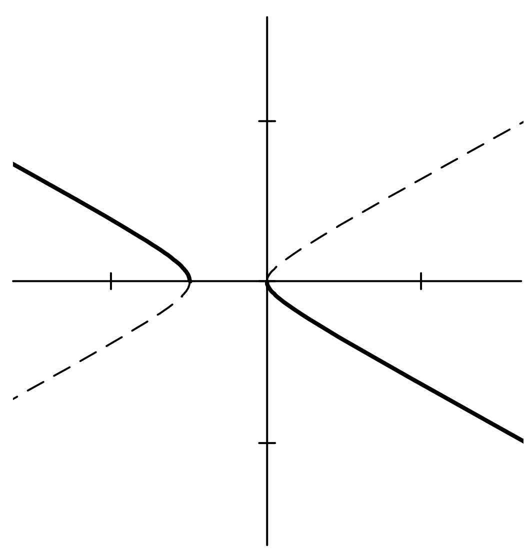




$\mathbf{G}(s) := \dfrac{s-12}{\left(s^2+4\right)\cdot(s+6)}$ $\mathbf{G}(0) = -0.5$

$siz := \left|\dfrac{\mathbf{G}(sz)}{sz}\right|$



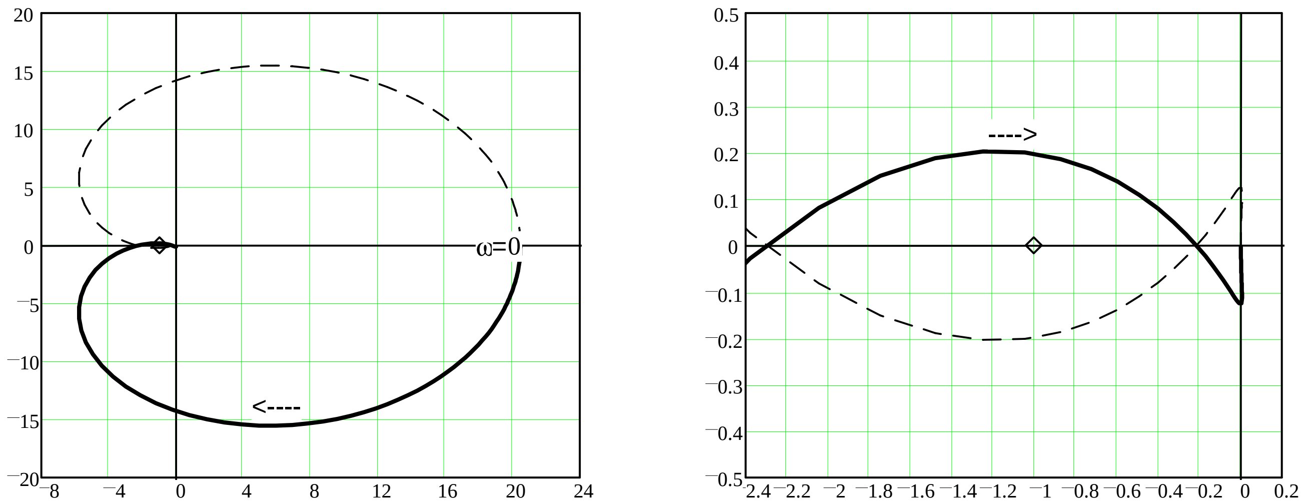


Name: ________________________________ ECE 3510 homework Nq1

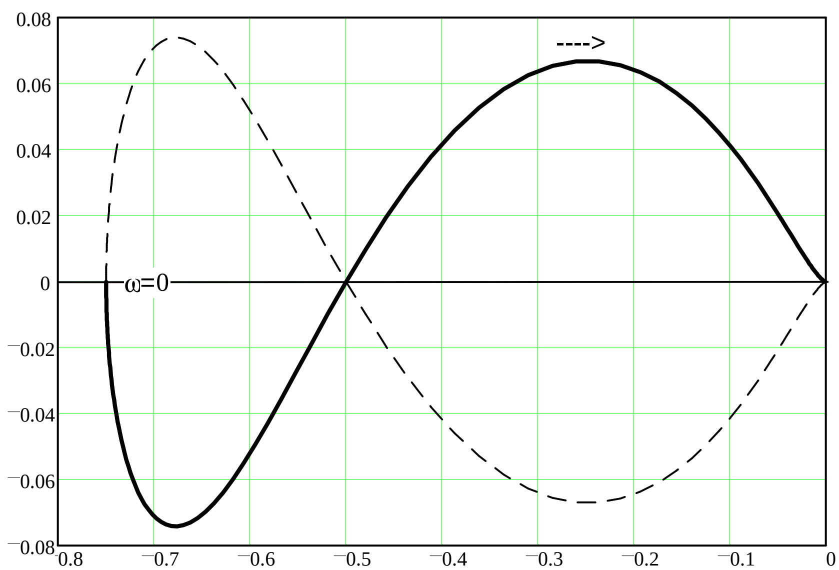
b

Due: Fri, 4/16/21

1. Similar to problem 5.4 in Bodson text.
   a) The Nyquist diagram of a stable system is shown below (or in text), with the overall diagram shown on the left and the detail around the (-1,0) point shown on the right. The solid line corresponds to $\omega > 0$, with the arrow giving the direction of increasing ω. The dashed line is the symmetric curve obtained for $\omega < 0$. Assuming that the transfer function of the system is multiplied by a gain k > 0, what is the set of values of k for which the system is stable in closed-loop?

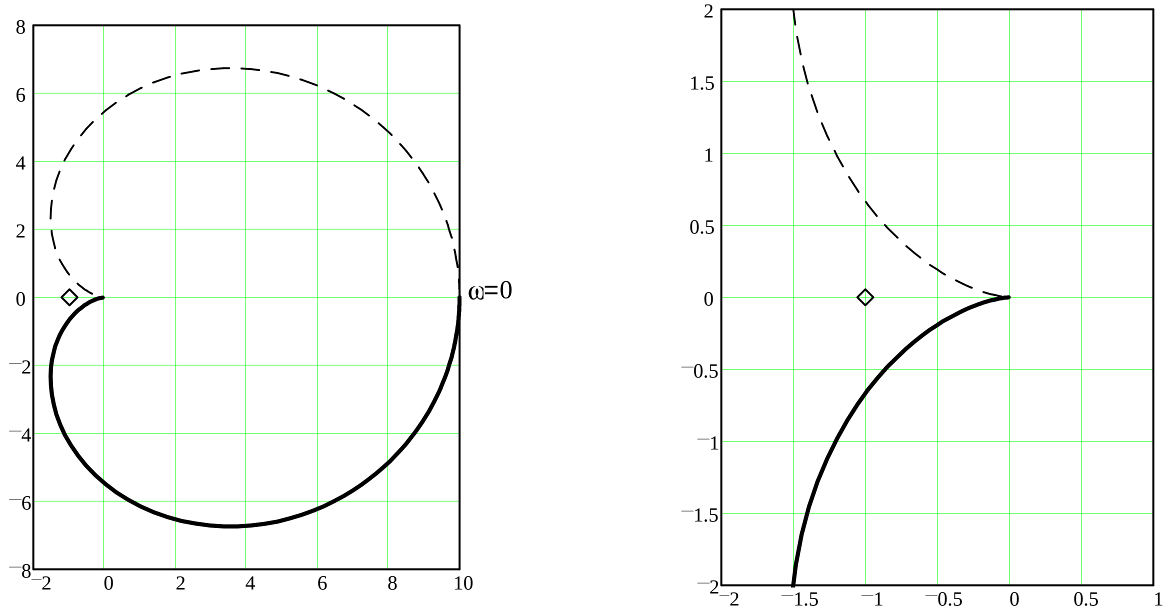


   b) Repeat part (a) for the system whose Nyquist curve is shown at below (or in text), given that the system has one unstable pole.

