

ECE 3510 Root Locus Additional rules

7. The breakaway (and arrival) points are solutions to:

$$\sum_{\text{all}} \frac{1}{(s + p_i)} = \sum_{\text{all}} \frac{1}{(s + z_i)}$$

Ex:

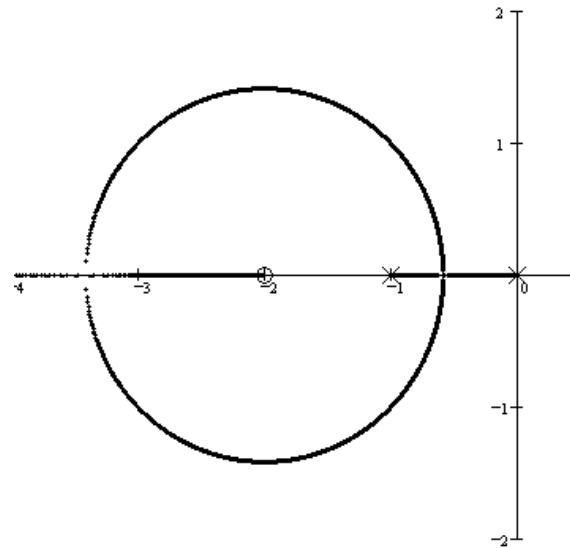
$$G(s) := \frac{s+2}{s(s+1)}$$

Solve: $\frac{1}{s} + \frac{1}{s+1} = \frac{1}{s+2}$

$$\frac{(s+1)+s}{s \cdot (s+1)} = \frac{1}{s+2}$$

$$(2 \cdot s + 1) \cdot (s + 2) = s \cdot (s + 1)$$

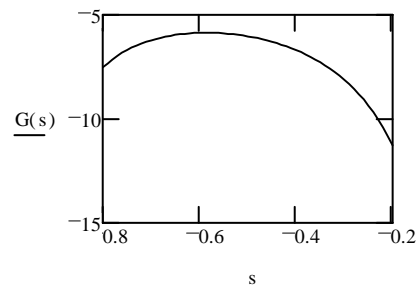
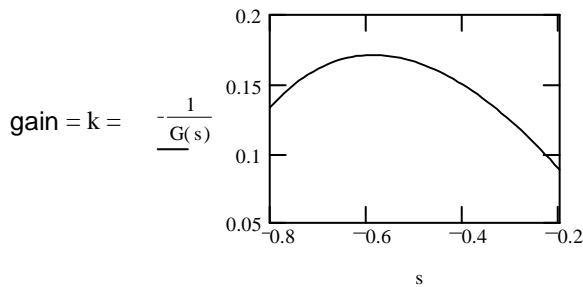
$$s^2 + 4 \cdot s + 2 = 0 \quad s = -3.414 \quad s = -0.586$$



Breakaway (and arrival) points from the real axis (σ_b) are also the solutions to: $\frac{d}{ds} G(s) = 0$ where s is a real number (on the real axis)

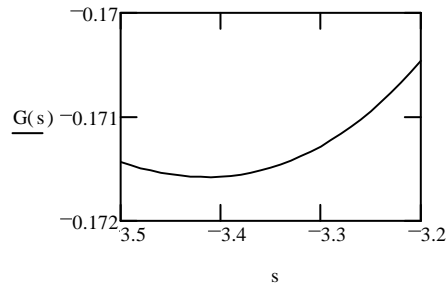
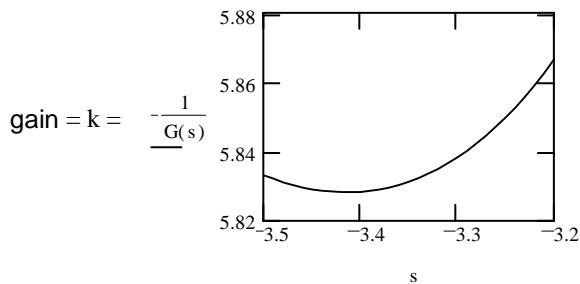
Why? Because gain = $k = -\frac{1}{G(s)}$ The breakaway point will be the point between -1 and 0 with the highest gain. That is also the point with the lowest $G(s)$ and highest $-G(s)$

Make some plots $s := -.8, -.79 \dots -.2$

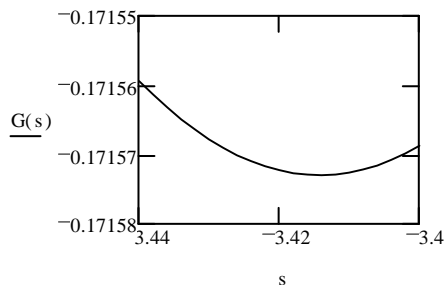


The breakin point will be the point between -4 and 2 with the lowest gain.

Make some plots $s := -3.5, -3.49 \dots -3.2$



$s := -3.44, -3.438 \dots -3.4$



Root Locus Examples, continued p2

Find the Break-in points for Basic Root Locus Examples, Example 7

$$7. G(s) = \frac{(s+5) \cdot (s+8)}{s^2 - 6 \cdot s + 13}$$

$$\text{Break-away points } \frac{1}{(s+5)} + \frac{1}{(s+8)} = \frac{1}{s-3-2 \cdot j} + \frac{1}{s-3+2 \cdot j} = \frac{2 \cdot s - 6}{s^2 - 6 \cdot s + 13} \quad \text{NOT } \neq \frac{1}{s^2 - 6 \cdot s + 13}$$

Note the way these poles are expressed

Guess -6.3 Use this guess in all but the closest poles and zeroes

$$\frac{1}{s+5} + \frac{1}{s+8} - \frac{2 \cdot (-6.3) - 6}{(-6.3)^2 - 6 \cdot (-6.3) + 13} = 0 \quad \text{Solutions: } \begin{pmatrix} 2.57 \\ -6.3 \end{pmatrix} \text{ guess was good}$$

Find the Departure angles from complex poles for Basic Root Locus Examples, Example 7

$$\text{atan}\left(\frac{2}{8+3}\right) = 10.305 \cdot \text{deg}$$

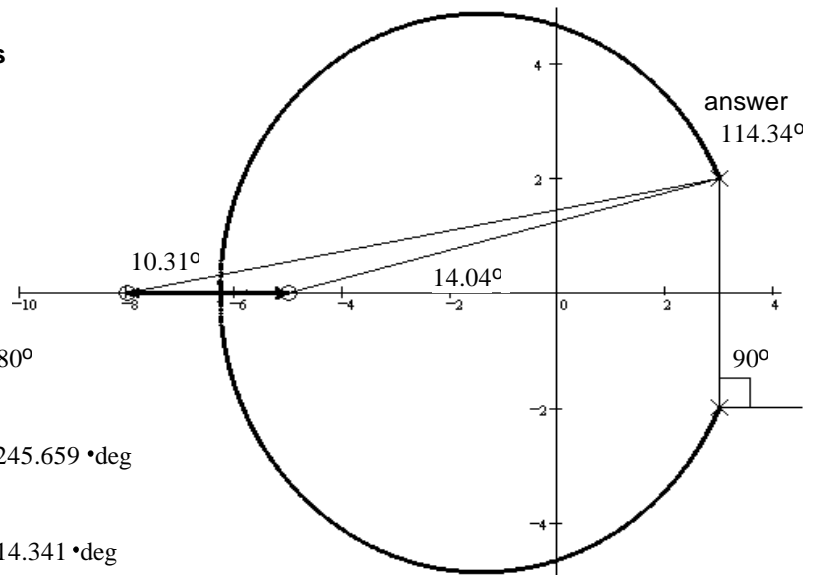
$$\text{atan}\left(\frac{2}{5+3}\right) = 14.036 \cdot \text{deg}$$

$$\text{atan}\left(\frac{2}{8+3}\right) + \text{atan}\left(\frac{2}{5+3}\right) - 90 \cdot \text{deg} - \theta = \pm 180^\circ$$

$$\text{atan}\left(\frac{2}{8+3}\right) + \text{atan}\left(\frac{2}{5+3}\right) - 90 \cdot \text{deg} - 180 \cdot \text{deg} = -245.659 \cdot \text{deg}$$

$$\text{atan}\left(\frac{2}{8+3}\right) + \text{atan}\left(\frac{2}{5+3}\right) - 90 \cdot \text{deg} + 180 \cdot \text{deg} = 114.341 \cdot \text{deg}$$

better answer



Finding the $j\omega$ crossing point using rule 9:

$$G(s) := \frac{(s+5) \cdot (s+8)}{s^2 - 6 \cdot s + 13}$$

$$\text{Try: } s := 5 \cdot j \quad \angle G(s) = \arg(G(5 \cdot j)) = -171.193 \cdot \text{deg}$$

$$\text{Try: } s := 4.5 \cdot j \quad \angle G(s) = \arg(G(s)) = 176.375 \cdot \text{deg} \quad \arg(G(s)) - 360 \cdot \text{deg} = -183.625 \cdot \text{deg}$$

$$\text{linear interpolation } 4.5 + \frac{183.625 - 180}{183.625 - 171.193} \cdot (5 - 4.5) = 4.646$$

$$\text{Try: } s := 4.646 \cdot j \quad \angle G(s) = \arg(G(s)) = -179.838 \cdot \text{deg}$$

$$\text{linear interpolation } 4.5 + \frac{183.625 - 180}{183.625 - 179.838} \cdot (4.646 - 4.5) = 4.64$$

$$\text{Try: } s := 4.64 \cdot j \quad \angle G(s) = \arg(G(s)) = -179.991 \cdot \text{deg} \quad \text{close enough}$$

Finding the $j\omega$ crossing gain using rule 8:

$$\text{Gain: } \frac{1}{|G(s)|} = \frac{(4.64 \cdot j)^2 - 6 \cdot (4.64 \cdot j) + 13}{(4.64 \cdot j + 5) \cdot (4.64 \cdot j + 8)} = \frac{\sqrt{[13 - (4.64)^2]^2 + (6 \cdot (4.64))^2}}{\sqrt{4.64^2 + 5^2} \cdot \sqrt{4.64^2 + 8^2}} = 0.462 \quad \text{to be stable: } k > 0.462$$

Root Locus Examples, continued p2

Root Locus Examples, continued p3

Find the Break-in points for Basic Root Locus Examples, Example 9

$$9 \quad G(s) = \frac{s + 12}{(s^2 + 4 \cdot s + 13) \cdot (s + 1) \cdot (s + 5)}$$

Break-away points

$$\frac{1}{(s + 12)} = \frac{1}{(s + 2 + 3 \cdot j)} + \frac{1}{(s + 2 - 3 \cdot j)} + \frac{1}{(s + 1)} + \frac{1}{(s + 5)} = \frac{(s + 2 - 3 \cdot j) + (s + 2 + 3 \cdot j)}{s^2 + 4 \cdot s + 13} + \frac{1}{(s + 1)} + \frac{1}{(s + 5)}$$

$$\frac{1}{(s + 12)} = \frac{2 \cdot s - 4}{s^2 + 4 \cdot s + 13} + \frac{1}{(s + 1)} + \frac{1}{(s + 5)}$$

Guess -4 Use this guess in all but the closest poles and zeroes

$$\frac{1}{(-4 + 12)} = \frac{2 \cdot s + 4}{s^2 + 4 \cdot s + 13} + \frac{1}{(-4 + 1)} + \frac{1}{(s + 5)}$$

$$0 = \frac{2 \cdot s + 4}{s^2 + 4 \cdot s + 13} + \frac{1}{(s + 5)} + \frac{1}{(-4 + 1)} - \frac{1}{(-4 + 12)}$$

