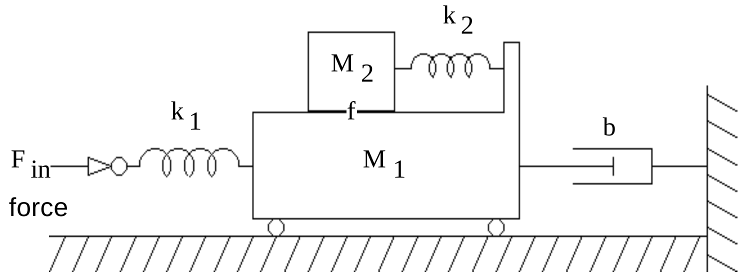
A.Stolp
2/14/06,
rev, 2/4/24

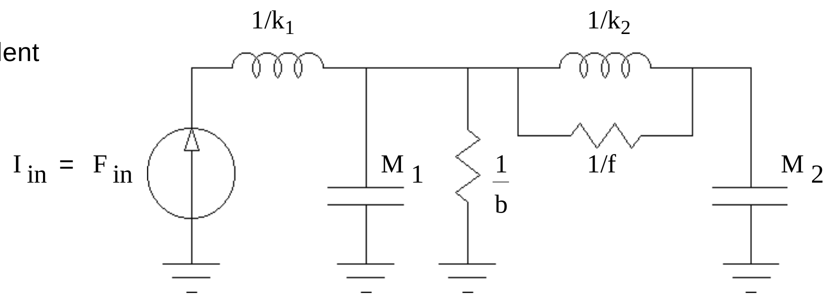
This method is based on the method of Linear Graphs. In Linear Graphs, all systems are reduced to a universal symbology peculiar to Linear Graphs (system graph) and then analyzed by methods very similar to those used with electric circuits. Since we, as electrical engineers, are already well versed in the symbology of electric circuits, my variation of linear graphs reduces mechanical systems to electric circuits and leaves them in that symbology for ready analysis by methods, tools and computer programs that we are already familiar with.

The objective is to be able to take a mechanical system like this:



And be able to draw an equivalent electrical circuit like this:

Because we know how to analyze the circuit:



Lecture: Skip forward to "Mechanical system with linear motion (translational)" Material below is for later reference.

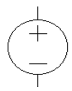
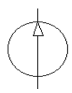
Across and Through Variables

	Across Variable	Through Variable
Electrical	V = voltage (volts) or (V)	I = current (Amps) or (A)
Mechanical translational	v = velocity $\left(\frac{\text{m}}{\text{sec}}\right)$	F = force (newtons) or (N) or $\left(\text{Kg}\cdot\frac{\text{m}}{\text{sec}^2}\right)$
Mechanical rotational	ω = angular velocity $\left(\frac{\text{rad}}{\text{sec}}\right)$	T = torque $(\text{N}\cdot\text{m})$
Fluid	P = pressure $\left(\frac{\text{N}}{\text{m}^2}\right)$ or (Pa)	Q = flow $\left(\frac{\text{m}^3}{\text{sec}}\right)$

Elements

	Dissipation	Across Variable Energy Storage	Through Variable Energy Storage
Electrical	R = resistance $\left(\frac{\text{V}}{\text{A}}\right)$ or (Ω)	C = capacitance $\left(\frac{\text{A}\cdot\text{sec}}{\text{V}}\right)$ or (F)	L = inductor $\left(\frac{\text{V}\cdot\text{sec}}{\text{A}}\right)$ or (H)
Mechanical translational	B = damping $\left(\frac{\text{N}\cdot\text{sec}}{\text{m}}\right)$	M = mass (Kg) or $\left(\frac{\text{N}\cdot\text{sec}^2}{\text{m}}\right)$	k = Spring constant $\left(\frac{\text{N}}{\text{m}}\right)$
Mechanical rotational	B = damping $(\text{N}\cdot\text{m}\cdot\text{sec})$ or $\left[\frac{\text{N}\cdot\text{m}}{\left(\frac{\text{rad}}{\text{sec}}\right)}\right]$	J = moment of inertia $\left(\text{Kg}\cdot\text{m}^2\right)$ or $\left(\frac{\text{N}\cdot\text{m}^3}{\text{sec}^2}\right)$	k = Spring constant $\left(\frac{\text{N}\cdot\text{m}}{\text{rad}}\right)$
Fluid	R_f = fluid resistance $\left(\frac{\text{N}\cdot\text{sec}}{\text{m}^5}\right)$	C_f = fluid capacitance $\left(\frac{\text{m}^5}{\text{N}}\right)$	I = fluid inertia $\left(\frac{\text{Kg}}{\text{m}^4}\right)$

