ECE 3510  Homework   RL1    Root Locus Plots by Hand          Due: ___________     b

Name ____________________________________

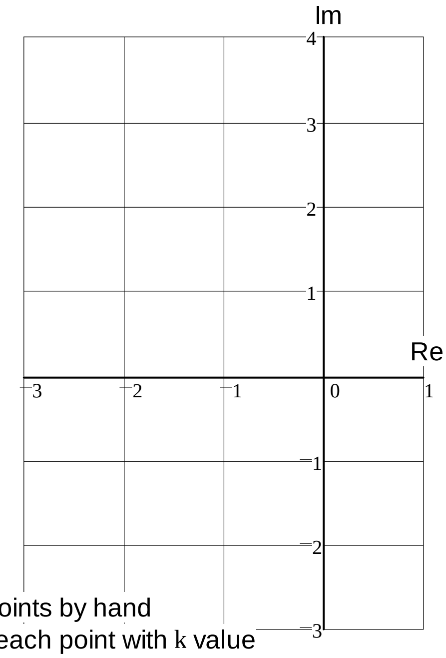
**Example 1 from p. 90 of Bodson text**

$G(s) = \dfrac{1}{s\cdot(s+2)}$  $p_1 := 0$  $p_2 := -2$  $H(s) = \dfrac{k\cdot\dfrac{1}{s\cdot(s+2)}}{1+k\cdot\dfrac{1}{s\cdot(s+2)}}\cdot\dfrac{s\cdot(s+2)}{s\cdot(s+2)}$

denominator: $s\cdot(s+2)+k \;=\; s^2+2\cdot s+k \;=\; 0$

$s_1(k) := \dfrac{-2-\sqrt{2^2-4\cdot k}}{2}$  $s_2(k) := \dfrac{-2+\sqrt{2^2-4\cdot k}}{2}$

| | | |
|---|---|---|
| $k := 0$ | $s_1(k) =$ ____________ | $s_2(k) =$ ____________ |
| $k := 0.1$ | $s_1(k) = -1.949$ | $s_2(k) = -0.051$ |
| $k := 0.2$ | $s_1(k) =$ ____________ | $s_2(k) =$ ____________ |
| $k := 0.5$ | $s_1(k) = -1.707$ | $s_2(k) = -0.293$ |
| $k := 0.8$ | $s_1(k) = -1.447$ | $s_2(k) = -0.553$ |
| $k := 1$ | $s_1(k) =$ ____________ | $s_2(k) =$ ____________ |
| $k := 2$ | $s_1(k) = -1-j$ | $s_2(k) = -1+j$ |
| $k := 5$ | $s_1(k) = -1-2j$ | $s_2(k) = -1+2j$ |
| $k := 10$ | $s_1(k) =$ ____________ | $s_2(k) =$ ____________ |
| $k := 100$ | $s_1(k) = -1-9.95j$ | $s_2(k) = -1+9.95j$ |




Plot points by hand
label each point with $k$ value

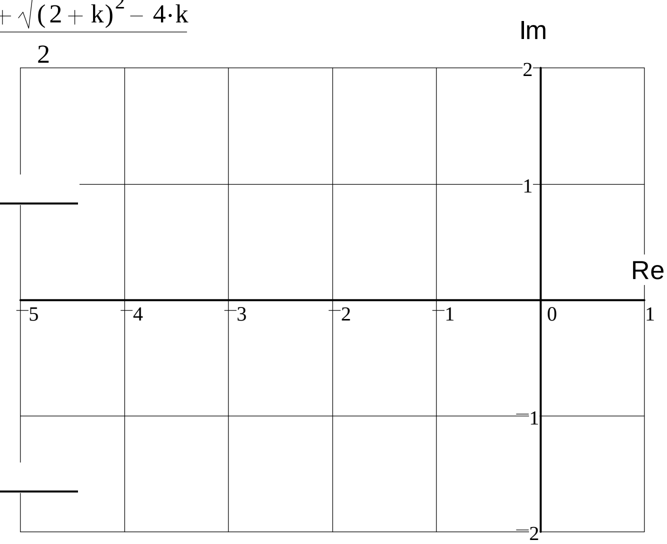
**Example 2 from p. 91 of text**

$G(s) = \dfrac{s+1}{s\cdot(s+2)}$  $z_1 := -1$  $p_1 := 0$  $p_2 := -2$  $H(s) = \dfrac{k\cdot\dfrac{s+1}{s\cdot(s+2)}}{1+k\cdot\dfrac{s+1}{s\cdot(s+2)}}\cdot\dfrac{s\cdot(s+2)}{s\cdot(s+2)}$

denominator: = ______________________________ = 0

$s_1(k) := \dfrac{-(2+k)-\sqrt{(2+k)^2-4\cdot k}}{2}$  $s_2(k) := \dfrac{-(2+k)+\sqrt{(2+k)^2-4\cdot k}}{2}$

| | | |
|---|---|---|
| $k := 0$ | $s_1(k) = -2$ | $s_2(k) = 0$ |
| $k := 0.1$ | $s_1(k) = -2.051$ | $s_2(k) = -0.049$ |
| $k := 0.2$ | $s_1(k) =$ ____________ | $s_2(k) =$ ____________ |
| $k := 0.5$ | $s_1(k) = -2.281$ | $s_2(k) = -0.219$ |
| $k := 0.8$ | $s_1(k) = -2.477$ | $s_2(k) = -0.323$ |
| $k := 1$ | $s_1(k) = -2.618$ | $s_2(k) = -0.382$ |
| $k := 2$ | $s_1(k) = -3.414$ | $s_2(k) = -0.586$ |
| $k := 5$ | $s_1(k) = -6.193$ | $s_2(k) = -0.807$ |
| $k := 10$ | $s_1(k) =$ ____________ | $s_2(k) =$ ____________ |
| $k := 100$ | $s_1(k) = -101.01$ | $s_2(k) = -0.99$ |