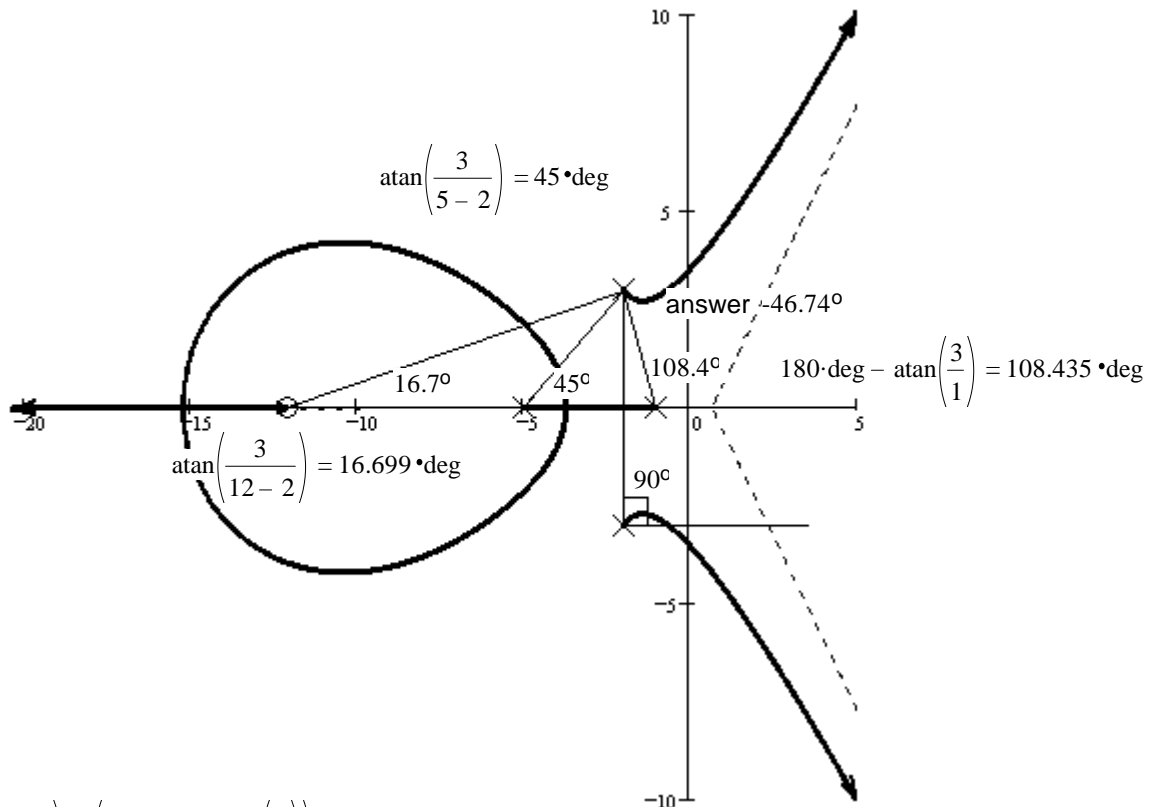
$$\text{Solve: } \begin{pmatrix} 2.105 \\ -3.648 \\ -0.912 \end{pmatrix}$$

$$0 = \frac{2 \cdot s + 4}{s^2 + 4 \cdot s + 13} + \frac{1}{(s + 5)} + \frac{1}{(-3.648 + 1)} - \frac{1}{(-3.648 + 12)}$$

$$\text{Solve: } \begin{pmatrix} 1.091 \\ -3.727 \\ -0.332 \end{pmatrix}$$

Close to actual answer of -3.712

Find the Departure angles from complex poles for Basic Root Locus Examples, Example 9

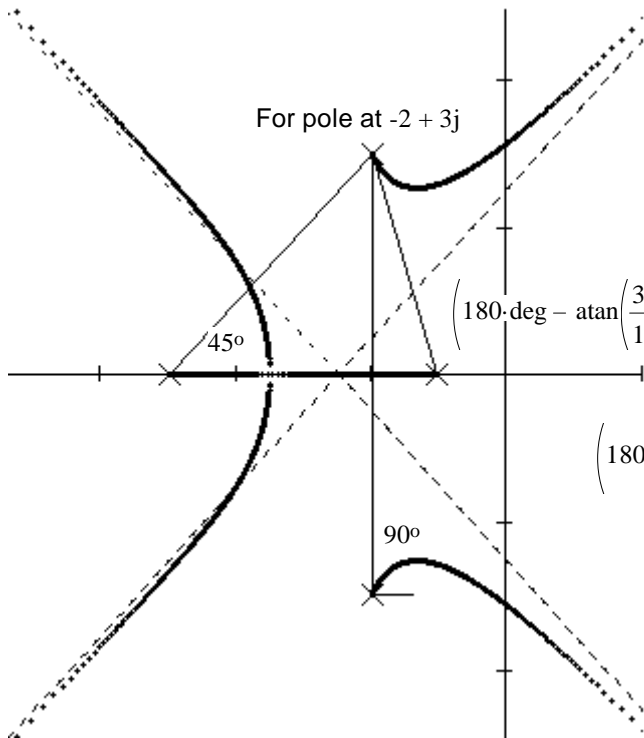


$$\text{atan}\left(\frac{3}{12-2}\right) - \text{atan}\left(\frac{3}{5-2}\right) - \left(180^\circ - \text{atan}\left(\frac{3}{1}\right)\right) - 90^\circ - \theta = \pm 180^\circ$$

$$\theta = 180^\circ + 16.699^\circ - 45^\circ - 108.435^\circ - 90^\circ = -46.736^\circ$$

ECE 3510 Root Locus Departure and Arrival Angles

$$G(s) := \frac{1}{(s^2 + 4s + 13) \cdot (s+1) \cdot (s+5)} = \frac{1}{(s+2+3j) \cdot (s+2-3j) \cdot (s+1) \cdot (s+5)}$$



$$s := -2 + 3j$$

$$\frac{1}{(s+2+3j) \cdot (s+1) \cdot (s+5)} = -5.5556 \cdot 10^{-3} + 0.0111i$$

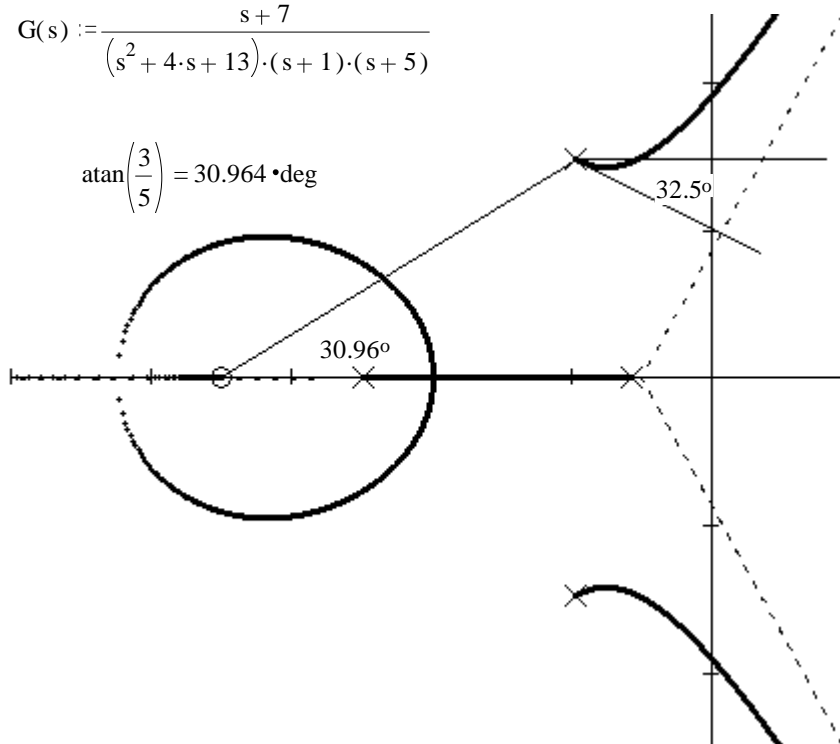
$$180\text{-deg} - \arg(-5.5556 \cdot 10^{-3} + 0.0111i) = 63.412 \text{ deg}$$

If you leave out 180° (not recommended)

$$\left(-\text{atan}\left(\frac{3}{1}\right)\right) + 90\text{-deg} + 45\text{-deg} = 63.435 \text{ deg} + \theta = 0^\circ, \pm 360^\circ$$

$$\theta = -63.435\text{-deg}$$

$$G(s) := \frac{s+7}{(s^2 + 4s + 13) \cdot (s+1) \cdot (s+5)}$$



$$s := -2 + 3j$$

$$\frac{s+7}{(s+2+3j) \cdot (s+1) \cdot (s+5)} = -0.0611 + 0.0389i$$

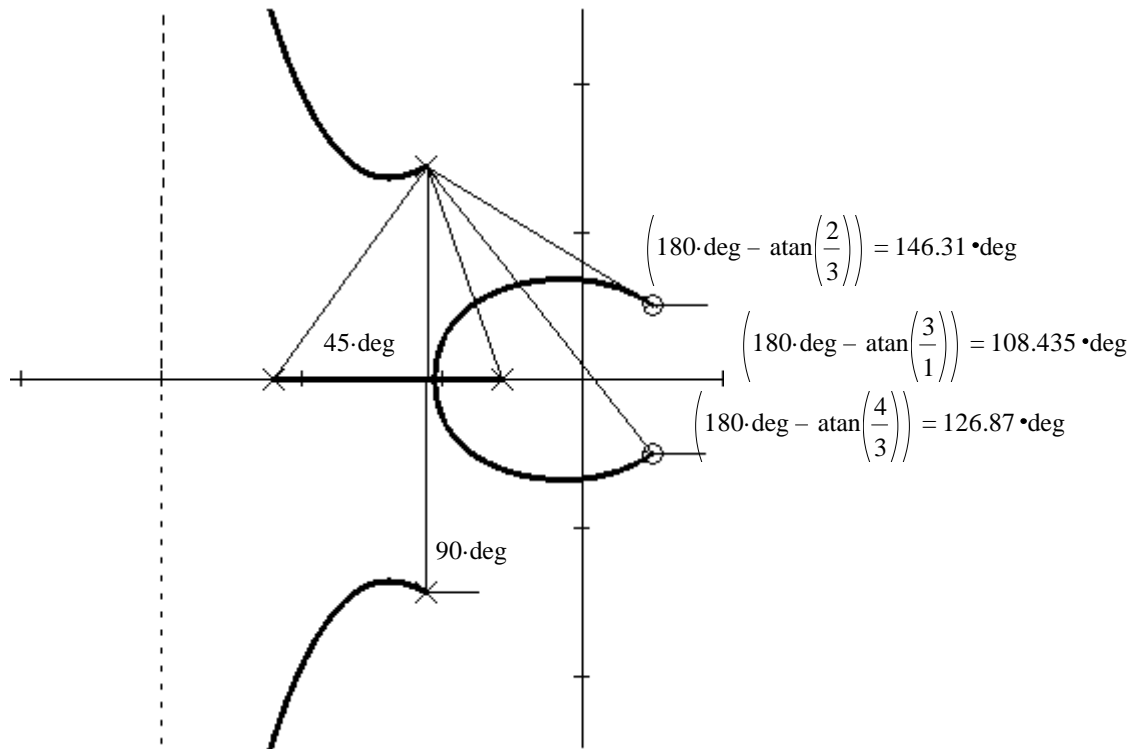
$$180\text{-deg} - \arg(-0.0611 + 0.0389i) = 32.483 \text{ deg}$$

$$\left(180\text{-deg} - \text{atan}\left(\frac{3}{1}\right)\right) + 90\text{-deg} + 45\text{-deg} - \text{atan}\left(\frac{3}{5}\right) = 212.471 \text{ deg} + \theta = \pm 180^\circ$$