Basic Electric Circuit Analysis

Element	Parts like resistors, capacitors, inductors & transformers
Wires and connections	Direct the current, but do not affect voltage
Circuit	Wires and elements connected to form loops
Voltage	Measured as a difference **across** an element
Current	Flows **through** a wire or element
Kirchhoff's Current Law (KCL)	Current in = current out of all elements, wires & connections
Kirchhoff's Voltage Law (KVL)	Voltage gains = voltage "losses" around any circuit loop
Node	Connected wires and connections which all have the same voltage
Ground	Zero-reference node for all other nodal voltages
Branch	Connected wires and elements which all have the same current
Power $P = V \cdot I$	Power = Across variable x Through variable
Voltage Source 	Constant voltage regardless of current in or out
Current Source 	Constant current regardless of voltage + or -

Passive Electrical Elements

Resistors

$V = I \cdot R$

Resistors dissipate power $P = V \cdot I = I^2 \cdot R = \dfrac{V^2}{R}$

Capacitors

$$C = \frac{Q}{V} \qquad \text{farad} = \frac{\text{coul}}{\text{volt}} = \frac{\text{amp} \cdot \text{sec}}{\text{volt}} \qquad v_C = \frac{1}{C} \cdot \int_{-\infty}^{t} i_C \, dt = \frac{1}{C} \cdot \int_0^t i_C \, dt + \overset{\text{initial voltage}}{v_C(0)} \qquad i_C = C \cdot \frac{d}{dt} v_C$$

Energy stored in electric field: $E_C = \frac{1}{2} \cdot C \cdot V_C^{\,2}$

Capacitor voltage **cannot** change instantaneously

Laplace: Impedance: $\mathbf{Z_C} = \dfrac{1}{C \cdot s}$

Inductors

$$\text{henry} = \frac{\text{volt} \cdot \text{sec}}{\text{amp}} \qquad i_L = \frac{1}{L} \cdot \int_{-\infty}^{t} v_L \, dt = \frac{1}{L} \cdot \int_0^t v_L \, dt + \overset{\text{initial current}}{i_L(0)} \qquad v_L = L \cdot \frac{d}{dt} i_L$$

Energy stored in magnetic field: $E_L = \frac{1}{2} \cdot L \cdot I_L^{\,2}$

Inductor current **cannot** change instantaneously

Laplace: Impedance: $\mathbf{Z_L} = L \cdot s$

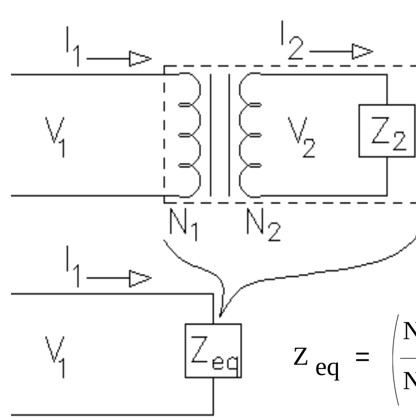
Transformers (ideal)

Ideal: $P_1 = P_2$ power in = power out

Turns ratio = $N = \dfrac{N_1}{N_2} = \dfrac{V_1}{V_2} = \dfrac{I_2}{I_1}$

Note: some books define the turns ratio as N_2/N_1

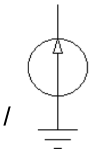
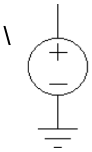
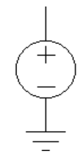
Equivalent impedance in primary: $\mathbf{Z_{eq}} = N^2 \cdot \mathbf{Z_2} = \left(\dfrac{N_1}{N_2}\right)^2 \cdot \mathbf{Z_2}$



$Z_{eq} = \left(\dfrac{N_1}{N_2}\right)^2 \cdot Z_2$

You can replace the entire transformer and load with ($\mathbf{Z_{eq}}$).
This "impedance transformation" can work across systems.