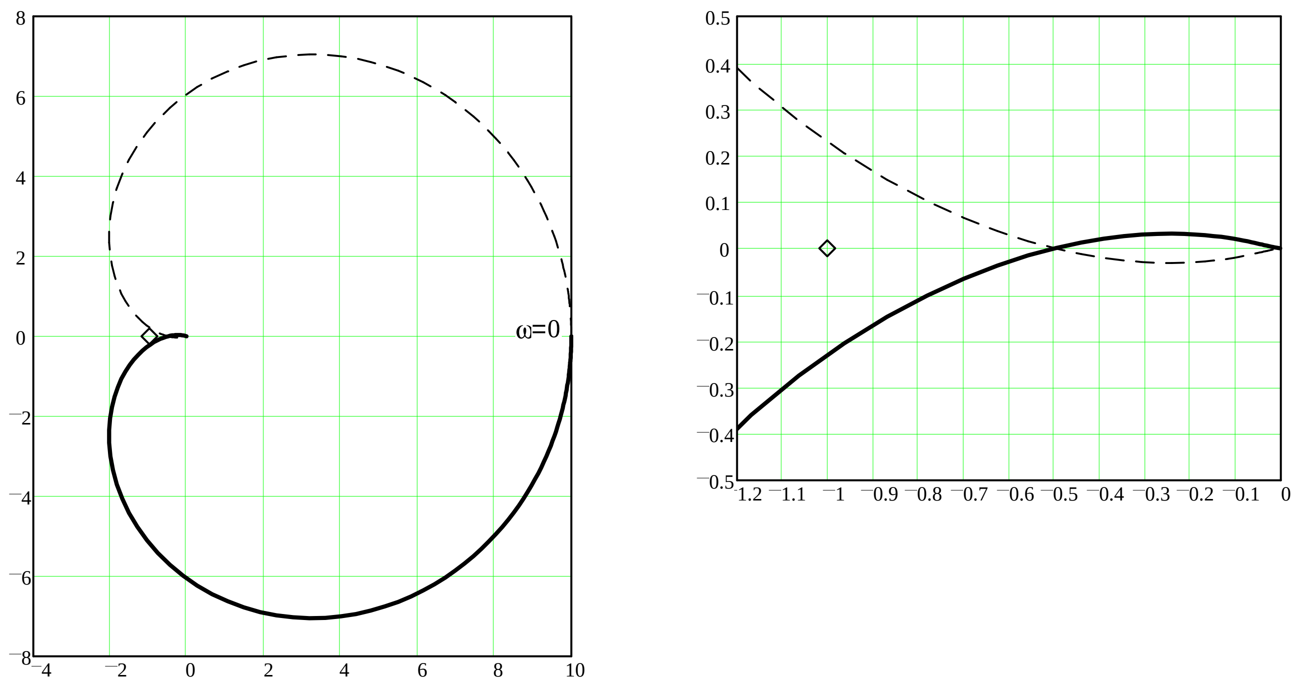


2. Problem 5.5 in Bodson text.

a) The Nyquist diagram for $P(s)=5(s+2)/(s+1)^3$ is shown below (or in text), with the overall diagram shown on the left and the detail around the (-1,0) point shown on the right. Indicate what the gain margin and the phase margins are (for the phase margin, show work on the drawing below). Compare the gain margin results with those predicted by a root-locus plot or the Routh-Hurwitz criterion.



b) Repeat part (a) for $P(s)=2(s+5)/(s+1)^3$ and the diagrams shown below.



Answers

1\. a) $k < 0.435$ or $k > 5$ b) $\frac{4}{3} < k < 2$

ECE 3510 Homework Nq1 p.2

2\. a) $GM = \infty$ $PM = 30 \cdot deg$

b) $GM = 2$ $PM = 12.2 \cdot deg$ GMs may be confirmed by either root-locus or Routh-Hurwitz

Name: ______________________________ ECE 3510 homework Nq2 c

Due: Tue, 4/20/21

1. For problem Nq1, 2a (5.5 in Bodson text): $\mathbf{P}(s) = \dfrac{5\cdot(s+2)}{(s+1)^3}$

a) Find the DC gain ($s = 0 = \omega$) from the transfer function and compare it to the $\omega = 0$ point on the Nyquist diagram.

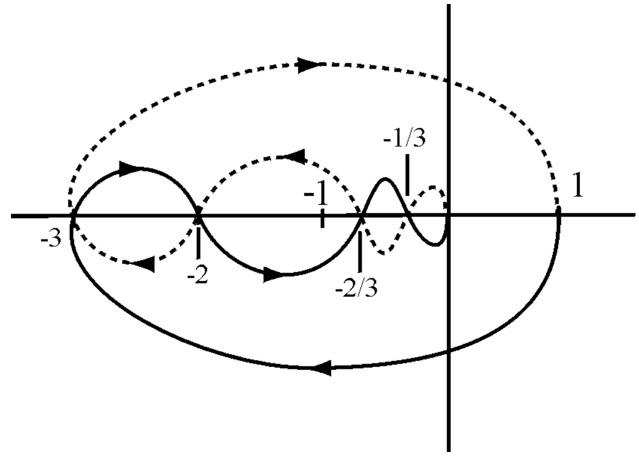
b) Find the final value ($\omega = \infty$) from the transfer function and compare it to the $\omega = \infty$ point on the Nyquist diagram.

c) Find the approach angle to the final value ($\omega = \infty$) from the transfer function and compare to the Nyquist diagram.

d) Reproduce the Nyquist diagram (left drawing). If you do this by hand, find and plot at least 3 more points (besides a & b, above) which will show the shape of the curve. You may also plot this diagram using a computer program of your choice.