Plot points by hand

**Example 3 from p92 of text**

$$G(s) = \frac{s+2}{s\cdot(s+1)} \qquad z_1 := -2 \qquad p_1 := 0 \qquad p_2 := -1$$

$$H(s) = \frac{k\cdot(s+2)}{s\cdot(s+1)+k\cdot(s+2)}$$

denominator: $s\cdot(s+1)+k\cdot(s+2) = 0$

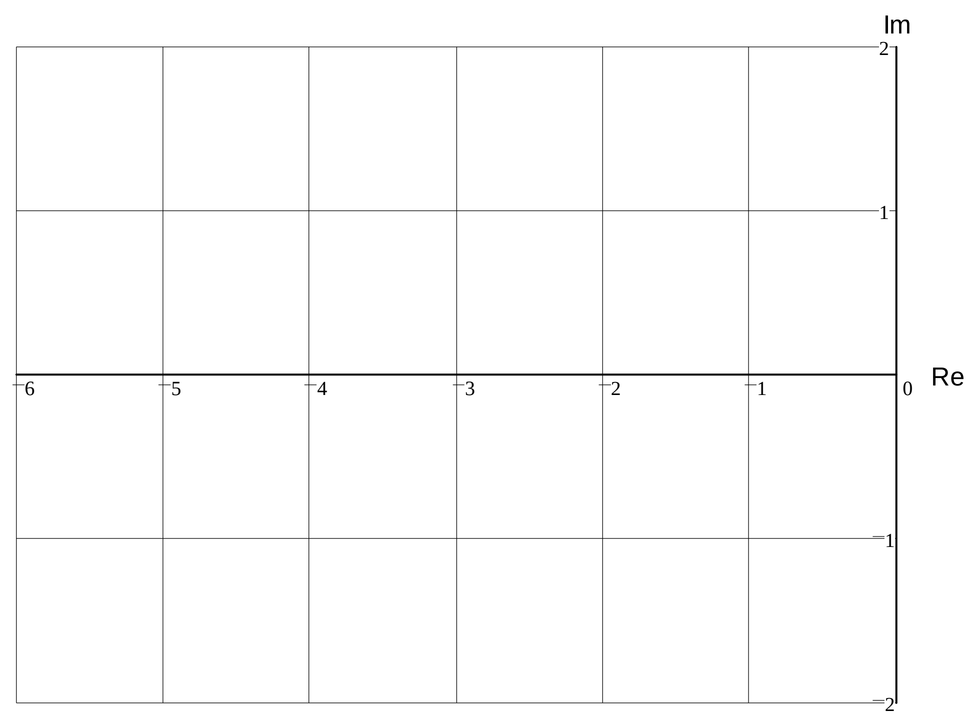
$$s^2+s+k\cdot s+2\cdot k = 0$$

$$s^2+(1+k)\cdot s+2\cdot k = 0$$

$s_1(k) =$ ____________________ $\qquad s_2(k) =$ ____________________

| | | |
|---|---|---|
| $k := 0$ | $s_1(k) =$ ________ | $s_2(k) =$ ________ |
| $k := 0.1$ | $s_1(k) = -0.87$ | $s_2(k) = -0.23$ |
| $k := 0.17157$ | $s_1(k) = -0.588$ | $s_2(k) = -0.584$ |
| $k := 0.172$ | $s_1(k) = -0.586 - 0.025j$ | $s_2(k) = -0.586 + 0.025j$ |
| $k := 0.2$ | $s_1(k) = -0.6 - 0.2j$ | $s_2(k) = -0.6 + 0.2j$ |
| $k := 0.5$ | $s_1(k) = -0.75 - 0.661j$ | $s_2(k) = -0.75 + 0.661j$ |
| $k := 0.8$ | $s_1(k) = -0.9 - 0.889j$ | $s_2(k) = -0.9 + 0.889j$ |
| $k := 1$ | $s_1(k) =$ ________ | $s_2(k) =$ ________ |
| $k := 3$ | $s_1(k) = -2 - 1.414j$ | $s_2(k) = -2 + 1.414j$ |
| $k := 5$ | $s_1(k) = -3 - j$ | $s_2(k) = -3 + j$ |
| $k := 5.827$ | $s_1(k) = -3.414 - 0.045j$ | $s_2(k) = -3.414 + 0.045j$ |
| $k := 7$ | $s_1(k) = -5.414$ | $s_2(k) = -2.586$ |
| $k := 10$ | $s_1(k) = -8.702$ | $s_2(k) = -2.298$ |
| $k := 100$ | $s_1(k) = -98.979$ | $s_2(k) = -2.021$ |

Plot points by hand below

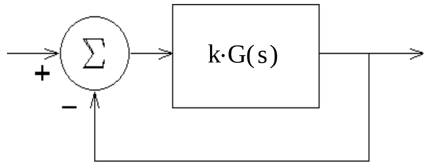


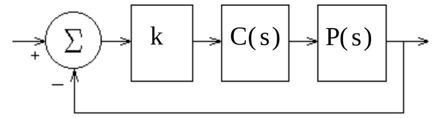

ECE 3510 Homework RL1 p.2

A. Stolp 2/16/06, rev. 2/26/23

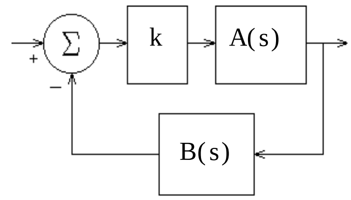
# ECE 3510 Root-Locus Plots

$G(s) = \dfrac{N_G}{D_G}$ = the Open-Loop (O-L) transfer function




$G(s) = C(s)\cdot P(s)$




$G(s) = A(s)\cdot B(s)$

The poles of the C-L transfer function solve: $1 + k\cdot G(s) = 0$

Any $s$ that makes $\underline{/\,G(s)} = 180^o$ will work for some $k$ and be a part of the Root Locus.