Mechanical system with linear motion (translational)

	Mechanical translational	**Electrical**	
Through Variable:	F = Force (N)	I = current (A)	Source: 
Across Variable:	v = velocity $\left(\frac{m}{sec}\right)$	V = voltage (V)	Source: 
$\int v\,dt$	x = displacement (m)	Source:  $v = \frac{d}{dt}x$ or $s \cdot X(s)$	
$\frac{V(s)}{s}$	X(s) = displacement (m·sec) (in freq domain)		

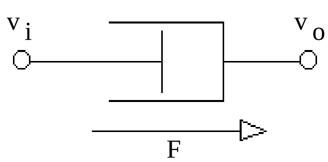
Dissipation element:

power

$$P = v \cdot F = \frac{F^2}{B}$$
$$= v^2 \cdot B$$

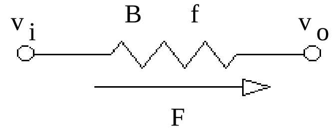
Damper or friction

B or f



Resistor

$\frac{1}{B}$ or $\frac{1}{f}$



Impedance

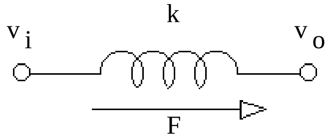
$\frac{1}{B}$ or $\frac{1}{f}$

Through variable energy storage:

$$E = \frac{1}{2} \cdot \frac{1}{k} \cdot F^2 \underset{(F=kx)}{=} \frac{1}{2} \cdot k \cdot x^2$$

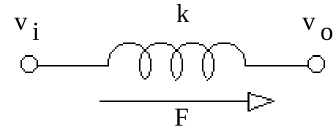
Spring

k



Inductor

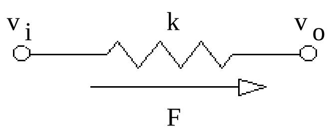
$\frac{1}{k}$



$\frac{s}{k}$

Springs are sometimes shown like this:

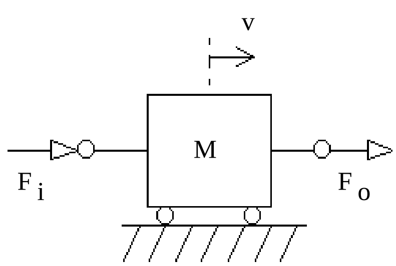
k



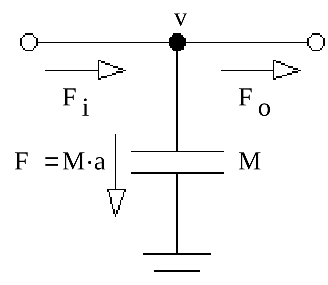
Across variable energy storage:

$$E = \frac{1}{2} \cdot M \cdot v^2$$

Mass

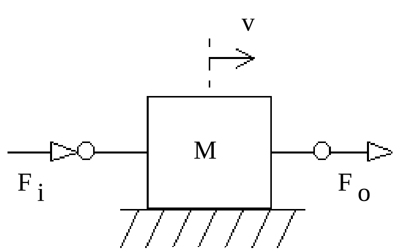


Capacitor hooked to ground



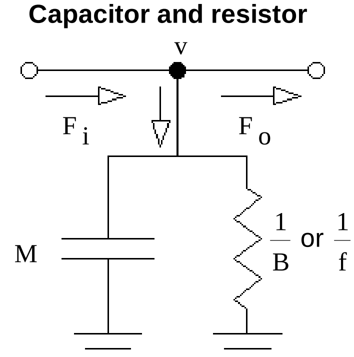
$\frac{1}{M \cdot s}$

Mass with friction



B or f
sliding friction

Capacitor and resistor



$$\frac{1}{\frac{1}{\left(\frac{1}{M \cdot s}\right)} + \frac{1}{\left(\frac{1}{B}\right)}}$$

$$\frac{1}{M \cdot s + B}$$

Do Examples 1 and 2

Transformers (ideal)

Two coils of wire that are magnetically coupled.