$$180 - 212.5 = -32.5 \text{ deg}$$

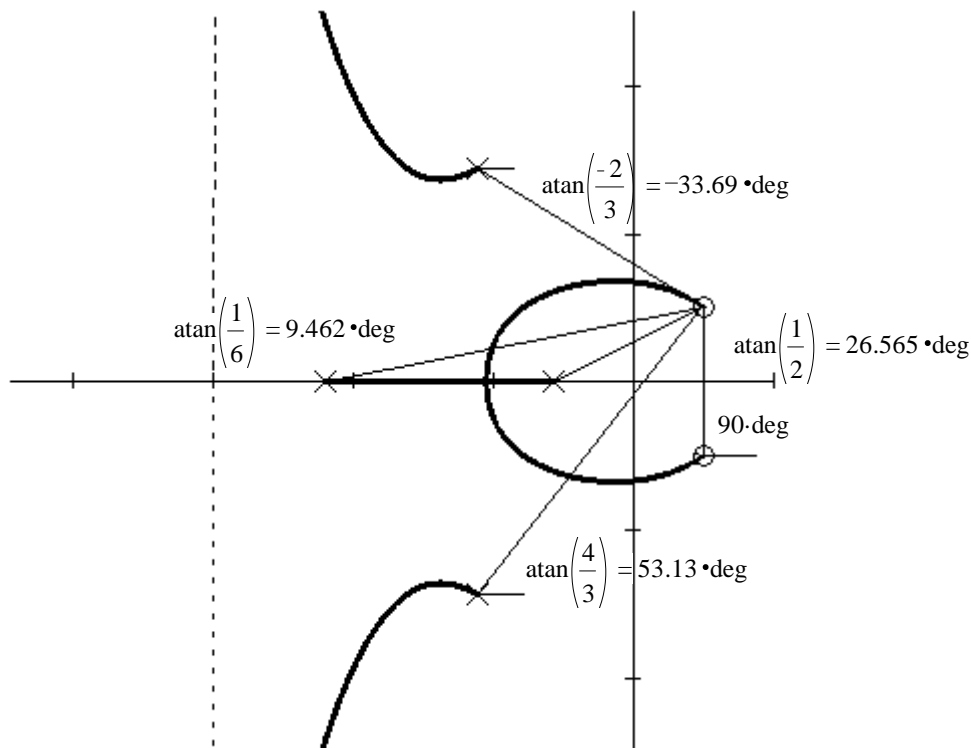
ECE 3510 Root Locus Departure and Arrival Angles p2

$$G(s) := \frac{s^2 - 2 \cdot s + 2}{(s^2 + 4 \cdot s + 13) \cdot (s + 1) \cdot (s + 5)}$$



$$\left(180\text{-deg} - \operatorname{atan}\left(\frac{3}{1}\right)\right) + 90\text{-deg} + 45\text{-deg} - \left(180\text{-deg} - \operatorname{atan}\left(\frac{2}{3}\right)\right) - \left(180\text{-deg} - \operatorname{atan}\left(\frac{4}{3}\right)\right) = -29.745 \text{ deg} + \theta = \pm 180^\circ$$

$$-(180 - 29.75) = -150.25 \text{ deg}$$



$$\operatorname{atan}\left(\frac{1}{2}\right) + \operatorname{atan}\left(\frac{-2}{3}\right) + \operatorname{atan}\left(\frac{1}{6}\right) + \operatorname{atan}\left(\frac{4}{3}\right) - 90\text{-deg} = -34.533 \text{ deg} \quad -\theta = \pm 180^\circ$$

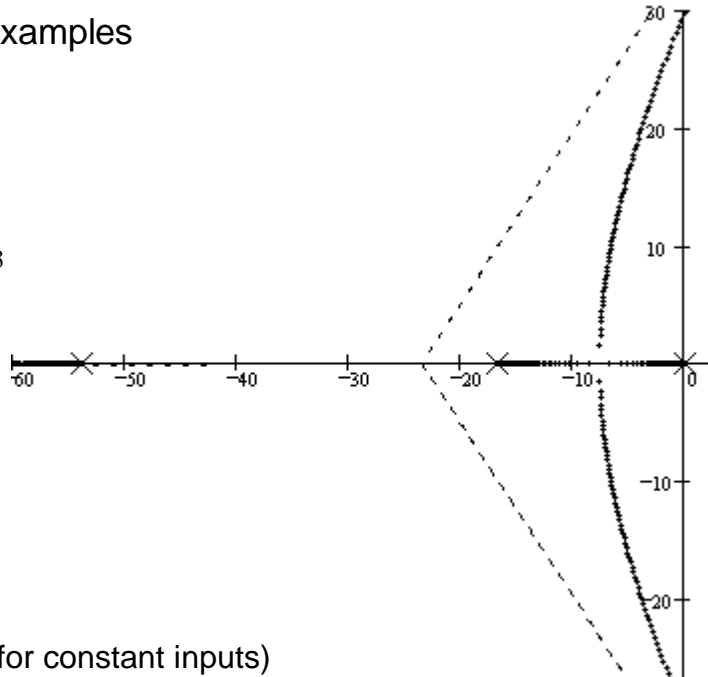
$$(180 - 34.5) = 145.5 \text{ deg}$$

ECE 3510 Root Locus Design Examples

Recall the simple crude servo from lab 1

$$G(s) := \frac{1643}{s \cdot (s + 16.64) \cdot (s + 53.78)}$$

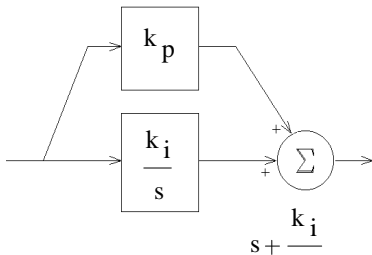
$$\sigma = \frac{-0 - 16.64 - 53.78}{3} = -23.473$$



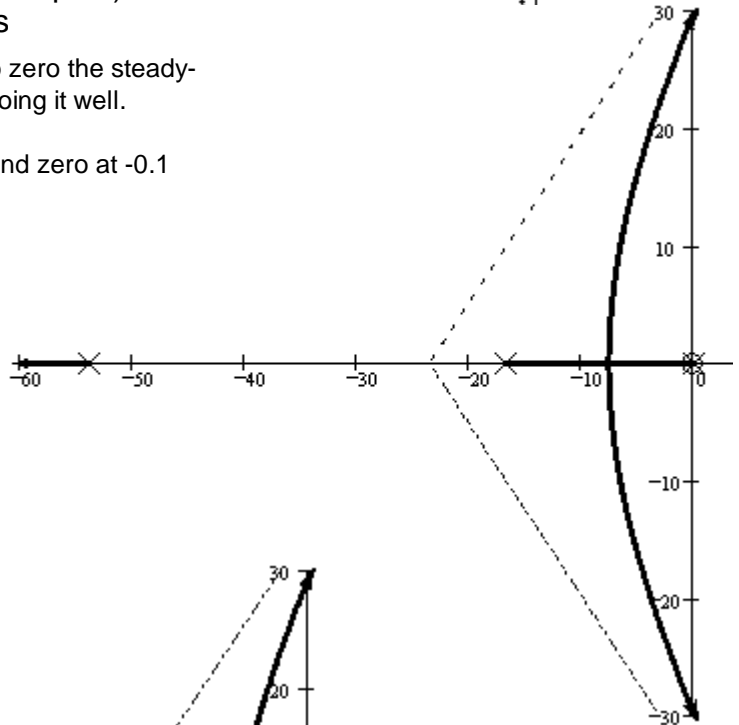
PI To eliminate steady-state error (for constant inputs) & perfect rejection of constant disturbances

Note: The DC motor has a pole at zero and should do zero the steady-state error by itself, but nonlinearities prevent it from doing it well.

$$G(s) := \frac{1643}{s \cdot (s + 16.64) \cdot (s + 53.78)} \cdot \frac{s + 0.1}{s} \quad \text{Add pole at 0 and zero at -0.1}$$



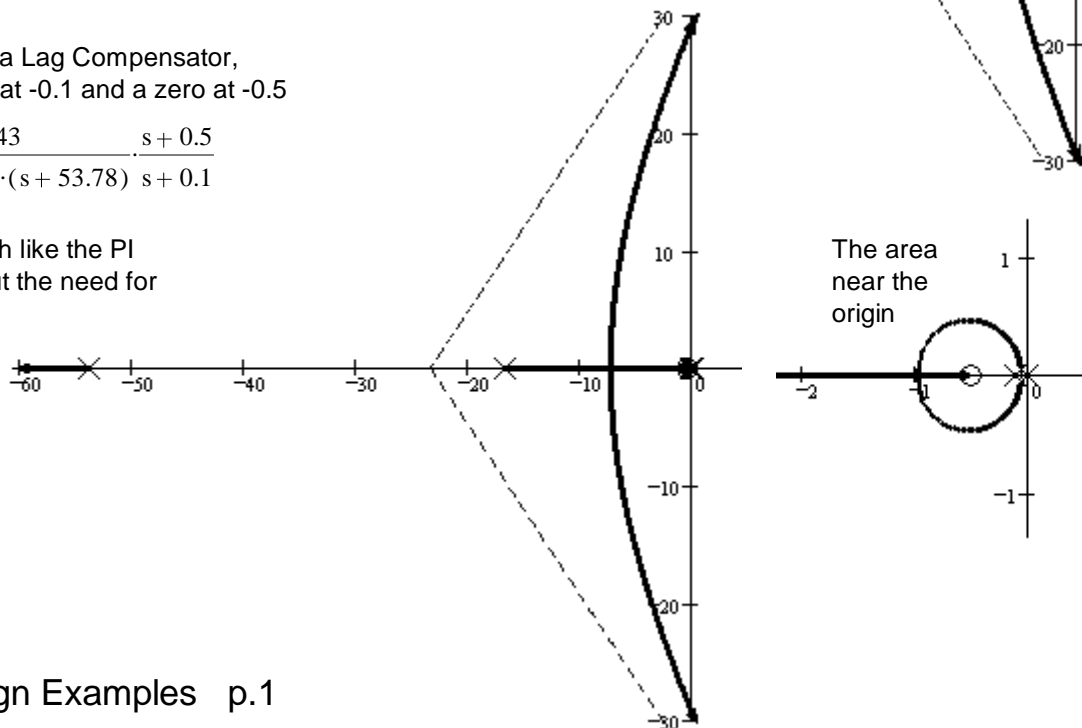
$$C(s) = k_p + \frac{k_i}{s} = k_p \cdot \frac{s + \frac{k_i}{k_p}}{s}$$



LAG An alternative is a Lag Compensator, here with a pole at -0.1 and a zero at -0.5

$$G(s) = \frac{1643}{s \cdot (s + 16.64) \cdot (s + 53.78)} \cdot \frac{s + 0.5}{s + 0.1}$$

This works very much like the PI controller, but without the need for active components.