## The Rules $(k \geq 0)$

1. Root-locus plots are symmetric about the real axis.
2. On the real axis, spaces left of an odd number of O-L poles and zeros are always part of the locus.
   (Essentially, every other space on the real axis (counting leftward) is part of the plot.)
3. Each O-L pole originates ($k = 0$) one branch. $(n)$

   Each O-L zero terminates ($k = \infty$) one branch. $(m)$

   All remaining branches go to $\infty$, one per asymptote. $(n - m)$

   They each approach their asymptotes as they go to $\infty$.
4. The origin of the asymptotes is the *centroid*.

$$\text{centroid} = \sigma = \frac{\sum_{\text{all}} \text{OLpoles} - \sum_{\text{all}} \text{OLzeros}}{n - m}$$

( # poles - # zeros)

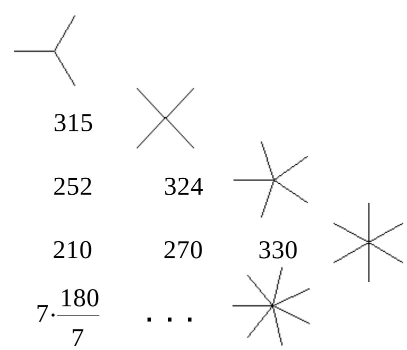
5. The angles of the asymptotes are:

$$i\cdot\frac{180}{n - m}$$

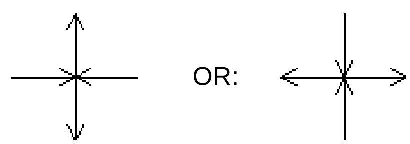
where $i = 1, 3, 5, 7, 9, \ldots$ full circle

Or figure for half circle and mirror around the real axis.

| n - m | angles (degrees) | | | | | |
|---|---|---|---|---|---|---|
| 2 | 90 | 270 | | | | |
| 3 | 60 | 180 | 300 | | | |
| 4 | 45 | 135 | 225 | 315 | | |
| 5 | 36 | 108 | 180 | 252 | 324 | |
| 6 | 30 | 90 | 150 | 210 | 270 | 330 |
| 7 | $\frac{180}{7}$ | $3\cdot\frac{180}{7}$ | $5\cdot\frac{180}{7}$ | $7\cdot\frac{180}{7}$ | . . . | |

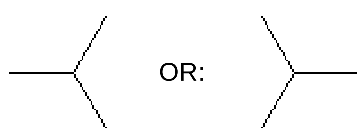


6. The angles of departure (and arrival) of the locus are almost always:



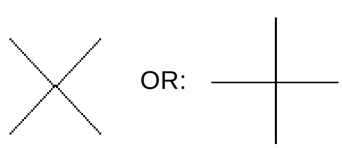

Only multiple poles result in different departure angles:
(or zeros)

triple poles:




Check real-axis rule, above

Quadruple poles:




Check real-axis rule, above

Multiple zeros attract branches from these same angles

Good guesses: Draw your break-outs midway between poles, and your break-ins midway between zeroes.
Draw circles centered approximately midway between poles and zeroes.

7. Gain at any point on the root locus: $k = -\dfrac{1}{G(s)} = \dfrac{1}{|G(s)|} = \dfrac{|D(s)|}{|N(s)|}$

8. Phase angle of $G(s)$ at any point $s$ on the root locus: $\arg(G(s)) = \arg(N(s)) - \arg(D(s)) = \pm 180^o \quad \pm 540^o \ldots$

Note: arg(x) is $\angle$(x) Or: $\arg\left(\dfrac{1}{G(s)}\right) = \arg(D(s)) - \arg(N(s)) = \pm 180^o \quad \pm 540^o \ldots$

Or: $\arg(-G(s)) = 0^o \quad \pm 360^o \ldots$

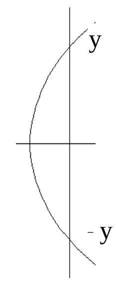
$$\sum_{\text{all zeroes}} (\text{angle of point } s \text{ relative to zero}) - \sum_{\text{all poles}} (\text{angle of point } s \text{ relative to pole}) = \pm 180^o \quad \pm 540^o \ldots$$

9. Gain at $j\omega$ crossing: Use Routh-Hurwitz test.

OR: a) Get a rough s (say y) value from your plot,

b) Check it (evaluate the angle of $G(jy)$) and iterate using rule 9,

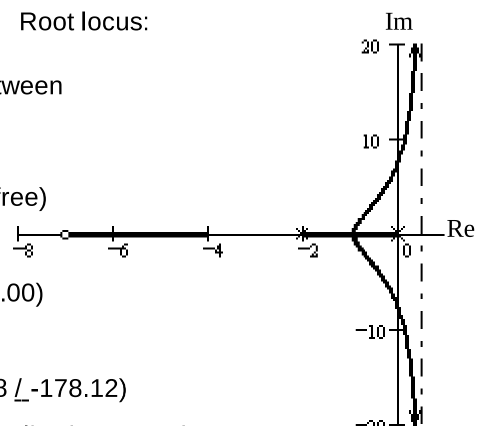
c) Find k using rule 8.




Calculator example: $G(s) = \dfrac{s+7}{s\cdot(s+2)\cdot(s+4)}$

Find the gain at $j\omega$ crossing:

Root locus:




Let's assume that the root locus crosses the $j\omega$ axis somewhere between 5 and 10. I first try 5, evaluating $\dfrac{1}{G(5j)}$ on my calculator

Note: I'm evaluating $1/G(s)$ so I'll end up with the gain value for free)

In a TI-86, I enter the following:

5.000->S:((0,S)*(2,S)*(4,S))/((7,S)) It returns: (20.04 $\angle$174.00)

Next I try:

10.00->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (89.98 $\angle$-178.12)

The first was a positive angle, and this is negative, yep, the answer lies between these two.

