ECE 3510 Control System Characteristics & Performance

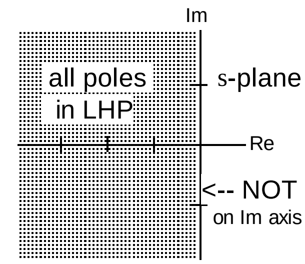
The objective of a control system is to allow an input to set or control an output.

Except in very rare circumstances:

The output should behave in a stable, predictable, manner regardless of noise, disturbances, perturbations, and system parameters that change over time. The output should track the input (or the meaning of the input) smoothly and quickly, adding as little as possible to the output that was not in the input.

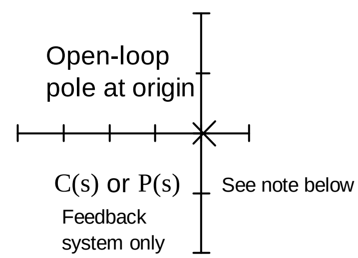
<u>A GOOD Control System</u> — <u>Corresponding system poles</u>

1. Is Stable. At the very minimum it needs to be BIBO stable. (Often the concept of stability is enlarged into areas we will consider minimal tracking.)



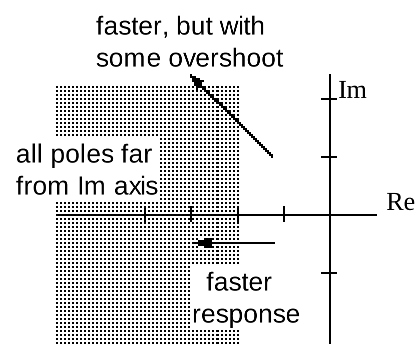
2. Tracks the input (or the meaning of the input) well.
   a) Converges to steady state values set by the input with little or no error.

   A pole at origin is an **integrator**.

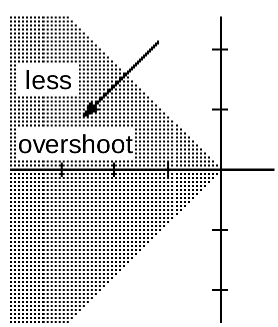


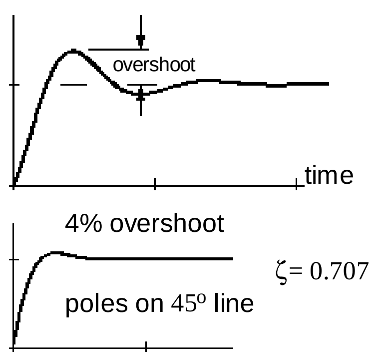
   b) Tracks varying inputs with little error.

      i) Responds quickly (fast response with little lag and short settling time).



      ii) Little or no overshoot.





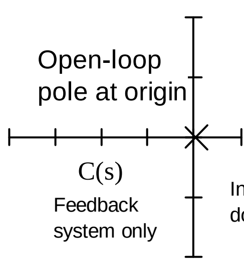
$$\%OS = 100\%\cdot e^{-\pi\cdot\frac{a}{b}}$$

See also Nise, Fig 4.19, p199 & earlier pages

$$\operatorname{atan}\left(\frac{-\pi}{\ln(2\cdot\%)}\right) = 38.8\cdot deg$$

$$\operatorname{atan}\left(\frac{-\pi}{\ln(8\cdot\%)}\right) = 51.2\cdot deg$$

3. Rejects disturbances



In reality, C(s) never just has a pole at zero by itself, that would slow down the system too much. Usually there will also be a nearby zero.

4. Be sufficiently insensitive to plant uncertainties and variations with time.

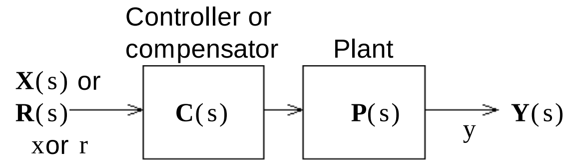
   Generally requires more loop gain and/or integrator (pole at origin) in C(s) or P(s)

5. Tolerant of noise, especially in the feedback measurements.

   Generally requires less gain and/or lower frequency response (slower).