Electrical transformers are only useful for AC, which is not true of mechanical transformers

Transformers are used to increase/decrease voltages/currents.

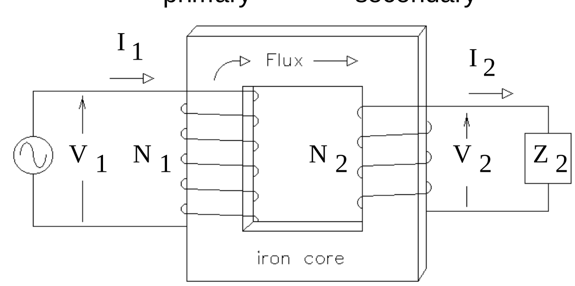
Ideal: $P_1 = P_2$

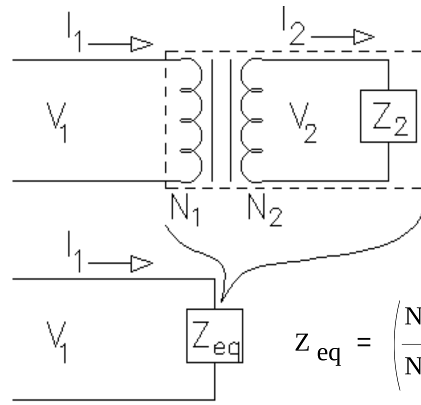
power in = power out

Turns ratio = $N = \dfrac{N_1}{N_2} = \dfrac{\mathbf{V_1}}{\mathbf{V_2}} = \dfrac{\mathbf{I_2}}{\mathbf{I_1}}$

Note: some books define the turns ratio as N_2/N_1

Iron-core transformer





$Z_{eq} = \left(\dfrac{N_1}{N_2}\right)^2 \cdot Z_2$

Equivalent impedance in primary: $\mathbf{Z_{eq}} = N^2 \cdot \mathbf{Z_2} = \left(\dfrac{N_1}{N_2}\right)^2 \cdot \mathbf{Z_2}$

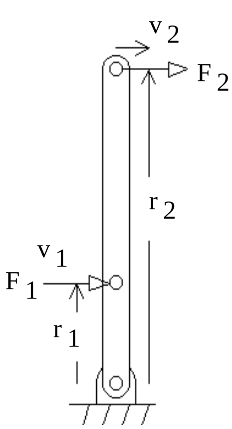
You can replace the entire transformer and load with ($\mathbf{Z_{eq}}$).
This "impedance transformation" can be very handy.

Transducers and Transformers

A transducer converts power from one type to another. We can model many of them with transformers. Transformers increase the through variable and correspondingly decrease the across variable or vice-versa.

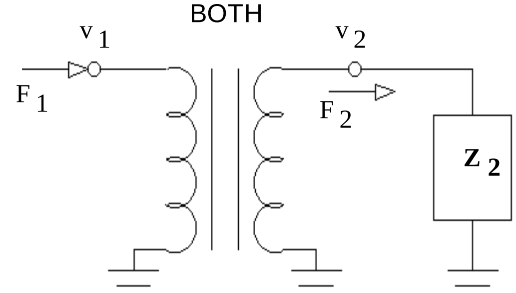
Mechanical "Transformers" (Translational motion)