The first was $6^o$ under $180^o$ and the second is $2^o$ over, interpolate: $5 + \dfrac{6}{6+2}\cdot 5 = 8.75$

Try: 8.75 8.750->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (67.43 $\angle$-178.78)

$$8.75 - \frac{180 - 178.78}{180 - 178.12}\cdot(10 - 8.75) = 7.939$$

Try: 7.9 7.900->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (54.01 $\angle$-179.52)

$$7.9 - \frac{.48}{1.22}\cdot(8.75 - 7.9) = 7.566$$

Try: 7.5 7.500->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (48.23 $\angle$-179.97)

$$7.5 - \frac{.03}{.48}\cdot(7.9 - 7.5) = 7.475$$

Try: 7.475 7.475->S:((0,S)*(2,S)*(4,S))/((7,S)) TI returns: (47.88 $\angle$179.99)

The root locus crosses at $\pm 7.475j$ and the gain is 48. $k = 48$

# ECE 3510 Root-Locus Plots **Additional Rules**

10. Breakaway points from the real axis ($\sigma_b$) are the solutions to: $\frac{d}{ds}G(s) = 0$
(and arrival)

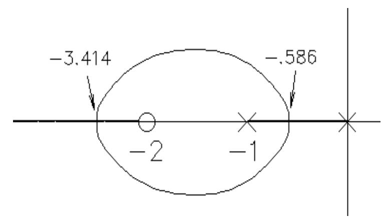
The breakaway points are also solutions to: $$\sum_{\text{all}} \frac{1}{(s + -p_i)} = \sum_{\text{all}} \frac{1}{(s + -z_i)}$$

IE: $$\frac{1}{(s+-p_1)} + \frac{1}{(s+-p_2)} + \frac{1}{(s+-p_3)} + \dots = \frac{1}{(s+-z_1)} + \frac{1}{(s+-z_2)} + \frac{1}{(s+-z_3)} + \dots$$

Not all of these solutions will actually be breakaway points, some may be eliminated by the real-axis rule, and some may be arrival points. Both of these methods can be messy and result in high-order equations to solve. The second example will show an iterative way to deal with the complexity.

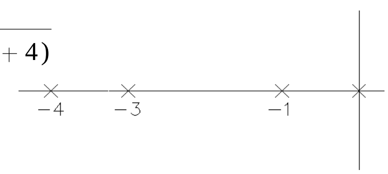
Example 1 $G(s) = \frac{s+2}{s\cdot(s+1)}$ Solve: $\frac{1}{s} + \frac{1}{s+1} = \frac{1}{s+2}$

$$\frac{(s+1)+s}{s\cdot(s+1)} = \frac{1}{s+2}$$




$(2\cdot s+1)\cdot(s+2) = s\cdot(s+1)$ $s^2 + 4\cdot s + 2 = 0$ $s = -3.414$ $s = -0.586$

Example 2 Iterative process, best shown by example: $G(s) = \frac{1}{s\cdot(s+1)\cdot(s+3)\cdot(s+4)}$




Find the breakaway point between 0 and -1.

Must solve: $$\frac{1}{s} + \frac{1}{(s+1)} + \frac{1}{(s+3)} + \frac{1}{(s+4)} = 0$$

Guess $s = -0.4$ and use that for all the s's except those closest to the breakaway you want to find.

Solve this instead: $$\frac{1}{s} + \frac{1}{(s+1)} + \frac{1}{(-0.4+3)} + \frac{1}{(-0.4+4)} = 0$$

$$\frac{1}{s} + \frac{1}{(s+1)} + \left(\frac{1}{2.6} + \frac{1}{3.6}\right) = 0$$

multiply by $s(s+1)$: $$\frac{s+1}{1} + \frac{s}{1} + s\cdot(s+1)\cdot\left(\frac{1}{2.6} + \frac{1}{3.6}\right) = 0$$

$$s^2 + 4.0194\cdot s + 1.5097 = 0$$

$$s = \frac{-4.0194 + \sqrt{4.0194^2 - 4\cdot 1.5097}}{2} = -0.419$$ Use this answer to try again

ignore the -3.6 solution for this answer.

$$\frac{1}{s} + \frac{1}{(s+1)} + \left(\frac{1}{2.581} + \frac{1}{3.581}\right) = 0$$

$$s^2 + 4\cdot s + 1.5 = 0$$

$$s = \frac{-4 + \sqrt{4^2 - 4\cdot 1.5}}{2} = -0.419$$ No significant change,
so this is the breakaway point

To find the breakaway point between -3 and -4: Guess $s = -3.6$

$$\frac{1}{-3.6} + \frac{1}{(-3.6+1)} + \frac{1}{(s+3)} + \frac{1}{(s+4)} = 0$$

solve for s: $s = -3.58$ Not much change,
so this is the breakaway point

Actually, it usually doesn't matter that much just where the breakaway points are.

11. Departure angle ($\theta_D$) from a complex pole ($p_c$).

Recall rule 9 (one of the most important rules):

for any point s on the root locus:

$$\arg(G(s)) = \arg(N(s)) - \arg(D(s)) = \pm 180^o \quad \pm 360^o \ldots$$

Note: arg(x) is /(x)

For multiple (r) poles:

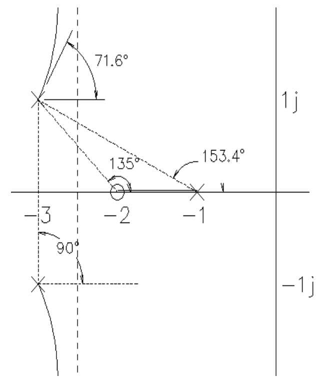
Divide the circle into r divisions: $\frac{360 \cdot deg}{r}$

and rotate all by $\frac{\theta_D}{r}$

Now imagine a point ε-distance away from the complex pole. That point would have an angle of $\theta_D$ with respect to the complex pole, but it's angle relative to all the other poles and zeros would be essentially the same as the complex pole.

$$\sum_{\text{all zeroes}} (\text{angle of point s relative to zero}) - \sum_{\text{all poles but } p_c} (\text{angle of point s relative to pole}) - \theta_D = \pm 180^o \quad \pm 540^o \ldots$$

Example: $G(s) := \frac{s+2}{(s+1)\cdot\left[(s+3)^2+1^2\right]}$ Find the departure angle from the pole at: $p_c := -3 + 1\cdot j$




$$135 - 153.4 - 90 - \theta_D = \pm 180^o \quad \pm 540^o \ldots$$

rearrange: $\theta_D = 180 - 90 - 153.4 + 135 = 71.6 \text{ deg}$

Mathmatically: $\theta_D = 180 \cdot deg + \arg\left[G(p_c)\cdot(s+p_c)\right]$

The O-L phase angle computed at the complex pole, but ignoring the effect of that complex pole.