Feedforward system

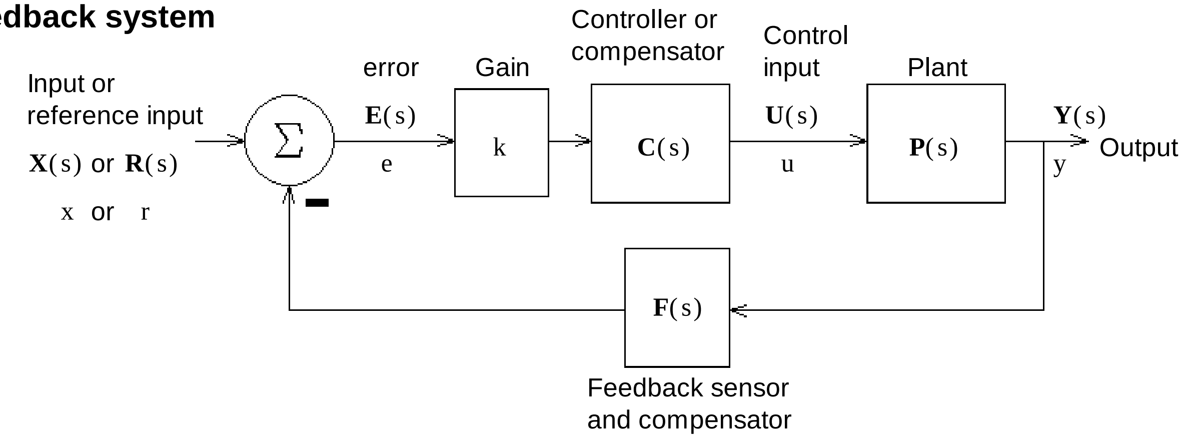
(prefiltering)



CANNOT stabilize an inherently unstable plant

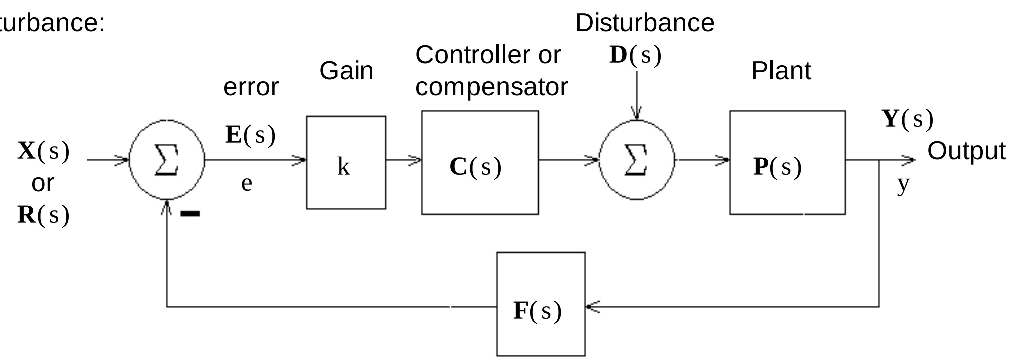
Feedforward compensation (prefiltering) is most often used in conjunction with feedback to improve the feedback system response.

Feedback system



Open-loop transfer function: $\mathbf{G}(s) = k \cdot \mathbf{C}(s) \cdot \mathbf{P}(s) \cdot \mathbf{F}(s)$ **Closed-loop** transfer function: $\mathbf{H}(s) = \dfrac{k \cdot (\mathbf{C}(s) \cdot \mathbf{P}(s))}{1 + k \cdot \mathbf{C}(s) \cdot \mathbf{P}(s) \cdot \mathbf{F}(s)}$

With disturbance:



Error Want this to go to zero

$$\mathbf{E}(s) = \frac{1}{1 + k \cdot \mathbf{C}(s) \cdot \mathbf{P}(s) \cdot \mathbf{F}(s)} \cdot \mathbf{R}(s) - \frac{\mathbf{P}(s) \cdot \mathbf{F}(s)}{1 + k \cdot \mathbf{C}(s) \cdot \mathbf{P}(s) \cdot \mathbf{F}(s)} \cdot \mathbf{D}(s) = \frac{1}{1 + k \cdot \mathbf{G}(s)} \cdot \mathbf{R}(s) - \frac{\mathbf{P}(s) \cdot \mathbf{F}(s)}{1 + k \cdot \mathbf{G}(s)} \cdot \mathbf{D}(s)$$