Root Locus Design Example p.2

Let's keep the pole at 0 and zero at -0.1 for elimination of steady-state errors and rejection of disturbances

$$\text{CL poles at } p := -7.06 + 7.06j \quad k = \frac{1}{|G(-7.06 + 7.06j)|} = 3.441$$

$$\text{and } -7.06 - 7.06j$$

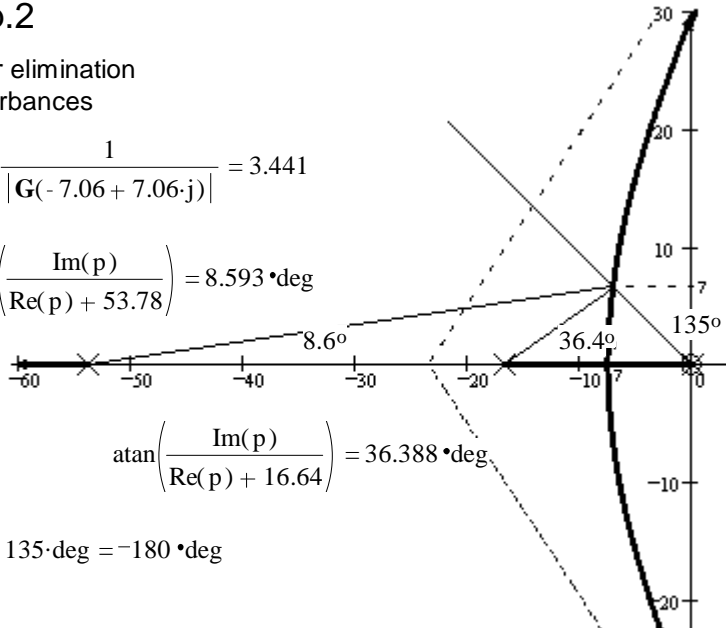
At gain of 3.44

$$\text{atan}\left(\frac{\text{Im}(p)}{\text{Re}(p) + 53.78}\right) = 8.593 \cdot \text{deg}$$

$$\text{atan}\left(\frac{\text{Im}(p)}{\text{Re}(p) + 16.64}\right) = 36.388 \cdot \text{deg}$$

This is a point in the root locus because:

$$-8.6 \cdot \text{deg} - 36.4 \cdot \text{deg} - 135 \cdot \text{deg} - 135 \cdot \text{deg} + 135 \cdot \text{deg} = -180 \cdot \text{deg}$$



PD or PID To Improve the dynamic response

Want to double the speed

Want poles to move to: $p := -14 + 14j$
 $-14 - 14j$

$$\text{atan}\left(\frac{\text{Im}(p)}{\text{Re}(p) + 53.78}\right) = 19.389 \cdot \text{deg}$$

$$\text{atan}\left(\frac{\text{Im}(p)}{\text{Re}(p) + 16.64}\right) = 79.321 \cdot \text{deg}$$

Unfortunately, this point is NOT on the root locus

$$-\text{atan}\left(\frac{\text{Im}(p)}{\text{Re}(p) + 53.78}\right) - \text{atan}\left(\frac{\text{Im}(p)}{\text{Re}(p) + 16.64}\right) - 135 \cdot \text{deg} = -233.71 \cdot \text{deg}$$

Maybe we could add a zero so that its angle is:

$$\theta_z := 233.71 \cdot \text{deg} - 180 \cdot \text{deg} \quad \theta_z = 53.71 \cdot \text{deg}$$

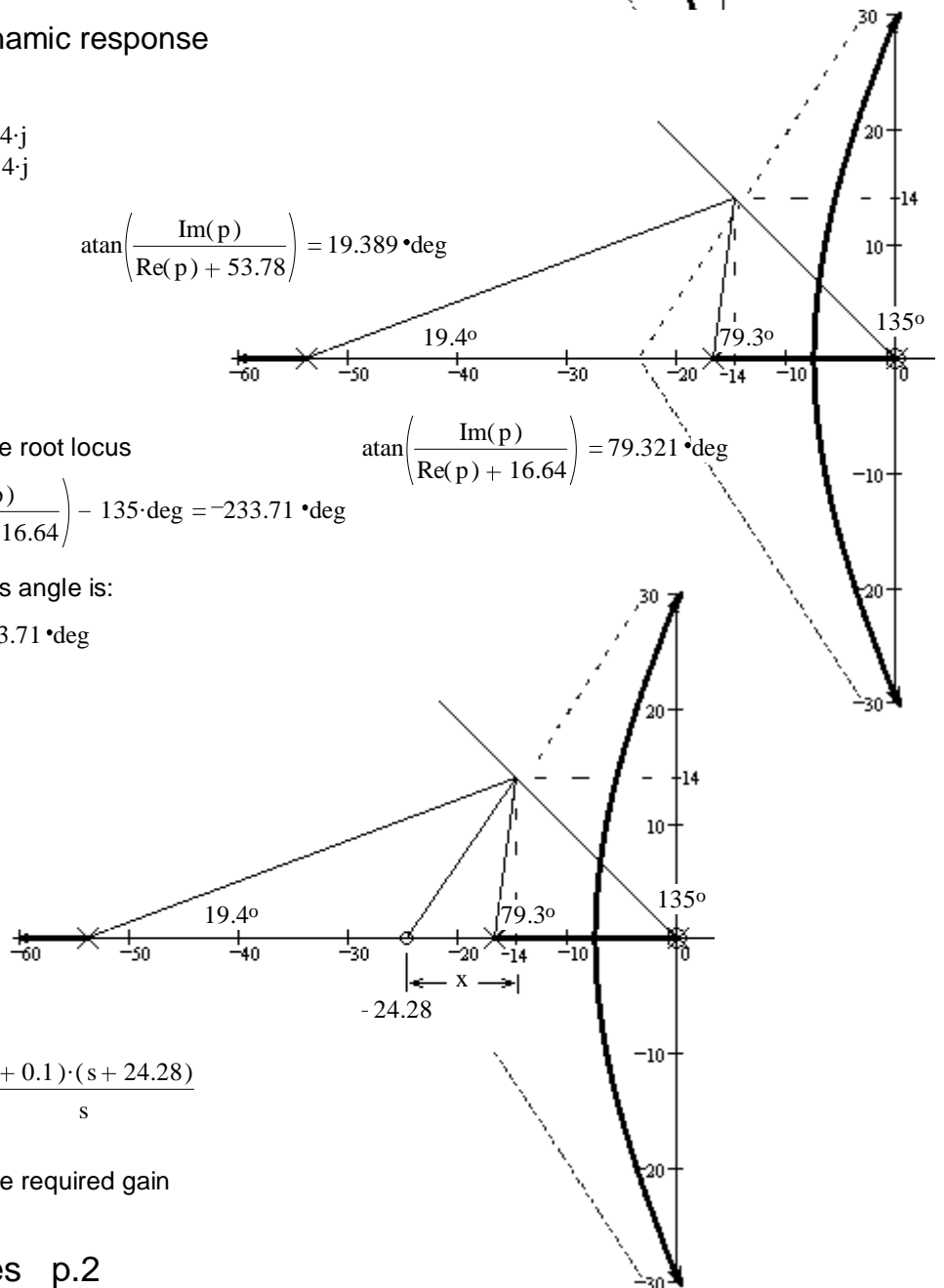
$$x = \text{Im}(p) \cdot \frac{1}{\tan(\theta_z)} = 10.28$$

$$z := \text{Re}(p) - \text{Im}(p) \cdot \frac{1}{\tan(\theta_z)}$$

$$z = -24.28$$

$$G(s) := \frac{1643}{s \cdot (s + 16.64) \cdot (s + 53.78)} \cdot \frac{(s + 0.1) \cdot (s + 24.28)}{s}$$

$$k = \frac{1}{|G(-14 + 14j)|} = 0.418 \text{ is the required gain}$$



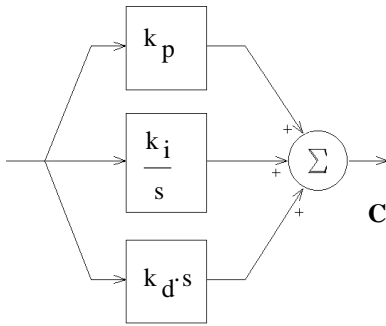
Root Locus Design Examples p.2

Root Locus Design Examples p.3

We have designed a our compensation with the following:

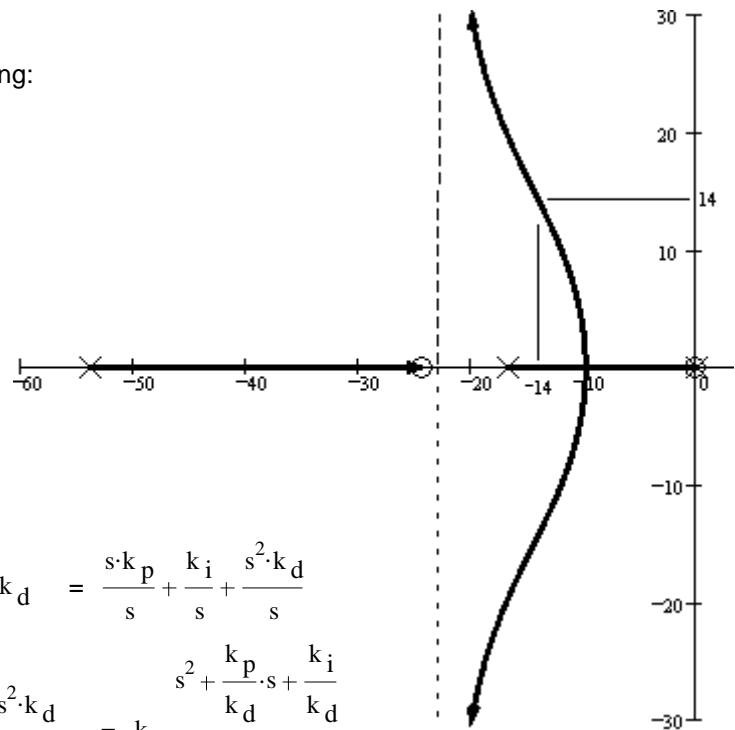
- A pole at the origin
- A zero at -0.1
- A zero at -24.28
- Gain of 0.418

Find the k_p , k_i , & k_d of a PID controller.



$$C(s) = k_p + \frac{k_i}{s} + s \cdot k_d = \frac{s \cdot k_p}{s} + \frac{k_i}{s} + \frac{s^2 \cdot k_d}{s}$$

$$= \frac{s \cdot k_p + k_i + s^2 \cdot k_d}{s} = k_d \cdot \frac{s^2 + \frac{k_p}{k_d} \cdot s + \frac{k_i}{k_d}}{s}$$



gain = $k_d := 0.418$

$$(s + 0.1) \cdot (s + 24.28) = s^2 + 24.38 \cdot s + 2.43$$

$$= s^2 + \frac{k_p}{k_d} \cdot s + \frac{k_i}{k_d}$$

$$\frac{k_i}{k_d} = 2.43 \quad k_i := k_d \cdot 2.43 \quad k_i = 1.016$$

$$\frac{k_p}{k_d} = 24.38 \quad k_p := k_d \cdot 24.38 \quad k_p = 10.191$$

Notice: $\frac{k_i}{k_p} = 0.1 \approx 0.1$

Notice that the proportional gain is actually almost 3 times higher than it was before. $3 \cdot 3.44 = 10.32$

LEAD An alternative to the differentiator is a Lead Compensator.

Instead of a single zero with: $\theta_z = 53.71 \cdot \text{deg}$

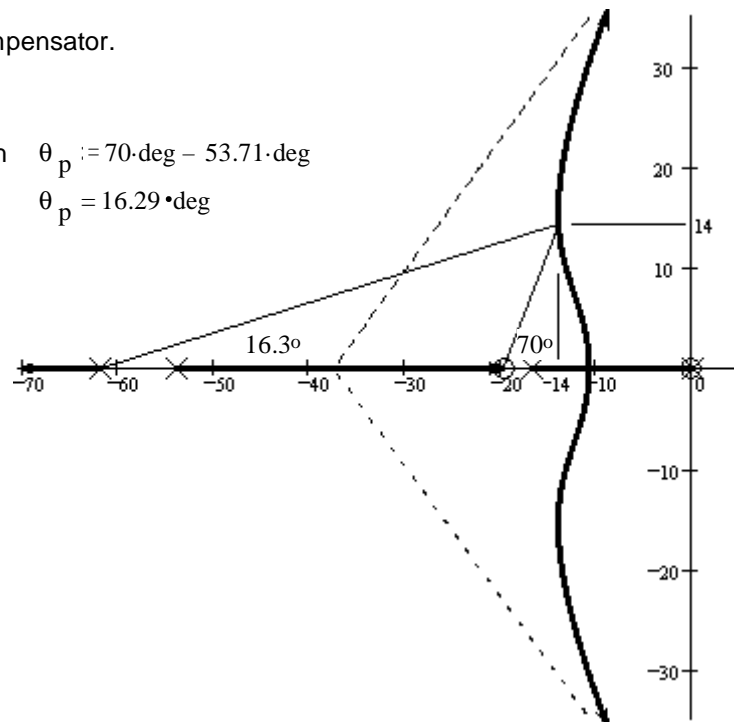
How about a zero with $\theta_z := 70 \cdot \text{deg}$ And a pole with $\theta_p := 70 \cdot \text{deg} - 53.71 \cdot \text{deg}$

$$x = \text{Im}(p) \cdot \frac{1}{\tan(\theta_z)} = 5.096$$

$$z := \text{Re}(p) - \text{Im}(p) \cdot \frac{1}{\tan(\theta_z)} \quad z = -19.096$$

$$x_p = \text{Im}(p) \cdot \frac{1}{\tan(\theta_p)} = 47.907$$

$$p := \text{Re}(p) - \text{Im}(p) \cdot \frac{1}{\tan(\theta_p)} \quad p = -61.907$$



This example is actually a PI-Lead controller

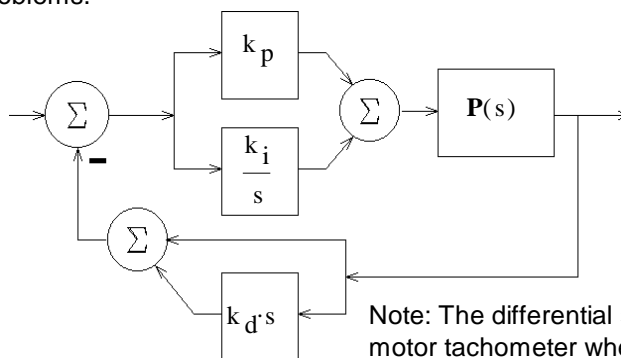
Problems with the differentiator

1. Tries to differentiate a step input into an impulse -- not likely.
 You'll have to consider how your differentiator will actually handle a step input and how your amplifier will saturate.
 If the differentiator and amplifiers saturate in such a way the the "area under the curve" approximates the impulse "area under the curve", then this may not be such a problem. It may not be as fast as predicted from the linear model, but it may be as fast as the system limits allow. (Pedal-to-the-metal.)
2. It's a high-pass filter and can accentuate noise.
 This is actually common to all compensators that speed up the response.
3. Requires active components and a power supply to build.
 Usually no big deal since your amplifier (source of gain) does too.
4. Is never perfect (always has higher-order poles), but then neither is anything else. Especially in mechanical systems, these poles usually are well beyond where they could cause problems.