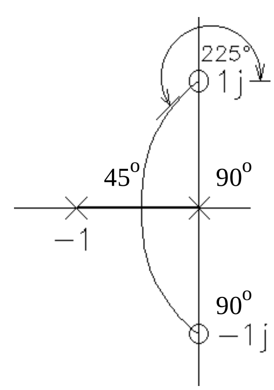
Our example: $\theta_D = 180 \cdot deg + \arg\left[\frac{p_c + 2}{(p_c+1)\cdot(p_c+3+1\cdot j)}\right] = 71.6 \cdot deg$

12. Arrival angle ($\theta_A$) to complex zero ($z_c$).

Exactly the same idea.

$$\sum_{\text{all zeroes but } z_c} (\text{angle of point s relative to zero}) + \theta_A - \sum_{\text{all poles}} (\text{angle of point s relative to pole}) = \pm 180^o \quad \pm 540^o \ldots$$

Example: $G(s) := \frac{s^2+1^2}{s\cdot(s+1)} = \frac{(s-1\cdot j)\cdot(s+1\cdot j)}{s\cdot(s+1)}$ Find the departure angle from the pole at: $z_c := 1\cdot j$




$$90 + \theta_A - 90 - 45 = \pm 180^o \quad \pm 540^o \ldots$$

rearrange: $\theta_A = 180 - 90 + 90 + 45 = 225 \quad \text{deg}$

Mathmatically: $\theta_A = 180 \cdot deg - \arg\left[\frac{G(z_c)}{(s+z_c)}\right]$

The O-L phase angle computed at the complex zero, but ignoring the effect of that complex zero.

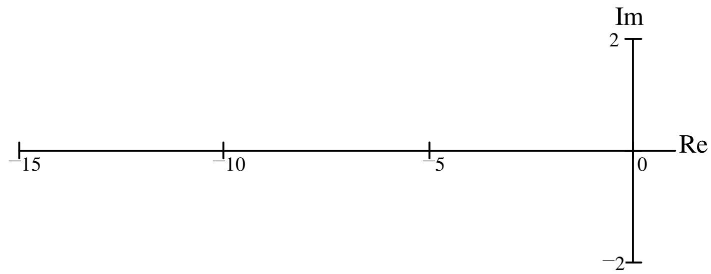
Our example: $\theta_A = 180 \cdot deg - \arg\left[\frac{1\cdot j + 1 \cdot j}{1\cdot j\cdot(1\cdot j+1)}\right] = 225 \cdot deg$

# ECE 3510 Basic Root Locus Examples

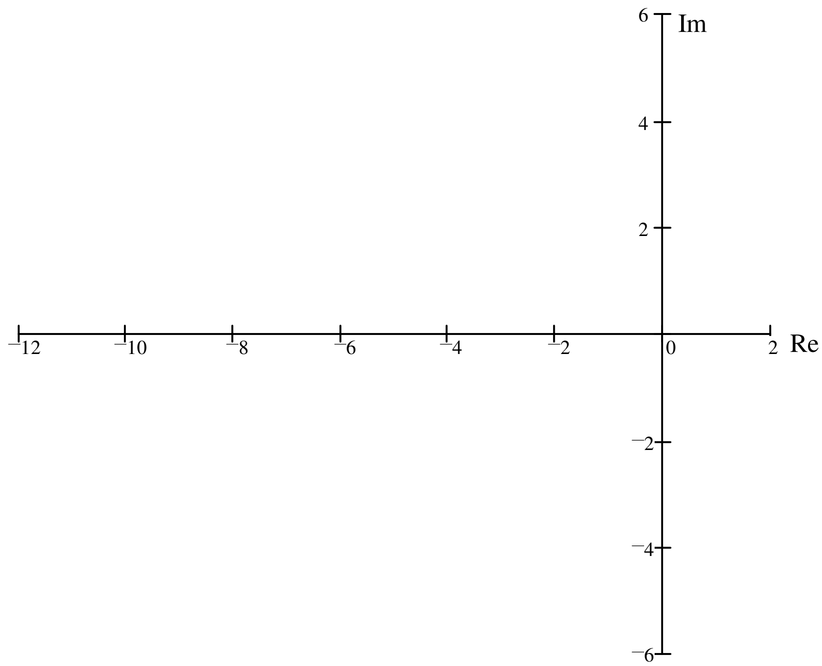


Sketch (by hand) the root-locus plots for the following open-loop transfer functions:
For these hand sketches, just use the rules on the first page of the notes
Mention the rules used and show work.

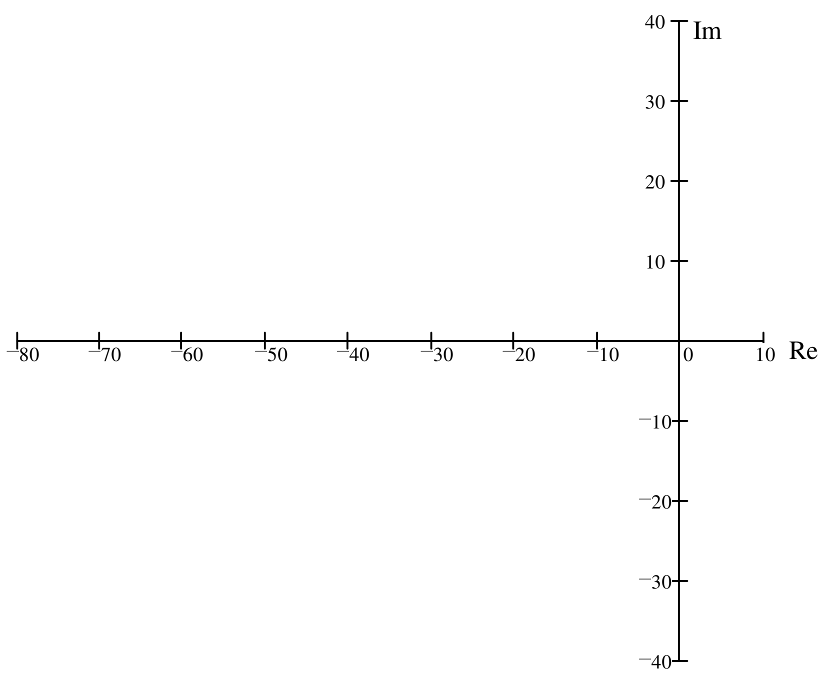
1. $\mathbf{G}(s) = \dfrac{s+6}{(s+1)\cdot(s+10)}$