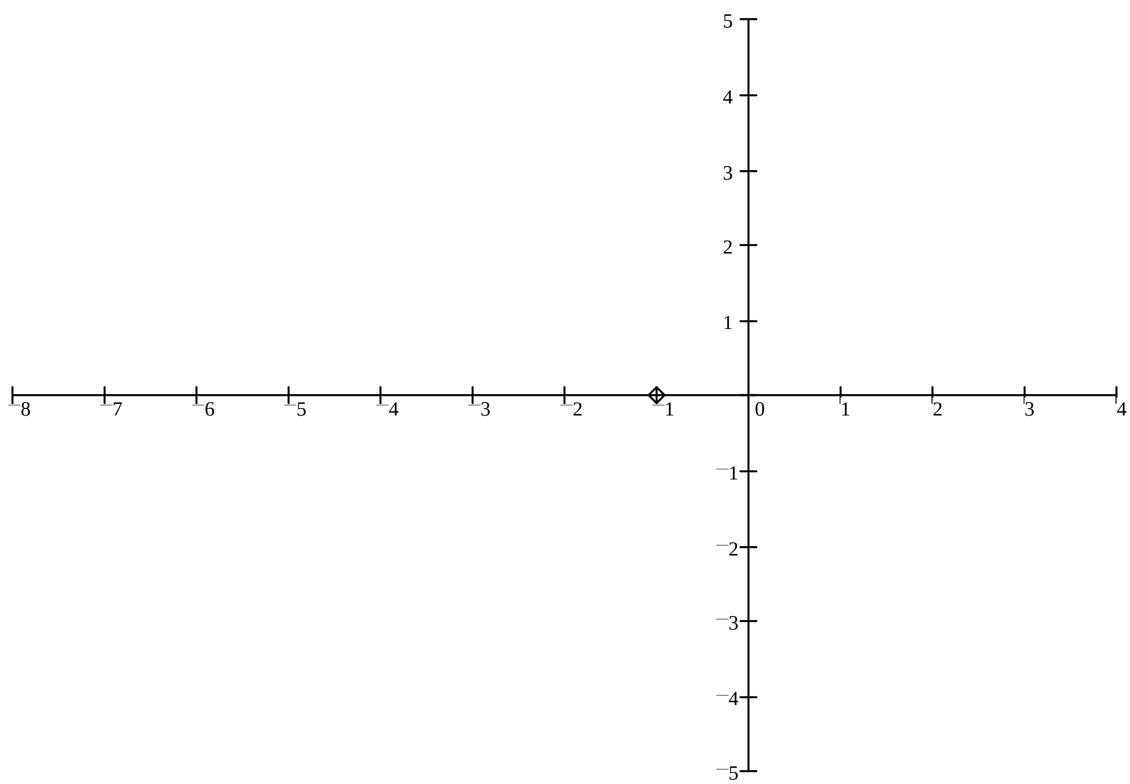
2. Problem 5.9 b - d in Bodson the text.

   b) Indicate whether the system whose Nyquist curve is shown is closed-loop stable, given that it is open-loop stable.



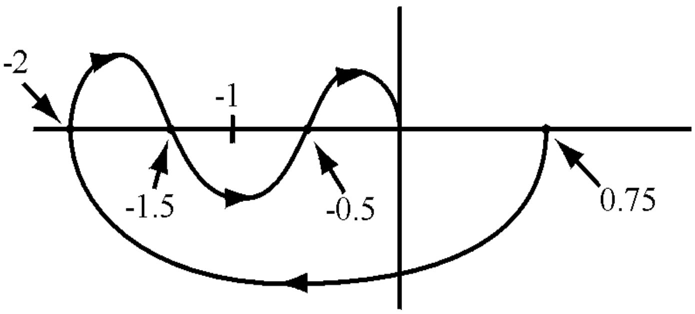
   c) What are the values of the gain $g > 0$ by which the open-loop transfer function of part (b) may be multiplied, with the closed-loop system being stable?

   d) Sketch an example of a Nyquist curve for a system which has three unstable open-loop poles, and which is closed-loop stable.