Alternatives:

1. Lag-Lead or PI-Lead compensation. This eliminates the differentiator, but it is still a high-pass filter that can be a noise problem and it could still saturate the amplifier if the input changes too rapidly.
 Be sure to check for saturation problems.
2. Place the differentiator in the feedback loop. The output of the plant is much less likely to be a step or to change so rapidly that it causes problems.

Differentiation in the feedback



Find the k_p , k_i , & k_d of this controller.

Note: The differential signal is often taken from a motor tachometer when the output is a position. Then you don't need a separate differentiator circuit, just a separate gain for that signal.

$$\begin{aligned}
 \mathbf{F(s) \cdot C(s)} &= \left(k_p + \frac{k_i}{s} \right) \cdot (1 + k_d \cdot s) = k_p \cdot k_d \cdot \left[\frac{s + \frac{k_i}{k_p}}{s} \right] \cdot \left(\frac{1}{k_d} + s \right) = k_p \cdot k_d \cdot \frac{\left(s + \frac{k_i}{k_p} \right) \cdot \left(s + \frac{1}{k_d} \right)}{s} \\
 \mathbf{C(s)} & \quad \mathbf{F(s)} \\
 \text{For our example:} & \quad = 0.418 \cdot \frac{(s + 0.1) \cdot (s + 24.28)}{s} \\
 k_d & := \frac{1}{24.38} \quad k_d = 0.041 \\
 k_p & := \frac{0.418}{k_d} \quad k_p = 10.191 \\
 k_i & := k_p \cdot 0.1 \quad k_i = 1.019
 \end{aligned}$$

In this case the open-loop zero in the feedback loop IS NOT in the closed-loop. This turns out to make the step response slower than predicted by the second-order approximation, but try a simulation, you may be able to use significantly more gain with no more overshoot. The differentiator in this position inhibits overshoot.

PI and PID Design Examples p.5

Ex.2, from S16 Exam 3 Consider the transfer function:

$$G(s) := \frac{s + 5}{(s + 1) \cdot (s^2 + 4 \cdot s + 20)}$$

a) Find the departure angle from a complex pole.

Angles:
 from pole at -1 $\theta_{p1} := 180 \cdot \text{deg} - \text{atan}\left(\frac{4}{1}\right)$ $\theta_{p1} = 104.036 \cdot \text{deg}$

from pole at -2-4j $\theta_{p2} := 90 \cdot \text{deg}$ $\theta_{p2} = 90 \cdot \text{deg}$

from zero at -5 $\theta_z := \text{atan}\left(\frac{4}{3}\right)$ $\theta_z = 53.13 \cdot \text{deg}$

$$\theta = 53.13 \cdot \text{deg} - 90 \cdot \text{deg} - 104.036 \cdot \text{deg} + 180 \cdot \text{deg} = 39.094 \cdot \text{deg}$$

b) Draw a root locus plot. Calculate the centroid and accurately draw the departure angle.

$$\sigma := \frac{5 - 1 - 2 - 2}{2} \quad \sigma = 0$$

c) Is there any decent place to locate the closed-loop poles? NO

d) You would like to place your closed-loop poles to get a settling time of 1/2 sec and 0.656% overshoot. Add the simplest possible compensator to accomplish this and calculate what the compensator should be.

2% settling time: $T_s = \frac{4}{a}$ $a = \frac{4}{\left(\frac{1}{2}\right)} = 8$

Overshoot: $OS = e^{-\frac{\pi \cdot a}{b}}$ $\%OS = 100\% \cdot e^{-\frac{\pi \cdot a}{b}}$

$$\frac{a}{b} = \frac{\ln(OS)}{-\pi} = \frac{\ln(0.00656)}{-\pi} = 1.6 \quad b = \frac{8}{1.6} = 5$$

Pole should be at -8 + 5j

Angles:

from pole at -1 $180 \cdot \text{deg} - \text{atan}\left(\frac{5}{7}\right) = 144.462 \cdot \text{deg}$

from pole at -2+4j $180 \cdot \text{deg} - \text{atan}\left(\frac{1}{6}\right) = 170.538 \cdot \text{deg}$

from pole at -2-4j $180 \cdot \text{deg} - \text{atan}\left(\frac{9}{6}\right) = 123.69 \cdot \text{deg}$

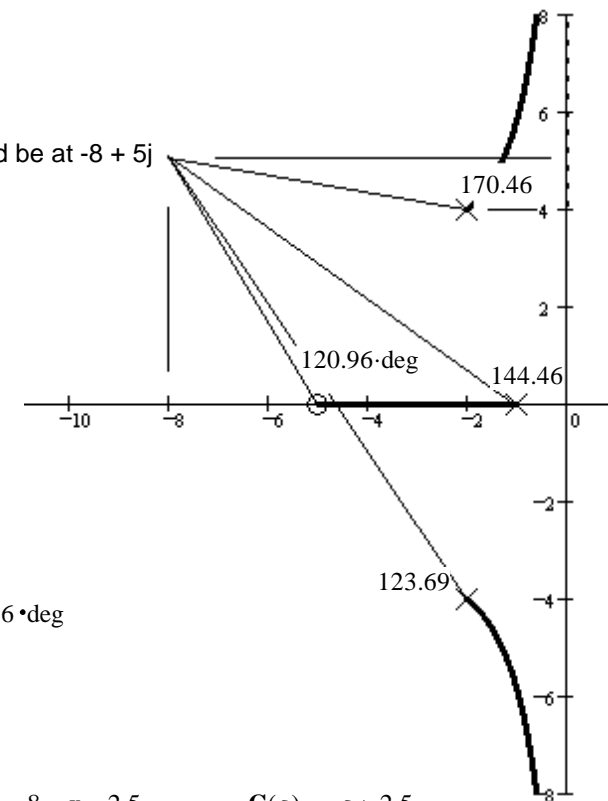
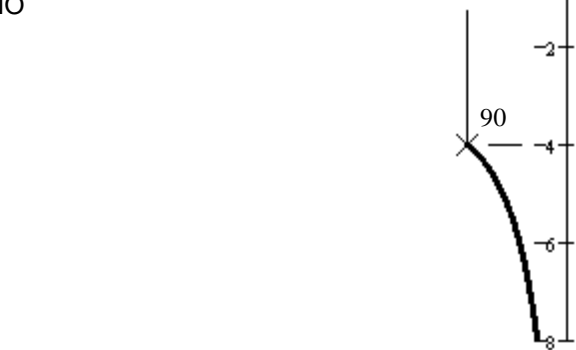
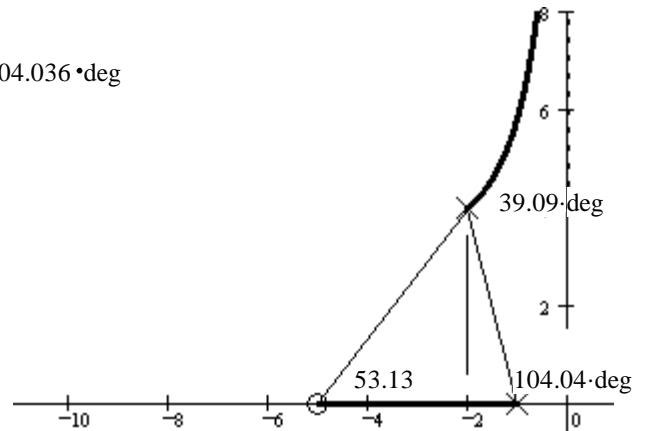
from zero at -5 $180 \cdot \text{deg} - \text{atan}\left(\frac{5}{3}\right) = 120.964 \cdot \text{deg}$

$$144.462 \cdot \text{deg} + 170.538 \cdot \text{deg} + 123.69 \cdot \text{deg} - 120.964 \cdot \text{deg} = 317.726 \cdot \text{deg}$$

$$\theta_z := 317.726 \cdot \text{deg} - 180 \cdot \text{deg} \quad \theta_z = 137.726 \cdot \text{deg}$$

$$\tan(137.726 \cdot \text{deg} - 90 \cdot \text{deg}) = 1.1 = \frac{x}{5} \quad x := 5 \cdot 1.1 \quad 8 - x = 2.5 \quad C(s) = s + 2.5$$

$$G_c(s) := \frac{(s + 5) \cdot (s + 2.5)}{(s + 1) \cdot (s^2 + 4 \cdot s + 20)} \quad s := -8 + 5 \cdot j \quad \text{Check: } \arg\left[\frac{(s + 5) \cdot (s + 2.5)}{(s + 1) \cdot (s^2 + 4 \cdot s + 20)}\right] = 180 \cdot \text{deg}$$



PI and PID Design Examples p.6

e) What is the gain? $k := \frac{1}{|G_c(s)|} = \left| \frac{(-8 + 5j + 1) \cdot [(-8 + 5j)^2 + 4 \cdot (-8 + 5j) + 20]}{(-8 + 5j + 5) \cdot (-8 + 5j + 2.5)} \right| = 13.059$

f) What is the steady-state error for a unit-step input?

$$G_c(s) := \frac{(s+5) \cdot (s+2.5)}{(s+1) \cdot (s^2 + 4s + 20)} \quad G_c(0) = \frac{(0+5) \cdot (0+2.5)}{(0+1) \cdot (0^2 + 4 \cdot 0 + 20)} = \frac{(5) \cdot (2.5)}{(1) \cdot (20)} = 0.625$$

$$G_c(0) = 0.625 \quad e_{\text{step}} = \frac{1}{1 + k \cdot 0.625} = 10.91\%$$

g) If this steady-state error was a little too big, what would be the very simplest way to reduce it?

turn up the gain

Ex.3, from S17 Exam 3

a) Sketch the root locus plot of, $G(s) := \frac{100}{(s+25) \cdot (s+40) \cdot (s+70)}$ $m := 0$
 $n := 3$

$$\sigma_C = \frac{-25 + -40 + -70}{n - m} = -45 \quad n - m = 3 \quad \text{so asymptotes are at } \pm 60^\circ \text{ \& } 180^\circ$$

The gain is set at 452, so that one of the closed-loop poles is at, $s := -24.48 + 27.2j$

Further calculations yield:

Settling time: 0.163·sec

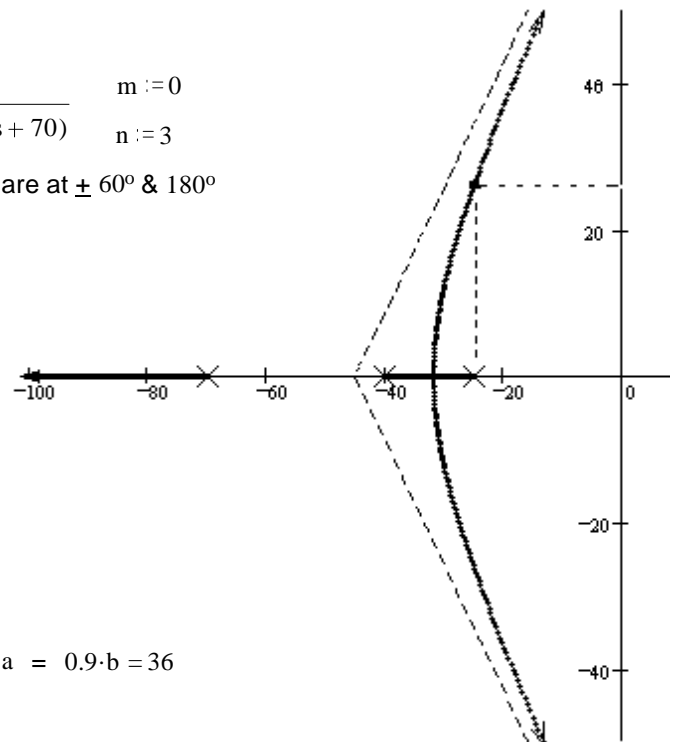
% overshoot: 5.92·%

Steady-state error to a unit-step input: 60.8%

b) You wish to increase the frequency of ringing to 40 rad/sec without changing the % overshoot at all. Where should the closed-loop pole be located?

$$\frac{a}{b} = \frac{24.48}{27.2} = 0.9 \quad \text{new } b := 40 \quad \text{new } a = 0.9 \cdot b = 36$$

New location: $s := -36 + 40j$



c) Add a LEAD compensator so that you will be able to place the closed-loop pole at the location found in b). Add the new zero at -30. Find the location of the new pole.

Angles:

from pole at -25

$$\theta_{25} := 180 \cdot \text{deg} - \text{atan}\left(\frac{40}{36 - 25}\right) \quad \theta_{25} = 105.376 \cdot \text{deg}$$

from pole at -40

$$\theta_{40} := \text{atan}\left(\frac{40}{40 - 36}\right) \quad \theta_{40} = 84.289 \cdot \text{deg}$$

from pole at -70

$$\theta_{70} := \text{atan}\left(\frac{40}{70 - 36}\right) \quad \theta_{70} = 49.635 \cdot \text{deg}$$

from new zero at -30

$$\theta_{30} := 180 \cdot \text{deg} - \text{atan}\left(\frac{40}{36 - 30}\right) \quad \theta_{30} = 98.531 \cdot \text{deg}$$

$$\theta_{25} + \theta_{40} + \theta_{70} - \theta_{30} + \theta_p = 180 \cdot \text{deg}$$

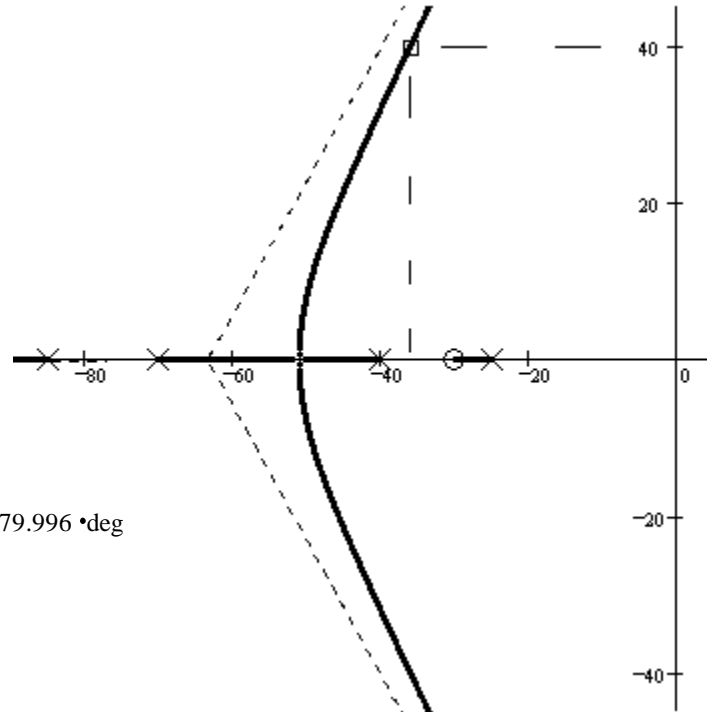
$$\theta_p := 180 \cdot \text{deg} - \theta_{25} - \theta_{40} - \theta_{70} + \theta_{30}$$

$$\theta_p = 39.23 \cdot \text{deg}$$

$$p := 36 + \frac{40}{\tan(\theta_p)} \quad p = 84.993 = 85$$

$$G_c(s) := \frac{100 \cdot (s + 30)}{(s + 25) \cdot (s + 40) \cdot (s + 70) \cdot (s + 85)}$$

$$\text{Check: } \arg \left[\frac{100 \cdot (s + 30)}{(s + 25) \cdot (s + 40) \cdot (s + 70) \cdot (s + 85)} \right] = -179.996 \cdot \text{deg}$$



d) With the compensator in place and a closed-loop pole at the location desired in part b)

i) What is the gain? $k := \frac{1}{|G_c(s)|} \quad k = 1369$

ii) What is the 2% settling time? Use the second-order approximation.

$$T_s = \frac{4}{36} = 0.111 \text{ sec}$$

iii) What is the steady-state error to a unit-step input?

$$G_c(0) = \frac{100 \cdot (0 + 30)}{(0 + 25) \cdot (0 + 40) \cdot (0 + 70) \cdot (0 + 85)} = 5.042 \cdot 10^{-4} \quad e_{\text{step}} = \frac{1}{1 + k \cdot G_c(0)} = 59.161 \cdot \%$$

e) Add another compensator: $C_2(s) := \frac{s + 2}{s}$ and maintain the gain of part d)

i) What is this type of compensator called and what is its purpose?

PI, used to eliminate steady-state error

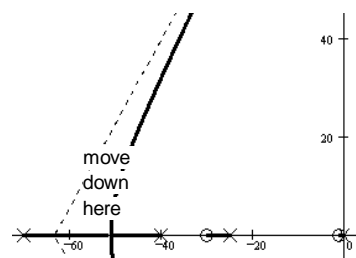
ii) Calculate what you need to show that this compensator achieved its purpose.

$$G_c(s) := \frac{100 \cdot (s + 30)}{(s + 25) \cdot (s + 40) \cdot (s + 70) \cdot (s + 85)} \cdot \frac{(s + 2)}{s}$$

$$G_c(0) = \infty \quad e_{\text{step}} = \frac{1}{1 + k \cdot \infty} = 0 \cdot \%$$

f) With both compensators in place, is there possibility for improvement (quicker settling time speed and/or lower ringing)? If yes, what would be the simplest thing to do? Justify your answer.

A quick sketch of the new root-locus shows that simply decreasing the gain would improve the system



Using 2nd-order approximation: $\frac{N(s)}{(s+a)^2 + b^2} = \frac{N(s)}{s^2 + 2 \cdot a \cdot s + a^2 + b^2} = \frac{N(s)}{s^2 + 2 \cdot \zeta \cdot \omega_n \cdot s + \omega_n^2}$

$\omega_n^2 = a^2 + b^2$ $\omega_n = \text{natural frequency}$

$\zeta \cdot \omega_n = a$

$\zeta = \frac{a}{\omega_n} = \frac{a}{\sqrt{a^2 + b^2}} = \text{damping factor}$ $\zeta = \sin\left(\text{atan}\left(\frac{a}{b}\right)\right)$

Overshoot: $OS = e^{-\frac{\pi \cdot a}{b}}$ $\%OS = 100\% \cdot e^{-\frac{\pi \cdot a}{b}}$ $\frac{a}{b} = \frac{\ln(OS)}{-\pi}$

angle of constant damping line: $90 \cdot \text{deg} + \text{atan}\left(\frac{a}{b}\right)$

2% settling time: $T_s = \frac{4}{a} = \frac{4}{\zeta \cdot \omega_n}$

Time of first peak: $T_p = \frac{\pi}{b}$

Static error constant (position): $K_p = \lim_{s \rightarrow 0} K \cdot C(s) \cdot G(s)$ $e_{\text{step}(\infty)} = e_{\text{step}} = \frac{1}{1 + K_p}$ Nise p378

Lag compensation improves K_p, K_v and K_a by $\frac{z_c}{p_c}$ IE: $K_{pc} \approx K_{puc} \cdot \frac{z_c}{p_c}$

Searching along a line of constant damping:

Try s values, choosing b: $s = -\left(\frac{a}{b} \cdot b\right) + b \cdot j$ Test: $\arg(G(s)) \pm 180^\circ$ or $-\text{Re}(G(s)) \gg \text{Im}(G(s))$

Linear interpolation: new b = $b_1 - \frac{b_2 - b_1}{\text{Im}(G(s_2)) - \text{Im}(G(s_1))} \cdot \text{Im}(G(s_1))$

Can also try "a" values with slight modification of the above.

Weird forms from Nise book:

$\sigma_d = a$ $\%OS = 100\% \cdot e^{\frac{-\zeta \cdot \pi}{\sqrt{1 - \zeta^2}}}$ p195

$\omega_d = b$ $\zeta = \frac{-\ln(OS)}{\sqrt{\pi^2 + (\ln(OS))^2}}$ p195 $T_p = \frac{\pi}{\omega_n \cdot \sqrt{1 - \zeta^2}}$ p194

p378 Static error constant (ramp): $K_v = \lim_{s \rightarrow 0} s \cdot K \cdot C(s) \cdot G(s)$ $e_{\text{ramp}} = \frac{1}{K_v}$
(velocity)

Static error constant (parabola): $K_a = \lim_{s \rightarrow 0} s^2 \cdot K \cdot C(s) \cdot G(s)$ $e_{\text{parabola}} = \frac{1}{K_a}$
(acceleration)

1. Problem 4.5 (p.119) in the Bodson text.

a) Sketch (by hand) the root-locus plot for the following open-loop transfer function:

Apply only the main rules (Section 4.4.2 in text or the first page of class notes)

$$G(s) = \frac{s \cdot (s + 1)}{(s + 2)^2 \cdot (s + 3)}$$

b) Repeat part a) for: $G(s) = \frac{(s + 3)}{s \cdot (s + 9)^3}$

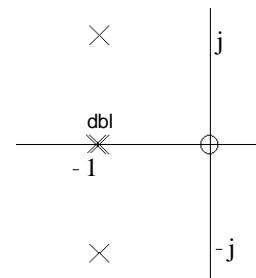
c) Repeat part a) for: $G(s) = \frac{(s + a)}{(s + b) \cdot (s^2 - 2 \cdot s + 2)}$ $a > 0$ $b > 0$ $k > 0$
 $a, b, \& k$ are all positive, real numbers

Also give condition(s) that a and b must satisfy for the closed-loop system to be stable for sufficiently high gain (k) (note that you do not need to apply the Routh-Hurwitz criterion, nor provide the range of k for which the system is closed-loop stable).

2. Problem 4.12 in the Bodson text.

Sketch the root-locus for the open-loop poles shown at right, using only the main rules.

There is a zero at $s = 0$, two poles at $s = -1$ and two poles at $s = -1 \pm j$.



The following review questions and problems come from the Nise 3rd Ed., starting on page 471, Or 4th Ed., starting p 473. If you are using the 4th Ed., clearly state that on your homework.