2. $\mathbf{G}(s) = \dfrac{20}{(s+2)\cdot(s+10)}$

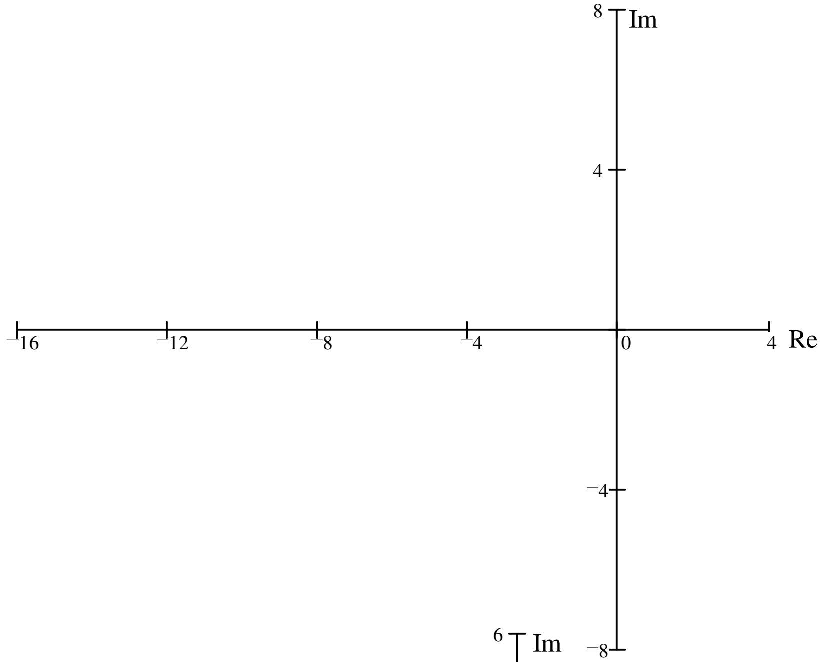



3. $\mathbf{G}(s) = \dfrac{1}{s\cdot(s+16.64)\cdot(s+53.78)}$

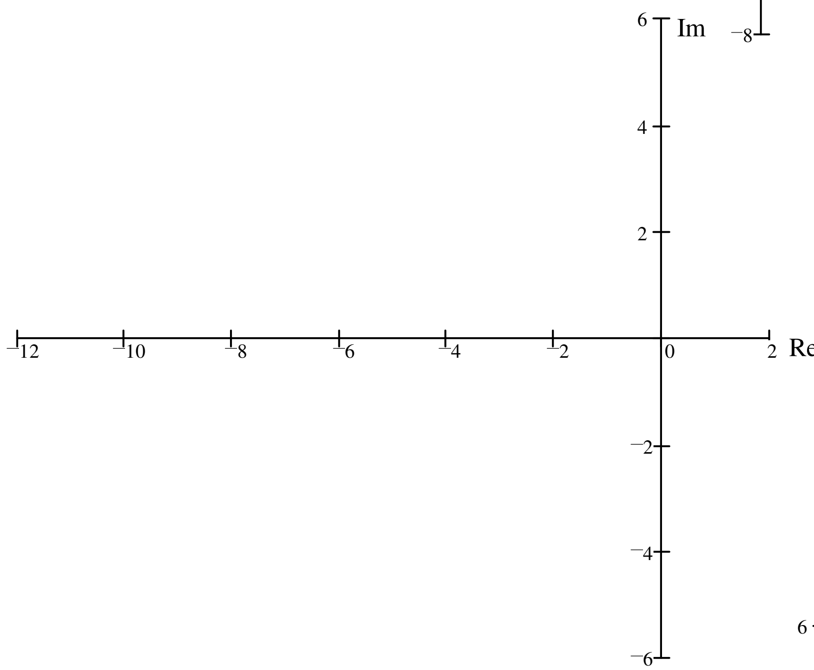


## Basic Root Locus Examples p2

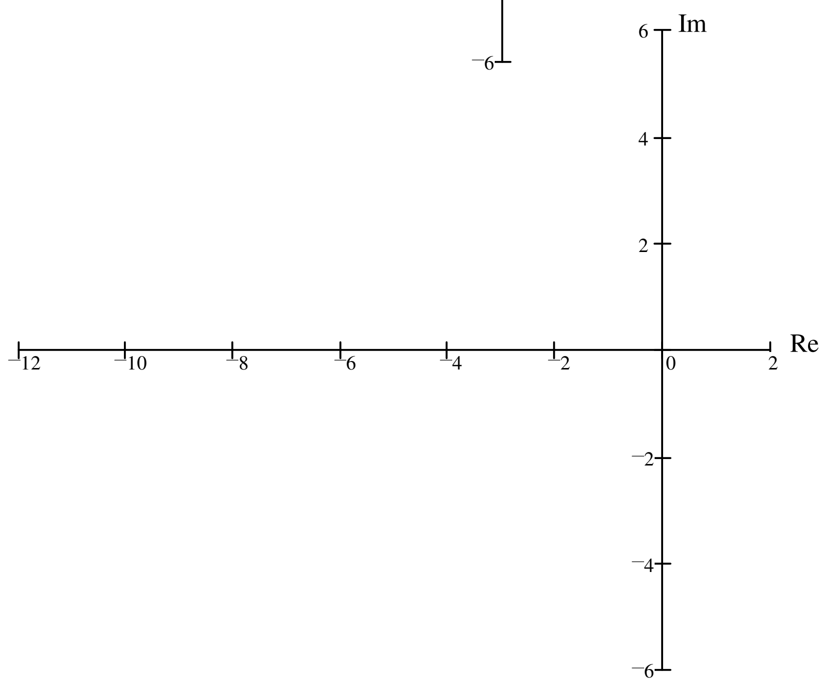
4. $\mathbf{G}(s) = \dfrac{4 \cdot s + 40}{s \cdot (s+2) \cdot (s+8) \cdot (s+4)}$