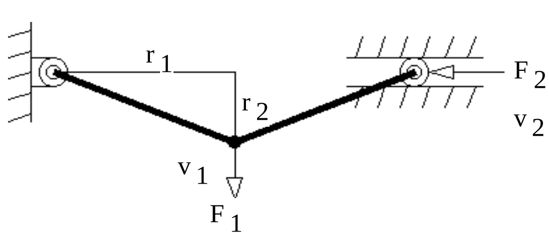
Levers



$N = \dfrac{r_1}{r_2} = \dfrac{v_1}{v_2} = \dfrac{F_2}{F_1}$

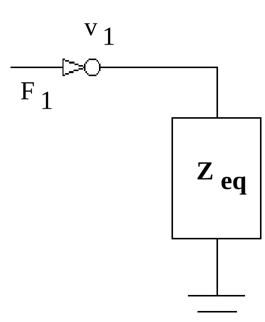
BOTH





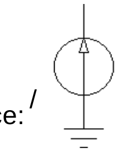
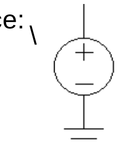
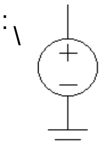
$N = \dfrac{r_1}{r_2} = \dfrac{v_1}{v_2} = \dfrac{F_2}{F_1}$

(not really this simple)



$\mathbf{Z_{eq}} = N^2 \cdot \mathbf{Z_2} = \left(\dfrac{r_1}{r_2}\right)^2 \cdot \mathbf{Z_2}$

Mechanical system with circular motion (rotational)

	Mechanical rotational	**Electrical**	
Through Variable:	T = Torque $(N \cdot m)$	I = current (A)	Source: 
Across Variable:	ω = angular velocity $\left(\frac{rad}{sec}\right)$	V = voltage (V)	Source: 
$\int \omega\, dt$	θ = angular displacement (rad)	Source:  $\omega = \frac{d}{dt}\theta$ or $s \cdot \theta(s)$	
$\frac{\omega(s)}{s}$	$\theta(s)$ = angular displacement $(rad \cdot sec)$ (in freq domain)		

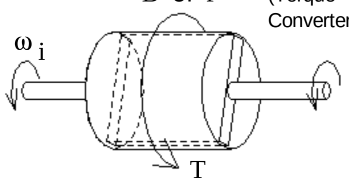
Dissipation element:

power

$$P = v \cdot T = \frac{T^2}{B}$$
$$= \omega^2 \cdot B$$

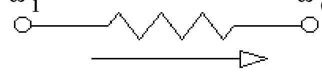
Damper or friction

B or f (Torque Converter)



Resistor

$\frac{1}{B}$ or $\frac{1}{f}$



Impedance

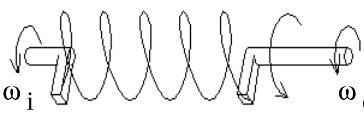
$\frac{1}{B}$ or $\frac{1}{f}$

Through variable energy storage:

$$E = \frac{1}{2} \cdot \frac{1}{k} \cdot T^2$$

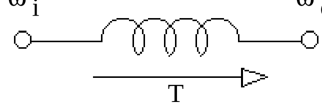
Springs

k



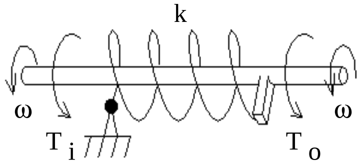
Inductor

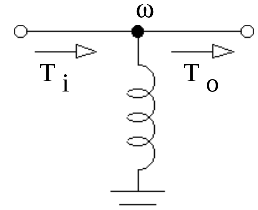
$\frac{1}{k}$



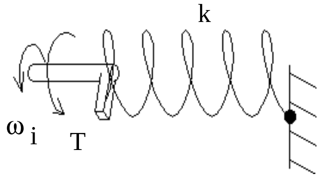
$\frac{s}{k}$

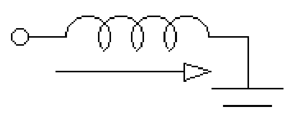
k





k

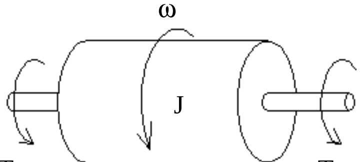




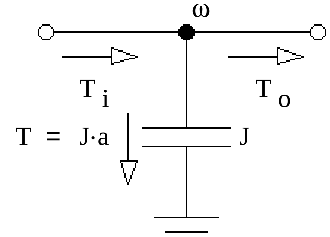
Across variable energy storage:

$$E = \frac{1}{2} \cdot J \cdot \omega^2$$

Moment of Inertia, J

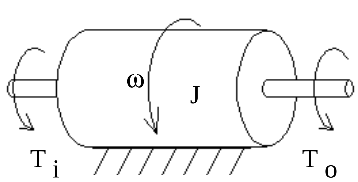


Capacitor hooked to ground



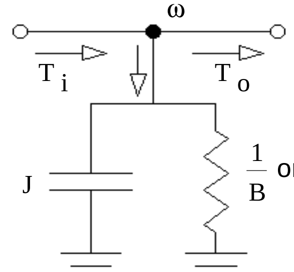
$\frac{1}{J \cdot s}$

J with friction



B or f
sliding friction

Capacitor and resistor



$\frac{1}{B}$ or $\frac{1}{f}$

$$\frac{1}{\frac{1}{\left(\frac{1}{J \cdot s}\right)} + \frac{1}{\left(\frac{1}{B}\right)}}$$

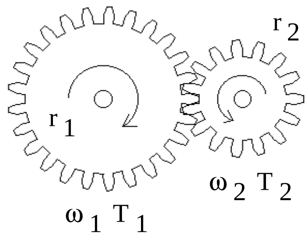
$$\frac{1}{J \cdot s + B}$$

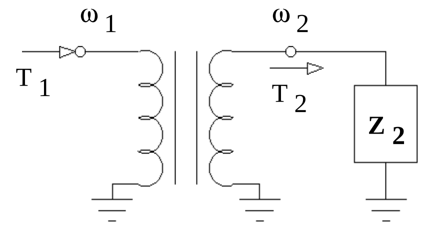
Do Example 6

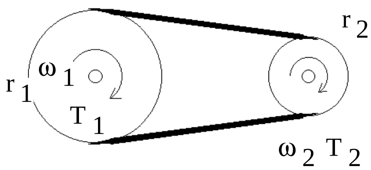
More Transducers and Transformers

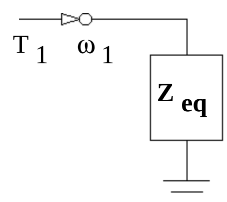
Belts, chains, & gears

r = pitch radius of pulley or of gears









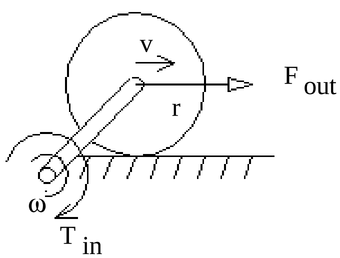
$$\mathbf{Z}_{eq} = N^2 \cdot \mathbf{Z}_2 = \left(\frac{r_2}{r_1}\right)^2 \cdot \mathbf{Z}_2$$

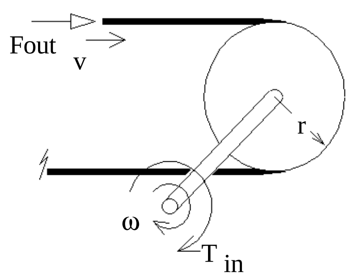
$$N = \frac{r_2}{r_1} = \frac{\omega_1}{\omega_2} = \frac{T_2}{T_1} = \text{gear tooth ratio} \left(\frac{N_2}{N_1}\right)$$

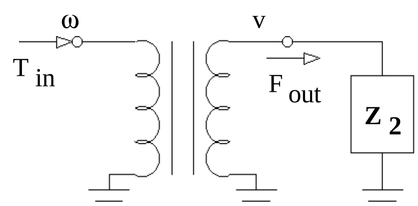
Transformation one type of motion or power to another requires a Transformer