3. Nise, Ch.8, review question 3, rephrased here: If $G(s_1) = 5 / \underline{180}^\circ$, is the point s_1 on the root locus?
If yes, what gain factor would place a closed-loop pole at s_1 ?
4. Nise, Ch.8, review questions 4, 6, 7, 8, 9, 10.
5. Nise, Ch.8, problem 3
6. a) Find the break-in point for Nise, problem 3a, above. Note, the math here may drive you nuts, but you may simply test and prove that a point you guess is correct.
b) Find the break-away point for Nise, problem 3d, above

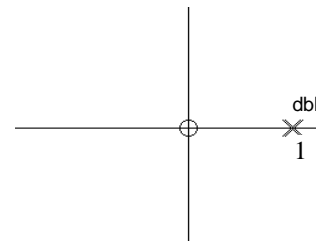
7. Problem 4.11 in the Bodson text. (Hint: do part c before b)

a) Sketch the root-locus for the open-loop poles shown at right.

There is one zero at $s = 0$ and two poles at $s = 1$.

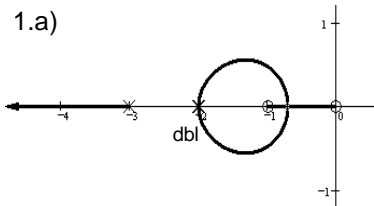
c) Give the location(s) of the break-away point(s) (or arrival) on the real axis.

b) Give the range of gain k ($k > 0$) for which the system is closed-loop stable, and give the locations of the $j\omega$ axis crossings.

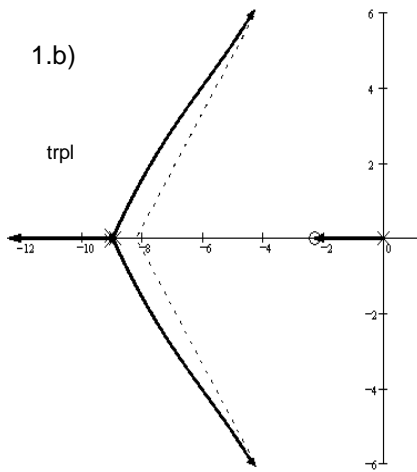


Answers

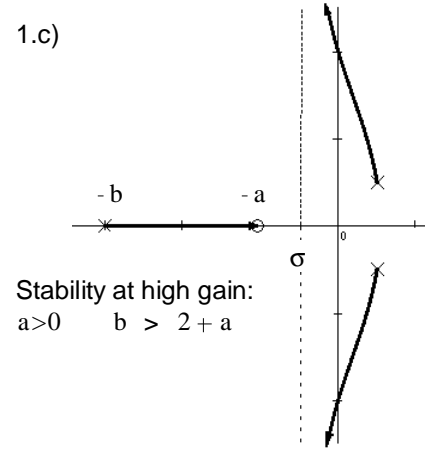
1.a)



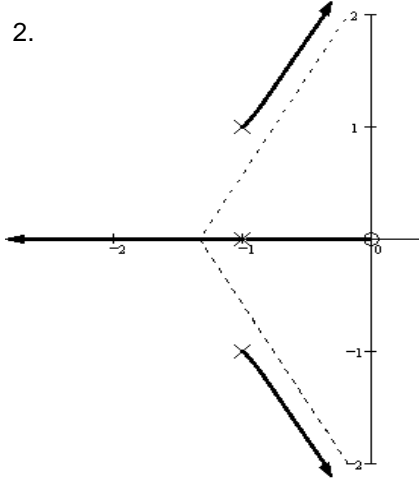
1.b)



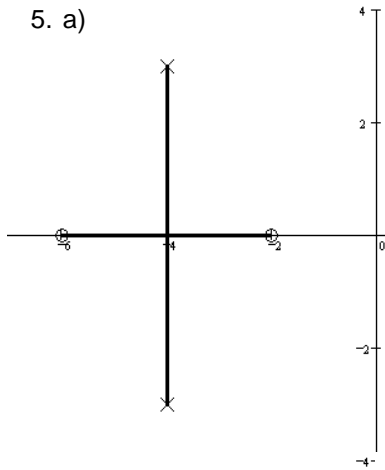
1.c)



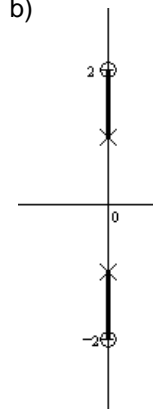
2.



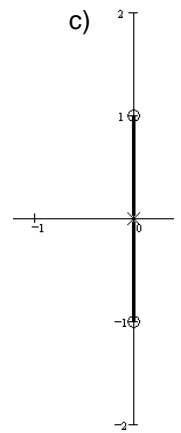
5. a)



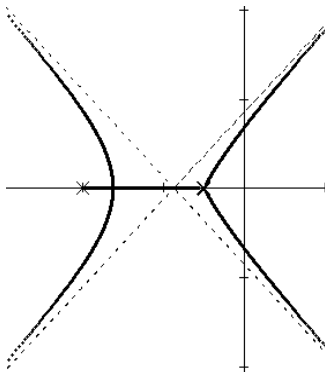
b)



c)



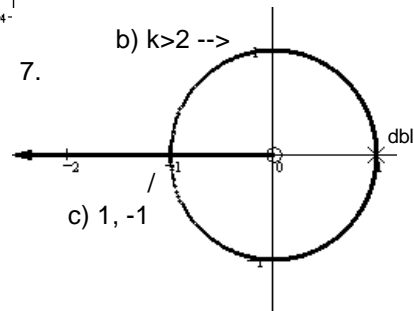
5. d)



6. a) -4

b) -3.25

b) $k > 2 \rightarrow$



c) 1, -1

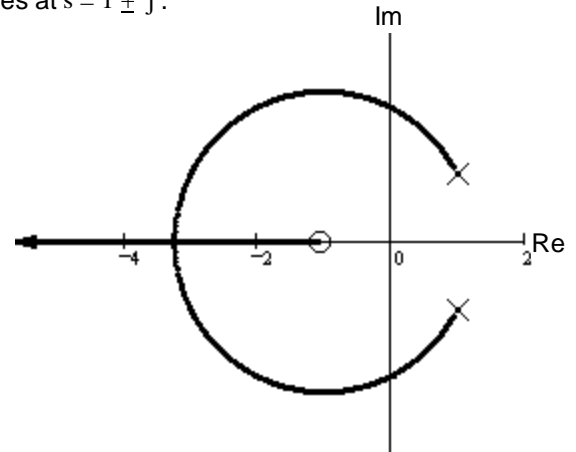
ANSWERS TO REVIEW QUESTIONS

- The plot of a system's closed-loop poles as a function of gain
- (1) Finding the closed-loop transfer function, substituting a range of gains into the denominator, and factoring the denominator for each value of gain.
(2) Search on the s-plane for points that yield 180 degrees when using the open-loop poles and zeros.
3. Yes, $K = 1/5$
4. No
5. At the zeros of $G(s)$ and the poles of $H(s)$
- (1) Apply Routh-Hurwitz to the closed-loop transfer function's denominator. (2) Search along the imaginary axis for $\angle G(s) = \pm 180^\circ$. (3) Use a computer with something like Matlab SISO tool to find the crossover point(s). (only need 2 ans)
- If any branch of the root locus is in the rhp, the system may be unstable.
If the gain places one of the closed-loop poles on that part of the branch, it will be unstable.
- If the branch of the root locus is vertical, the settling time remains constant for that range of gain on the vertical section.
- The natural frequency is the distance of a pole from the origin. If a region of the root locus is circular and the center of the circle is at the origin, then the natural frequency would not change over that region of gain.
- Determine if there are any break-in or breakaway points
- (1) Poles must be at least five times further from the imaginary axis than the dominant second order pair,
(2) Zeros must be nearly canceled by higher order poles.
- Number of branches, symmetry, starting and ending points
- The zeros of the open loop system help determine the root locus. The root locus ends at the zeros.
Thus, the zeros are the closed-loop poles for high gain.

1. A root - locus is sketched at right.

The open - loop transfer function has one zero at $s = -1$ and two poles at $s = 1 \pm j$.

$$G(s) = \frac{s+1}{s^2 - 2s + 2} = \frac{s+1}{(s-1-j)(s-1+j)}$$



a) Find the departure angle from the complex pole $1 + j$.

b) It looks like the root-locus crosses the $j\omega$ axis at 2

Determine if this is true.

Clearly show your work, guesses don't count.

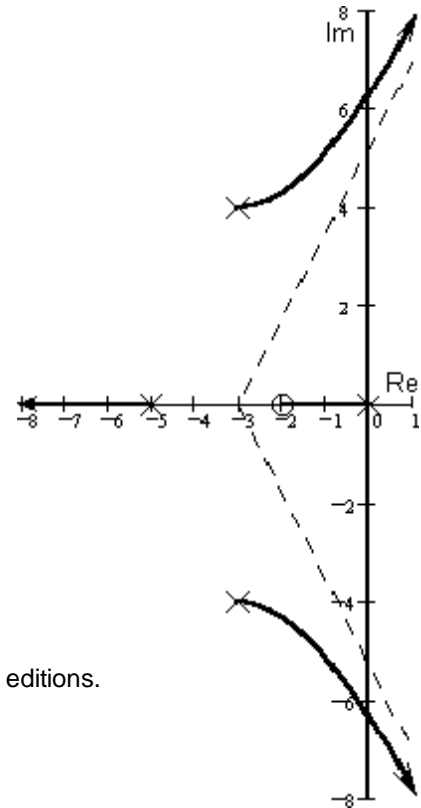
c) Regardless of what you found in part b), continue to assume that the root-locus crosses the $j\omega$ axis at 2.

Give the range of gain k for which the system is closed-loop stable.

2. A root - locus is sketched at right.

$$G(s) = \frac{3 \cdot (s+2)}{s \cdot (s+5) \cdot (s^2 + 6s + 25)}$$

Find the departure angle from the complex pole $-3 + 4j$.



3. Problem 4.13 in the Bodson text.

4. a) Nise, Ch.8, problem 4. Note: the answers are different for the 3rd & 4th editions.

(6th edition p.6 is like 4th ed. p.4)

b) Also find the point where the root locus crosses the imaginary axis.

c) Find the range of gain for which the system is "stable".

d) Find the arrival angle at the top zero (departure of top pole in 4th Ed.).

Answers

1. a) 117·deg

b) YES

c) $k > 2$

2. 3.73·deg

3. 206.6·deg

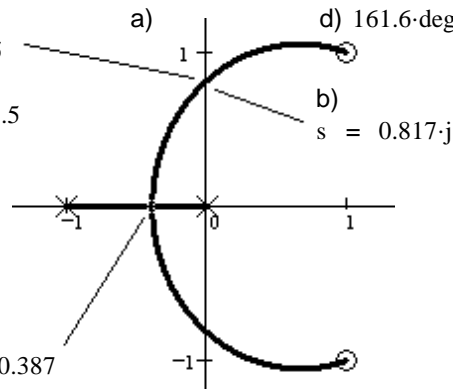
45·deg

4. 3rd Ed.

$k = 0.5$

c) $k < 0.5$

a) -0.387



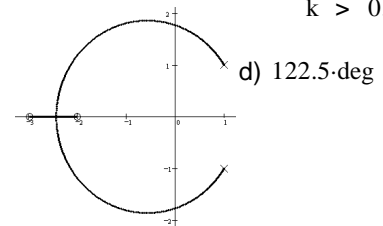
4. 4th Ed.

breakaway: -2.434

crossing: $s = 1.773·j$

$k = 0.4$

$k > 0.4$



These problems should be done using MATLAB or some other program that creates root-locus plots. Refer to a help sheet attached to this homework to get started. You will need to compare shapes of root-locus (RL) plots and speeds of step responses. Unless you manually set your plot limits, Matlab will automatically scale on your plots. This can make it very hard to compare them or to determine the "45°" line.

1. a) Homework RL2 problem 1c. $G(s) := \frac{1}{s \cdot (s+2) \cdot (s+4)}$ Experiment with moving the pole at -4.
- Describe what happens when you move it to the left.
 - Describe what happens when you move it to the right, especially when you move it right of -2. Put the pole back at -4.
- b) Describe what happens when you add a fourth pole left of the $j\omega$ axis. Try different positions. Erase the added pole.
- c) Describe what happens when you add a zero left of the $j\omega$ axis. Try different positions.