$$= \frac{\mathbf{D}_{\mathbf{C}}(s) \cdot \mathbf{D}_{\mathbf{P}}(s) \cdot \mathbf{D}_{\mathbf{F}}(s)}{\mathbf{D}_{\mathbf{C}}(s) \cdot \mathbf{D}_{\mathbf{P}}(s) \cdot \mathbf{D}_{\mathbf{F}}(s) + k \cdot \mathbf{N}_{\mathbf{C}}(s) \cdot \mathbf{N}_{\mathbf{P}}(s) \cdot \mathbf{N}_{\mathbf{F}}(s)} \cdot \mathbf{R}(s) - \frac{\mathbf{N}_{\mathbf{P}}(s) \cdot \mathbf{N}_{\mathbf{F}}(s) \cdot \mathbf{D}_{\mathbf{C}}(s)}{\mathbf{D}_{\mathbf{C}}(s) \cdot \mathbf{D}_{\mathbf{P}}(s) \cdot \mathbf{D}_{\mathbf{F}}(s) + k \cdot \mathbf{N}_{\mathbf{C}}(s) \cdot \mathbf{N}_{\mathbf{P}}(s) \cdot \mathbf{N}_{\mathbf{F}}(s)} \cdot \mathbf{D}(s)$$

For constant input and constant disturbance

$$e(\infty) = \frac{\mathbf{D}_{\mathbf{C}}(0) \cdot \mathbf{D}_{\mathbf{P}}(0) \cdot \mathbf{D}_{\mathbf{F}}(0)}{\mathbf{D}_{\mathbf{C}}(0) \cdot \mathbf{D}_{\mathbf{P}}(0) \cdot \mathbf{D}_{\mathbf{F}}(0) + k \cdot \mathbf{N}_{\mathbf{C}}(0) \cdot \mathbf{N}_{\mathbf{P}}(0) \cdot \mathbf{N}_{\mathbf{F}}(0)} \cdot R_m - \frac{\mathbf{N}_{\mathbf{P}}(0) \cdot \mathbf{N}_{\mathbf{F}}(0) \cdot \mathbf{D}_{\mathbf{C}}(0)}{\mathbf{D}_{\mathbf{C}}(0) \cdot \mathbf{D}_{\mathbf{P}}(0) \cdot \mathbf{D}_{\mathbf{F}}(0) + k \cdot \mathbf{N}_{\mathbf{C}}(0) \cdot \mathbf{N}_{\mathbf{P}}(0) \cdot \mathbf{N}_{\mathbf{F}}(0)} \cdot D_m$$

Perfect tracking of DC input:

$\mathbf{C}(s)$ or $\mathbf{P}(s)$ has a pole at the origin (integrator)

Perfect rejection of constant disturbance:

$\mathbf{C}(s)$ has a pole at the origin (integrator)

Note: Nise devotes an entire chapter (7) to steady-state errors from different types of inputs. Refer to that chapter to learn much more than we cover in this class. The short version is: The more integrators, the better.

ECE 3510 Routh-Hurwitz Lecture

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2/23/06,
2/20/09,
2/16/11

Routh-Hurwitz Stability test

Denominator of transfer function or signal: $a_n \cdot s^n + a_{n-1} \cdot s^{n-1} + a_{n-2} \cdot s^{n-2} + a_{n-3} \cdot s^{n-3} \quad \cdots \quad a_1 \cdot s + a_0$

Usually of the **Closed-loop transfer function denominator** to test fo BIBO stability

Test denominator for poles in CRHP (RHP including imaginary axis)

1. For all poles to be in the LHP, all coefficients must be > 0

   For a second-order denominator, that is enough, skip the next step.

2. If all coefficients are > 0 & order > 2, then:
   Create Routh-Hurwitz array:

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$	0
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$	0
	③ & divide ⑥					
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$	0
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$	0
.	.	.	.			
.	.	.	.			
s^0	z_1	0				

$$b_1 = \frac{a_{n-1} \cdot a_{n-2} - a_n \cdot a_{n-3}}{a_{n-1}} \qquad b_2 = \frac{a_{n-1} \cdot a_{n-4} - a_n \cdot a_{n-5}}{a_{n-1}} \qquad b_3 = \frac{a_{n-1} \cdot a_{n-6} - a_n \cdot a_{n-7}}{a_{n-1}}$$

$$c_1 = \frac{b_1 \cdot a_{n-3} - a_{n-1} \cdot b_2}{b_1} \qquad c_2 = \frac{b_1 \cdot a_{n-5} - a_{n-1} \cdot b_3}{b_1} \qquad c_3 = \frac{b_1 \cdot a_{n-7} - a_{n-1} \cdot b_4}{b_1}$$

Look at first column:

All positive = All roots left of imaginary axis

If any negative or 0, then there are poles on the Imaginary axis or in the RHP (Right-Half Plane)

Count sign reversals down the first column

Sign reversals = number of poles on the Imaginary axis or in the RHP (Right-Half Plane)

0 can be replaced by $-\varepsilon$ to see if there are any other sign reversals

The transfer functions of C(s) and P(s) are given below. In each case determine if the steady-state error will go to zero and whether disturbances will be completely rejected. Be sure to check for closed-loop stability when needed.