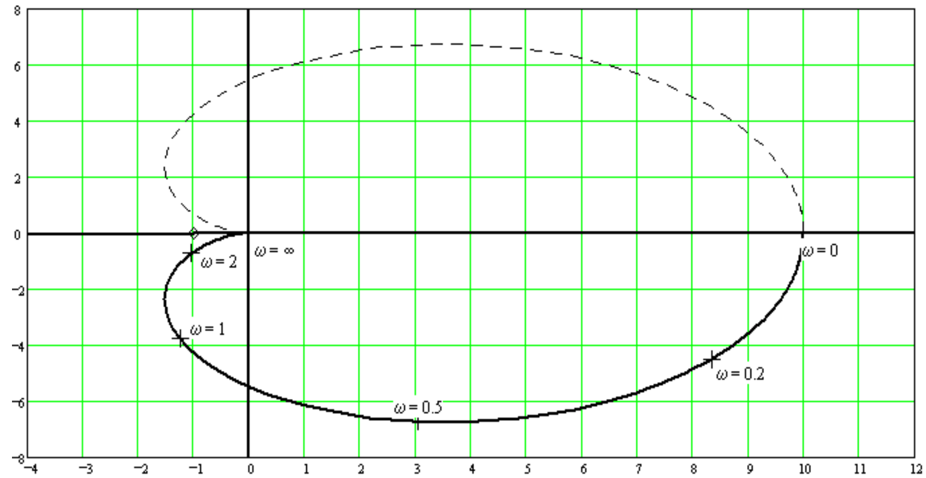
3. Problem 5.13 in the text.
   a) Consider the Nyquist diagram of a transfer function $\mathbf{G}(s)$ shown at right. Only the portion for $\omega > 0$ is plotted.

      Assume that $\mathbf{G}(s)$ has no poles in the open right-half plane, and that a gain K is cascaded with $\mathbf{G}(s)$. Find the ranges of positive K for which the closed-loop system is stable.



Answers

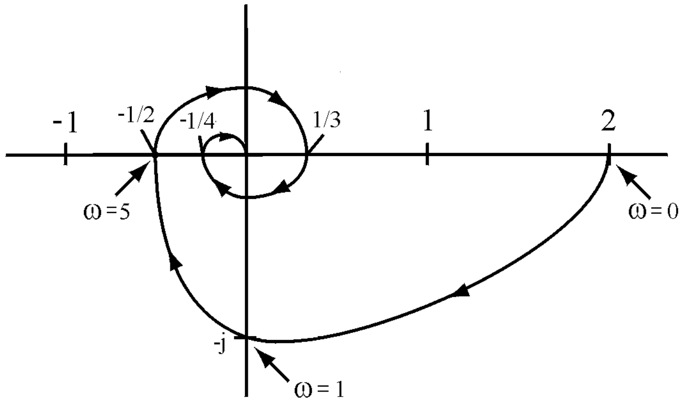
1. a) $\mathbf{P}(0) = 10$
   b) $\mathbf{P}(\infty) = 0$
   c) $-180 \cdot \text{deg}$
   d) Extra points shown are for
      $s = 0.2j$,
      $s = 0.5j$,
      $s = 1j$,
      and $s = 2j$



2. b) yes   c) $0 < g < \frac{1}{3}$, $\frac{1}{2} < g < \frac{3}{2}$ or $g > 3$ d) Need 3 CCW encirclements of -1
3. $k < \frac{1}{2}$, $\frac{2}{3} < k < 2$
4. a) yes   b) GM $\simeq 2 \ (6 \cdot \text{dB})$ PM $\simeq 90 \cdot \text{deg}$ c) 4   d) 4 , $3 \cdot \cos(t - 90 \cdot \text{deg})$, $-2 \cdot \cos(5 \cdot t)$
   e) $\frac{4}{3}$, $\frac{3 \cdot \sqrt{2}}{2} \cdot \cos(t - 45 \cdot \text{deg})$, $-4 \cdot \cos(5 \cdot t)$

 Turn Over, More on Next Page ========>>>

4. Problem 5.11 in the text.
   All parts of this problem refer to the system whose Nyquist curve is shown at right (only the portion for $\omega > 0$ is plotted). Recall that the Nyquist curve represents the frequency response of the open-loop system, or $\mathbf{G}(j\omega)$. If $\mathbf{G}(s)$ is the open-loop transfer function. The closed-loop transfer function is $\mathbf{G}(s)/(1 + \mathbf{G}(s))$.



   a) Knowing that the closed-loop system is stable, can one say for sure that the open-loop system is stable?

   b) Given the closed-loop system is stable, estimate the gain margin and the phase margin of the closed-loop system.

   c) How many unstable poles does the closed-loop system have if the open-loop gain is multiplied by 5?

   d) Give the steady-state response $y_{ss}(t)$ of the open-loop system to an input $x(t) = 2$.

   Repeat for $x(t)=3\cos(t)$

   and $x(t)=4\cos(5t)$.

   e) Repeat part (d) for the closed-loop system.
   Hint: remember that the output of the closed-loop system is $\text{input}\cdot\dfrac{\mathbf{G}(s)}{1+\mathbf{G}(s)}$

   $x(t)=3\cos(t)$

   $x(t)=4\cos(5t)$.