For the remaining problems:

Print one or more plots for **each** problem. Each plot should be labeled clearly on the same page as the plot. If you added poles and/or zeros (a compensator) make sure it is clear which one(s) were added or what your $C(s)$ is. Plot of the root locus should always show the point(s) on the root locus you determined to be the "best" position for the CL pole(s) with squares (like Matlab does). Indicate the gain (k) used. "Best" = Fastest step response to get to and stay within 4% of the desired final response (usually 1 for step response). Limit overshoot to about 4%. To determine speed information, refer to the step response curve. Prints of those curves are nice, but not required. Be careful, if Matlab changes the scales on your plots it can be very hard to compare them.

2. a) Homework RL2 problem 1c (see above for $G(s)$). Add a compensator to your system. This compensator will add one pole and one zero to the open-loop (OL) transfer function. The new pole must lie somewhere between -10 and +2, you choose where. Same goes for the zero. Choose the locations of your added OL pole and OL zero and gain (k) to get the "best" response.

b) Homework RL2 problem 1d. $G(s) := \frac{s+7}{s \cdot (s+2) \cdot (s+4)}$ Just find the "best" gain and print an RL plot.

- c) Homework RL2 problem 1d. Repeat part a) above for this system, only this time your added pole and zero are limited to -16 to +2.

3. a) Homework RL3 problem 1a $G(s) = \frac{s \cdot (s+1)}{(s+2)^2 \cdot (s+3)}$ Just print an RL plot. Note that the step response goes to zero, why?

b) Homework RL3 problem 1b. $G(s) = \frac{(s+3)}{s \cdot (s+9)^3}$ Just find the "best" gain and print an RL plot.

- c) Homework RL3 problem 1b. Add a compensator. Your compensator may have up to 2 poles (0, 1, or 2) and they may be complex. Same for the zeros. All must lie between -20 and +2 and $-12j$ and $+12j$. Choose the best possible poles and/or zeros, find the best gain and plot like you did for 1b.

4. Homework RL4 problem 1. $G(s) = \frac{s+1}{s^2 - 2 \cdot s + 2}$

Confirm the departure angles (With a small gain, place a CL pole very close to $1 + 1j$ and figure out the angle from its location.). Find the true Im-axis crossing point and the gain at that point. Find the break in point.

5. a) Enter in the plant $G(s)$ of the crude servo: $G(s) = \frac{1643}{s \cdot (s+16.64) \cdot (s+53.78)}$
Add a PI compensator (a pole at 0 and a zero at -0.1).

Find the gain for a 0.707 damping factor. Record the time it takes the step response to reach 1 the first time. Save this: Compensators → Store/Retrieve etc..

- b) Add a zero at -16.64 and repeat, saving under a different name.
- c) Adjust the zero location further to the right while pushing the gain and watching the step response to find an even better combination. Again record the time it takes the step response to reach 1 the first time and save. Print the Root locus **and** the step response for this case.
- d) You will be using this device to build widgets. The time per widget is 10 times the time you recorded in each case. Your profit per widget is 1 cent. Your fixed costs per 8-hour day is \$250. Describe the results to your company of each of the 3 compensators above.

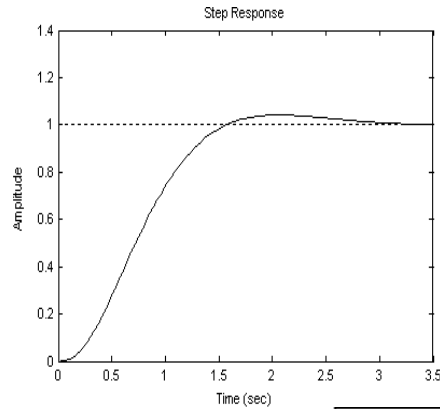
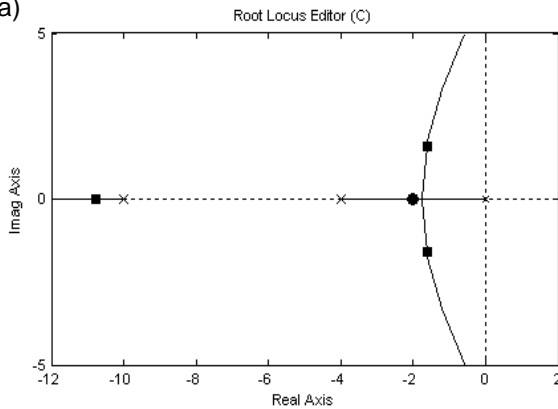
ECE 3510 homework RL5 p2

6. Create the most interesting root locus plot that you can with no more than 10 poles and 10 zeros. Have some fun with this. You can go over 10 poles and zeros if you want, but the fewer the better to make the figure you want.

Answers

1. a) i) If the pole at -4 is moved further to the left:
 - Curved part of RL straightens out because the centroid moves to the left.
 - The gain required to reach any given damping factor goes up.
 - The breakaway point moves closed to -1
- ii) If the pole at -4 is moved to the right the opposite effects occur, plus, if moved right of zero at -2 effects get worse, fast.
- iii) Adding a pole makes matters worse. It makes the curves bend down and makes a new curve bend off to the left.
- iv) Adding a zero makes matters better. Especially if placed on top of the pole at -2. Further to the right of that and it tends to make a dip in the step response. If that can be tolerated, then increased speeds are possible with increased gain.

2. a)



$$C(s) := \frac{s + 2}{s + 10}$$

Gain is about 56

Your answer may be different, especially, your zero may be to the right of -2.

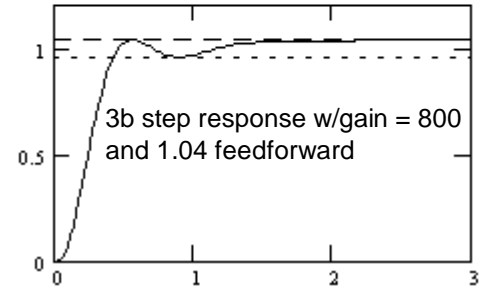
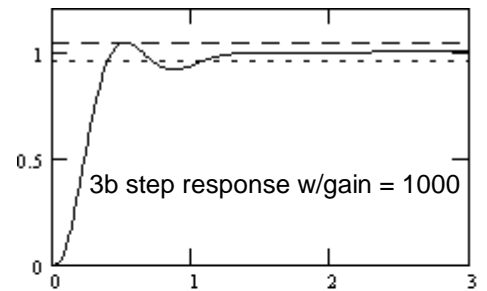
b) See answers for RL2, gain is about 0.92 c) you're on your own

3. a) See answers for RL3 System has a zero at the origin.

b) See answers for RL3

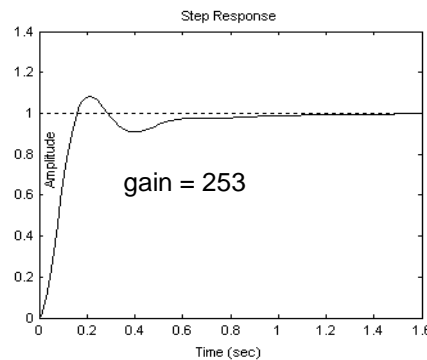
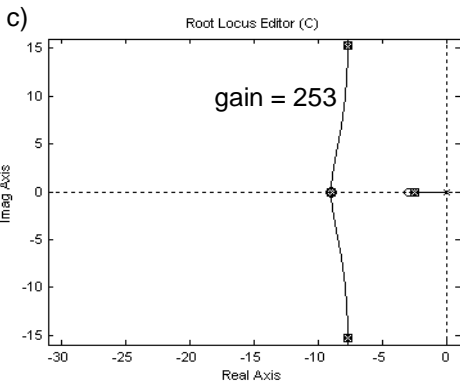
Gain of 346 puts the CL poles on the 45° line, BUT, because of the zero at about -1.3, a gain of 1000 is almost twice as fast.

A gain of 800 with a feedforward gain of 1.04 will make it 4 times faster, but will result in a long-term +4% error in the output. If only we could limit the feedforward gain to the first part of the response. Ahhh... maybe that explains the weird feedback / feedforward scheme in lab 5b.



$$C(s) := \frac{s + 9}{1} \quad \text{Your answer may be different}$$

With gain = 253, overshoot is too high, gain = 215 meets requirements.



4. My close point: $s_1 := .945 + 1.1j \quad \arg[s_1 - (1 + j)] = 118.811 \cdot \text{deg} \quad \text{close to } 117 \text{ deg}$

Place CL pole at crossing, gain = 2

Move CL pole to break in point, location shows at -3.23, at a gain of 8.47.

- 4.d) a. You go broke at \$168/day. b. You make \$101/day.
- c. You make \$161/day. Your answers may be quite different

Matlab's SISO tool (single-input/single-output tool)

The version I used was R2016b, but this may still be of some help for other versions.

3/9/19

The SISO tool can be used to draw and manipulate root-locus plots of single-input / single-output systems. It is part of the Matlab Control System Toolbox.

To use the SISO tool, you first need to create the open-loop transfer function "object" in Matlab. There are several ways to do this, but I recommend this way:

1. Define the variable "s" as a special TF model `s = tf('s');`
This only needs to be done once, after that any other expression of s will automatically be interpreted by Matlab as a transfer function.
2. Enter your transfer function as a rational expression in s For example, $G = s / (s^2 + 2*s + 10);$
Now G is a "transfer function object" of the transfer function

$$G(s) = \frac{s}{s^2 + 2s + 10}$$

Now type: `sisotool(G)`

3. You can close the Bode plot views by: View -> Float . Now you can close the two Bode windows, leaving the Root Locus and Step Response windows. View -> Left/Right will return to larger windows.
4. The little red squares (or circles) on the root locus (RL) plot show the locations of the closed-loop (CL) poles. You can grab any of these and move them to a new location. Note how the other(s) move as well. Note the effects on the step response. Also note the information given at the bottom of the window as you grab the CL pole.
5. If you choose the ROOT LOCUS EDITOR tab, you can add a pole or zero to the real axis by: click X or O button -> click on plot where you want the pole or zero. You can later drag it left and right. You can erase it with the eraser tool. You can add complex poles or zeros using the buttons that look like x/x or o/o fractions.
6. If you right-click on the root locus plot, you can open "Edit Compensator" and see the gain as well as add, delete or change compensator poles or zeros. (Right-click within the "Dynamics" window to get options.) Note: Each time you add a pole or zero it adds it at -1. Then you have to select it and change the position. The new pole will not automatically be selected. (Big pain in posterior.)
7. If you've added any poles or zeroes, then you may have noticed that the format of the compensator is a bit weird. They are in a $(1 - s/p)$ or $(1 - s/z)$ form. To fix this and get gain numbers that will match mine, Choose the CONTROL SYSTEM tab -> Preferences -> Options, click the Zero/pole/gain option to change the format of the poles and zeros from $(1 + s/p)$ to $(s + p)$.
8. If you move any compensator poles or zeros into the right-half plane the gain suddenly becomes negative. I don't know why or how to fix this, just keep it in mind. If you figure out how to eliminate this weirdness, let me know.
9. In order to effectively evaluate changes, you will need to inhibit Matlab's constant rescaling of the plot: right-click anywhere on root-locus plot area -> Properties -> Limits -> uncheck the Autoscale boxes and set limits to match the aspect ratio of your window.

You can't modify the Plant poles and zeros (at least as far as I know).

Play with this until you are ready to start the Homework.

Start Homework RL5. Read up through problem 1a and refer back to homework RL2. Back in the Matlab Command window, type: `G = 1` to create a very simple transfer function. That way all your poles and zeros can be manipulated in the SISO tool. type: `sisotool(G)`. Reset your preferences as above (items 3, 7, & 9). (SISOtool used to have a way to import a new G, I don't see that option anymore.) Add poles at 0, -2, and -4 so that $C(s) = 1 \times 1/s(s+2)(s+4)$. You are now ready to work problem 1a.

More information

Google, sisotool help and Matlab help.