		0 steady-state error?	Reject Disturbances?
		$e_{ss}(\infty) = 0?$	$e_{ss}(\infty) = 0$ for disturbance?
a) $C(s) = \dfrac{s+2}{s^2+5\cdot s+4}$	$P(s) = \dfrac{s+1}{s^2+5\cdot s+15}$	No no pole at zero	No no pole at zero
b) $C(s) = \dfrac{s+5}{s^2+4\cdot s+3}$	$P(s) = \dfrac{s+1}{s^2+2\cdot s}$	Yes (Tentative answer) $P(s)$ has pole at zero	No $C(s)$ has no pole at zero

For a): No stability test needed to answer those questions

Must test for stability: Closed loop transfer function $= \dfrac{C(s)\cdot P(s)}{1+C(s)\cdot P(s)} = \dfrac{N_C(s)\cdot N_P(s)}{D_C(s)\cdot D_P(s)+N_C(s)\cdot N_P(s)}$

Closed loop denominator $= D_C(s)\cdot D_P(s)+N_C(s)\cdot N_P(s)$

$$C(s)\cdot P(s) = \frac{s+5}{\left(s^2+4\cdot s+3\right)}\cdot\frac{s+1}{\left(s^2+2\cdot s\right)}$$

Closed loop denominator $= \left(s^2+4\cdot s+3\right)\cdot\left(s^2+2\cdot s\right)+(s+5)\cdot(s+1)$

$$D_H(s) = s^4+6\cdot s^3+12\cdot s^2+12\cdot s+5$$

Routh-Hurwitz Stability test

Test denominator for poles in CRHP (RHP including imaginary axis)

1. All coefficients must be > 0

   For a second-order denominator, that is enough

2. Create Routh-Hurwitz array:

$$D_H(s) = s^4+6\cdot s^3+12\cdot s^2+12\cdot s+5$$

(RH Ex.1)

s^4	1	12	5	0
s^3	6	12	0	
s^2	$\dfrac{6\cdot 12-1\cdot 12}{6}=10$	$\dfrac{6\cdot 5-1\cdot 0}{6}=5$		
s^1	$\dfrac{10\cdot 12-6\cdot 5}{10}=9$	$\dfrac{10\cdot 0-6\cdot 0}{10}=0$		
s^0	$\dfrac{9\cdot 5-10\cdot 0}{9}=5$			

Look at first column: All positive, so
All roots left of imaginary axis, so tentative answers above are correct

If any were negative or 0, then
$D_H(s)$ would have poles on the Imaginary axis or in the RHP (Right-Half Plane)

Alternatively, check the actual roots

Using your calculator, find the roots of:

$$0 = s^4+6\cdot s^3+12\cdot s^2+12\cdot s+5$$

Roots: $\begin{bmatrix} -1 \\ -3.359 \\ -0.82-0.903\cdot j \\ -0.82+0.903\cdot j \end{bmatrix}$

roots all negative, stable

So tentative answers above are correct

More Routh-Hurwitz method examples

RH Ex.2 Given a cloosed-loop denominator: $D(s) = s^4 + 10 \cdot s^3 + 3 \cdot s^2 + 5 \cdot s + 2$ Are all the poles in the OLHP?

s^4	1	3	2	0
s^3	10	5	0	
s^2	$\frac{10 \cdot 3 - 1 \cdot 5}{10} = 2.5$	$\frac{10 \cdot 2 - 1 \cdot 0}{10} = 2$		
s^1	$\frac{2.5 \cdot 5 - 10 \cdot 2}{2.5} = -3$	0		
s^0	$\frac{-3 \cdot 2 - 1 \cdot 0}{-3} = 2$			

Two sign reversals = two problem poles, in the RHP   NO

Actual roots: $\begin{bmatrix} 0.062 + 0.732 \cdot j \\ 0.062 - 0.732 \cdot j \\ -0.381 \\ -9.743 \end{bmatrix}$ Two roots positive