5. $\mathbf{G}(s) = \dfrac{3 \cdot s + 18}{s \cdot (s+4) \cdot (s+10)}$

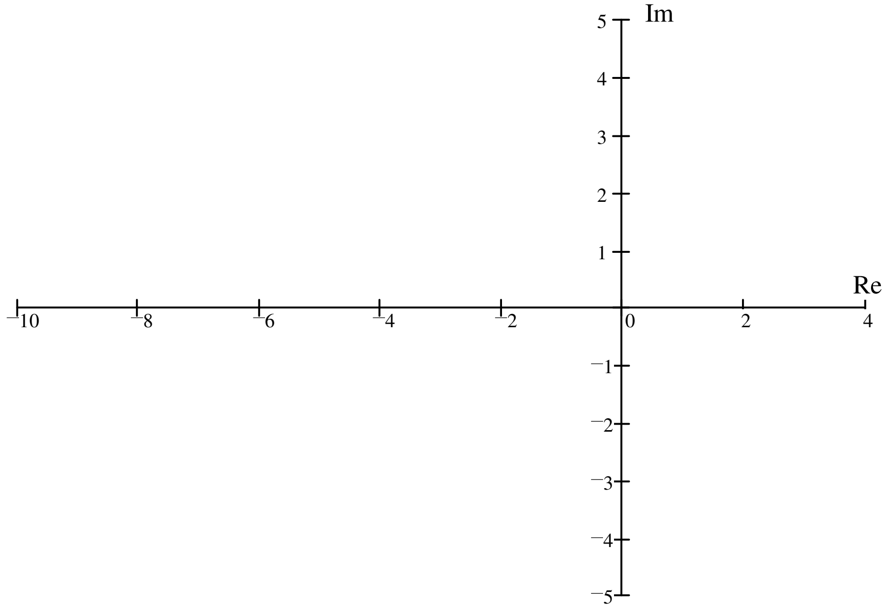



6 $\mathbf{G}(s) = \dfrac{s + 8}{(s+1) \cdot (s+3)^3}$




Basic Root Locus Examples p2

# Basic Root Locus Examples p3




7. $\mathbf{G}(s) = \dfrac{(s+5)\cdot(s+8)}{s^2 - 6\cdot s + 13}$

$m := 2$

$n := 2$

$n - m = 0$

no asymptotes:

8. $\mathbf{G}(s) = \dfrac{1}{\left(s^2 + 4\cdot s + 13\right)\cdot(s+1)\cdot(s+5)}$

$m := 0$

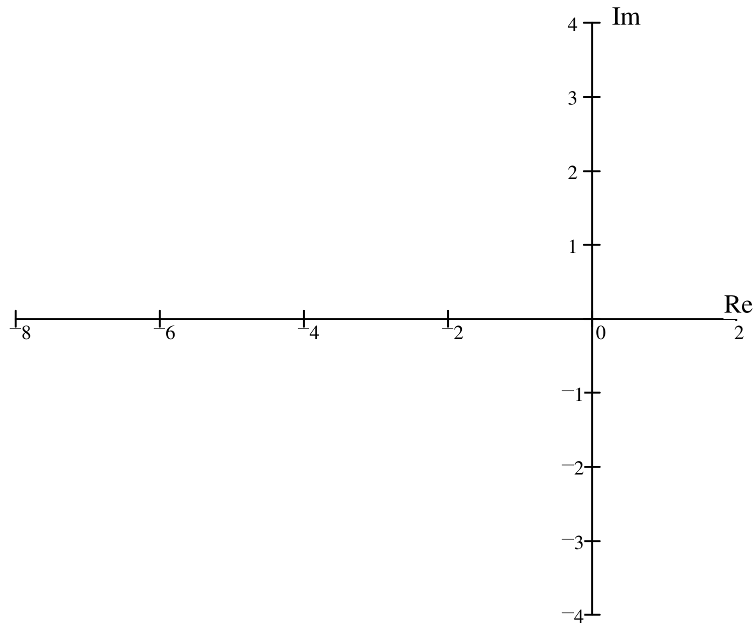
$n := 4$

$n - m = 4$

asymptotes: 

centroid:

$$\sigma_C = \frac{-2 + -2 + -1 + -5}{4} = -2.5$$

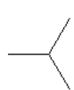



9 $\mathbf{G}(s) = \dfrac{s + 12}{\left(s^2 + 4\cdot s + 13\right)\cdot(s+1)\cdot(s+5)}$

$m := 1$

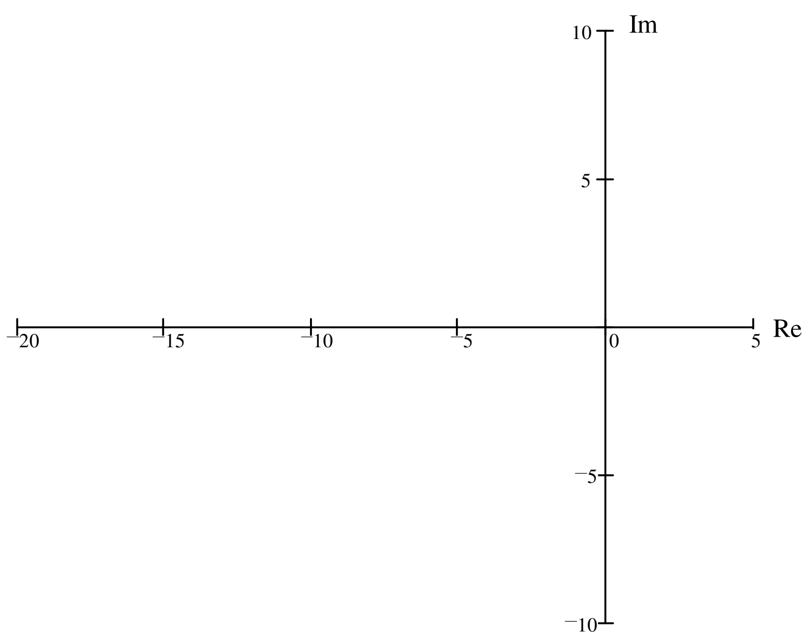
$n := 4$

$n - m = 3$

asymptotes: 

centroid:

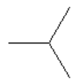
$$\sigma_C = \frac{-2 + -2 + -1 + -5 - -12}{3} = 0.667$$



10 $\mathbf{G}(s) = \dfrac{s+3}{(s+6)^3\cdot(s+12)}$ $\quad m := 1$

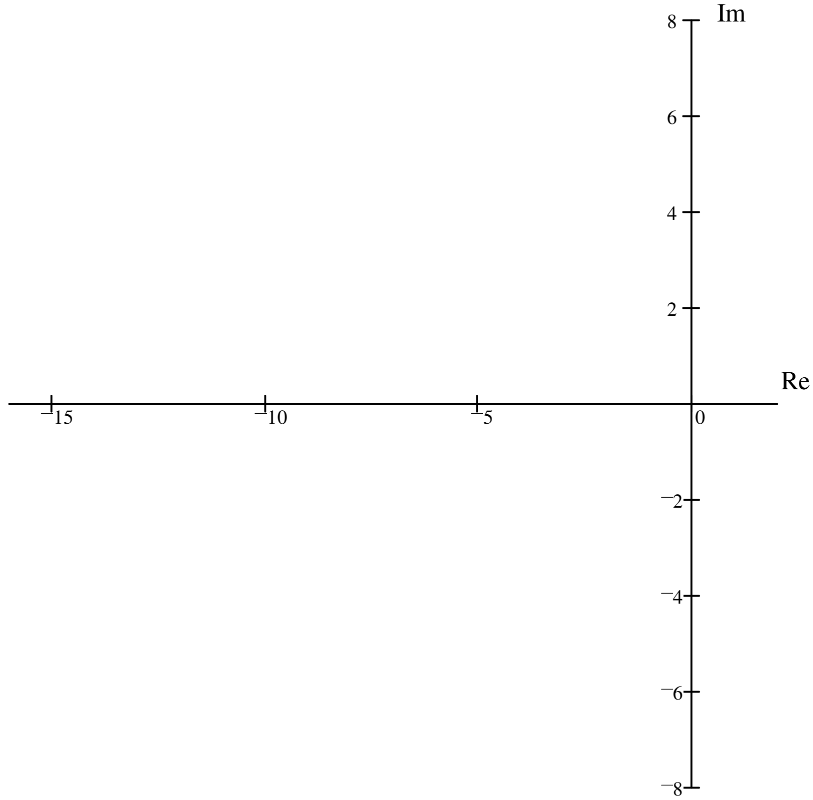
$n := 4$

$n - m = 3$

asymptotes: 

centroid:

$$\sigma_C = \frac{(3\cdot(-6) + -12) - -3}{3} = -9$$

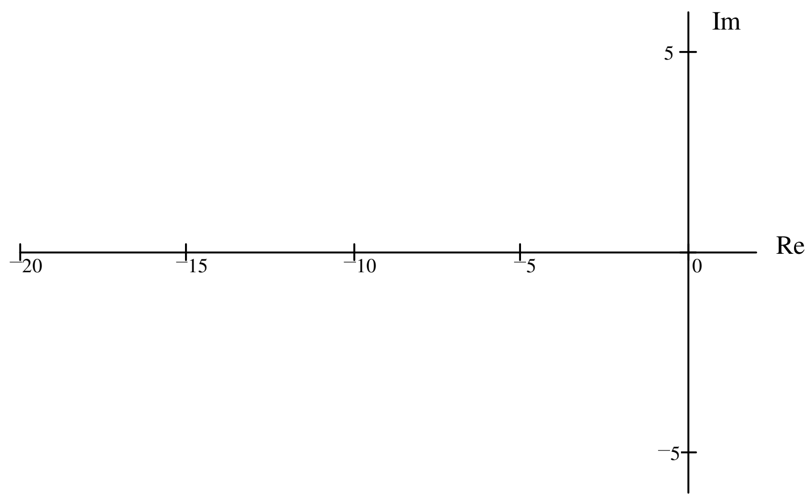



11 $\mathbf{G}(s) = \dfrac{(s+3)\cdot(s+12)}{(s+6)^3}$ $\quad m := 2$

$n := 3$

$n - m = 1$

no asymptotes

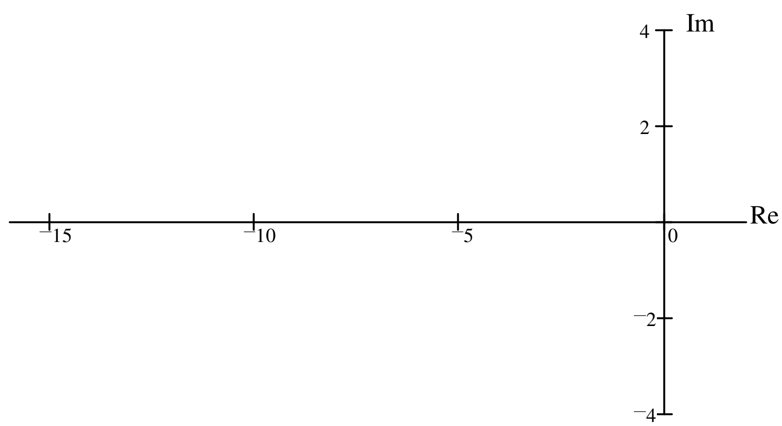



12 $\mathbf{G}(s) = \dfrac{(s+3)\cdot(s+12)^2}{(s+6)^3}$ $\quad m := 3$

$n := 3$

$n - m = 0$

no asymptotes




**Basic Root Locus Examples  p4**