RH Ex.3

$C(s) = \frac{3 \cdot s^2 + 8}{s^3 + 2 \cdot s^2 + 4 \cdot s}$ $P(s) = \frac{2}{s^2 + 3}$ (Notice that the Plant is not inherently stable)

$C(s) \cdot P(s) = \frac{3 \cdot s^2 + 8}{\left(s^3 + 2 \cdot s^2 + 4 \cdot s\right)} \cdot \frac{2}{\left(s^2 + 3\right)}$ Closed loop denominator $= \left(s^3 + 2 \cdot s^2 + 4 \cdot s\right) \cdot \left(s^2 + 3\right) + \left(3 \cdot s^2 + 8\right) \cdot 2$

$= s^5 + 2 \cdot s^4 + 7 \cdot s^3 + 12 \cdot s^2 + 12 \cdot s + 16$

Routh-Hurwitz array:

s^5	1	7	12	0
s^4	2	12	16	
s^3	$\frac{2 \cdot 7 - 1 \cdot 12}{2} = 1$	$\frac{2 \cdot 12 - 1 \cdot 16}{2} = 4$	0	
s^2	$\frac{1 \cdot 12 - 2 \cdot 4}{1} = 4$	$\frac{1 \cdot 16 - 2 \cdot 0}{1} = 16$		
s^1	$\frac{4 \cdot 4 - 1 \cdot 16}{4} = 0$ ($-\varepsilon$)	0		
s^0	$\frac{-\varepsilon \cdot 16}{-\varepsilon}$			

\ Problem, some root(s) in CRHP
Consider this $-\varepsilon$ & you get 2 sign changes, 2 unstable poles

\ Doesn't make sense to progress to the next row if all you want to know is stability, but if you count above as $-\varepsilon$, this answer would come out +, indicating two problem poles

Actual roots: $\begin{bmatrix} 2 \cdot j \\ -2 \cdot j \\ -1.651 \\ -0.175 - 1.547 \cdot j \\ -0.175 + 1.547 \cdot j \end{bmatrix}$ First 2 roots are on imaginary axis, unstable

Closed-loop transfer-function denominator	Transfer function stable?	
a) $D(s) = s^5 + 3 \cdot s^4 - 18 \cdot s^3 + 3 \cdot s^2 + s + 2$	No	The third coefficient is negative, there must be root(s), & thus poles, in the closed right half plane.
b) $D(s) = s^6 + 3 \cdot s^4 + 18 \cdot s^3 + 3 \cdot s^2 + s + 2$	No	The s^5 coefficient is zero, there must be root(s) in the closed right half plane.
c) $D(s) = s^6 + 3 \cdot s^5 + 18 \cdot s^4 + 3 \cdot s^3 + s^2 + 2 \cdot s$	No	The last coefficient is zero, there must be root(s) in the closed right half plane.
d) $D(s) = \left(s^2 + 2 \cdot s + 5\right) \cdot \left(s^2 + 4 \cdot s + 4\right)$ (Example 1 in text)	Yes	Neither factor has unstable poles so together they also have none. Don't multiply and complicate matters
e) $D(s) = \left(s^2 - 2 \cdot s + 5\right) \cdot \left(s^2 + 4 \cdot s + 4\right)$ (Example 2 in text)	No	First factor has at least one unstable pole, so together they also have at least one. Don't multiply and complicate matters
f) $D(s) = s^5 + 4 \cdot s^4 + 2 \cdot s^3 + 6 \cdot s^2 + 2 \cdot s + 1$		Can't tell without the full array

RH Ex.4

s^4	1	2	2	0
s^3	4	6	1	0
s^2	$\dfrac{4 \cdot 2 - 1 \cdot 6}{4} = 0.5$	$\dfrac{4 \cdot 2 - 1 \cdot 1}{4} = 1.75$		
s^1	$\dfrac{0.5 \cdot 6 - 4 \cdot 1.75}{0.5} = -8$	$\dfrac{0.5 \cdot 1 - 4 \cdot 0}{0.5} = 1$		
s^0	$\dfrac{-8 \cdot 1.75 - 0.5 \cdot 1}{-8} = 1.813$			

\ Problem, some root(s) in CRHP

\ No need to progress to the next row if all you want to know is stability, but this extra steps can tell you there are two problem poles

Actual roots: $\begin{bmatrix} -3.855 \\ -0.187 - 0.4 \cdot j \\ -0.187 + 0.4 \cdot j \\ 0.114 - 1.147 \cdot j \\ 0.114 + 1.147 \cdot j \end{bmatrix}$ Last two roots are in the RHP

No Not stable

RH Ex.5 Use the Routh-Hurwitz method to determine the value range of K that will produce a stable system.

$D(s) = s^4 + 2 \cdot s^3 + 1 \cdot s^2 + s + K$ Characteristic equation of a feedback sytem.

s^4	1	1	K	0
s^3	2	1	0	
s^2	$\dfrac{2 \cdot 1 - 1 \cdot 1}{2} = 0.5$	$\dfrac{2 \cdot K - 1 \cdot 0}{2} = K$		
s^1	$\dfrac{0.5 \cdot 1 - 2 \cdot K}{0.5} = 1 - 4 \cdot K$	0		
s^0	$\dfrac{(1 - 4 \cdot K) \cdot K - 0.5 \cdot 0}{1 - 4 \cdot K} = K$			

$K > 0$ $\quad 0 = 1 - 4 \cdot K$ $\quad K < \dfrac{1}{4} = 0.25$

$0 < K < 0.25$

RH Ex.6 Use the Routh-Hurwitz method to determine the value range of K that will produce a stable system.

$D(s) = s^4 + 4\cdot K\cdot s^3 + 12\cdot s^2 + 2\cdot K\cdot s + K$

s^4	1	12	K	0
s^3	$4\cdot K$	$2\cdot K$	0	
s^2	$\frac{4\cdot K\cdot 12 - 1\cdot 2\cdot K}{4\cdot K} = 11.5$	$\frac{4\cdot K\cdot K - 1\cdot 0}{4\cdot K} = K$		
s^1	$\frac{11.5\cdot 2\cdot K - 4\cdot K\cdot K}{11.5} = 2\cdot K - \frac{4}{11.5}\cdot K^2$	0		
s^0	$\frac{\left(2\cdot K - \frac{4}{11.5}\cdot K^2\right)\cdot K}{2\cdot K - \frac{4}{11.5}\cdot K^2} = K$			

$K > 0$ This could have been seen from the original expression

$0 < 2 - \frac{4}{11.5}\cdot K$ $\quad K < 2\cdot\frac{11.5}{4} = 5.75$

$0 < K < 5.75$

RH Ex.7 Use the Routh-Hurwitz method to determine if all the poles are to the left of - 5.

$D(s) = s^3 + 44\cdot s^2 + 320\cdot s + 648$ Characteristic equation of a feedback sytem.

Replace all occurances of s with (s - 5)

$(s-5)^3 + 44\cdot(s-5)^2 + 320\cdot(s-5) + 648$

$\left(s^3 - 15\cdot s^2 + 75\cdot s - 125\right) + 44\cdot\left(s^2 - 10\cdot s + 25\right) + 320\cdot(s-5) + 648$

$s^3 - 15\cdot s^2 + 75\cdot s - 125 + 44\cdot s^2 - 44\cdot 10\cdot s + 44\cdot 25 + 320\cdot s - 320\cdot 5 + 648 \quad = \quad s^3 + 29\cdot s^2 - 45\cdot s + 23$

No, this has a negative coefficient

RH Ex.7b Are all the poles are to the left of - 4?

Replace all occurances of s with (s - 4)

$(s-4)^3 + 44\cdot(s-4)^2 + 320\cdot(s-4) + 648$

$\left(s^3 - 12\cdot s^2 + 48\cdot s - 64\right) + 44\cdot\left(s^2 - 8\cdot s + 16\right) + 320\cdot(s-4) + 648$

$s^3 - 12\cdot s^2 + 48\cdot s - 64 + 44\cdot s^2 - 44\cdot 8\cdot s + 44\cdot 16 + 320\cdot s - 320\cdot 4 + 648 \quad = \quad s^3 + 32\cdot s^2 + 16\cdot s + 8$

s^3	1	16	0
s^2	32	8	0
s^1	$\frac{32\cdot 16 - 1\cdot 8}{32} = 15.75$	$\frac{32\cdot 0 - 1\cdot 0}{32} = 0$	
s^0	$\frac{15.75\cdot 8 - 32\cdot 0}{15.75} = 8$		

Look at first column: All positive, so all roots are indeed left of -4.

Actual roots of:

$0 = s^3 + 44\cdot s^2 + 320\cdot s + 648 \quad \begin{pmatrix} -35.5 \\ -4.25 - 0.438\cdot j \\ -4.25 + 0.438\cdot j \end{pmatrix}$ Sure enough, all roots are left of -4, and not all left of -5