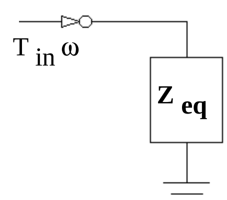
Tires, racks, & conveyors

r = radius of wheel or pitch radius of belt pulley









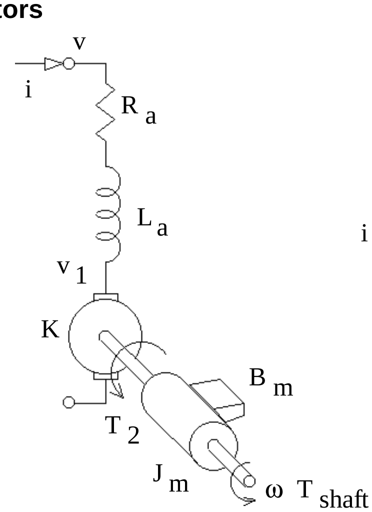
$$\mathbf{Z}_{eq} = N^2 \cdot \mathbf{Z}_2 = \left(\frac{1}{r}\right)^2 \cdot \mathbf{Z}_2$$

$$N = \frac{1}{r} = \frac{\omega}{v} = \frac{F}{T}$$

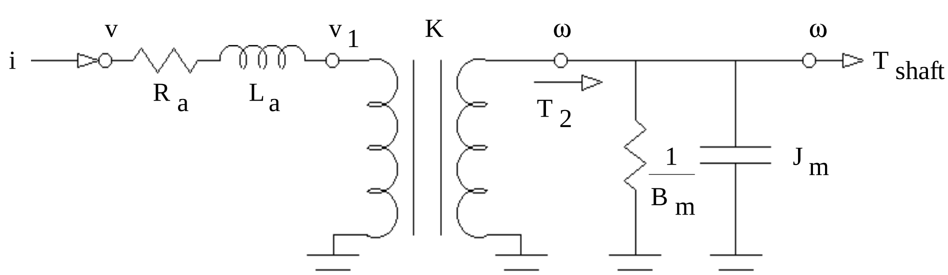
Note: $N = r$ if the input is linear motion and output is rotational.

Do Example 6

DC Motors

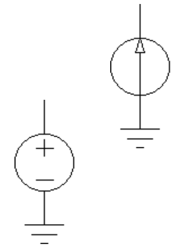


$$N = K = \frac{v_1}{\omega} = \frac{T_2}{i}$$



Fluid (hydraulic) system

Sources

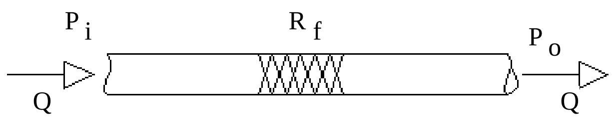


	Fluid	**Electrical**
Through Variable:	Q = volumetric flow rate $\left(\frac{m^3}{sec}\right)$	I = current (A)
Across Variable:	P = Pressure $\left(\frac{N}{m^2}\right)$ or (Pa)	V = voltage (V)

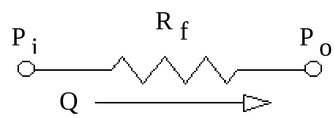
Dissipation element:
power

$$P = P \cdot Q = \frac{Q^2}{R_f} = P^2 \cdot R_f$$

Fluid resistance



Resistor



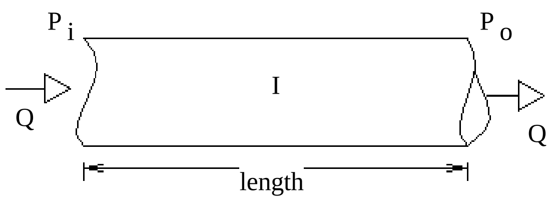
Impedance

R_f

Through variable
energy storage:

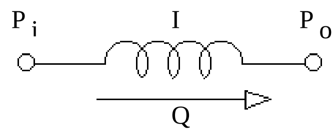
$$E = \frac{1}{2} \cdot I \cdot Q^2$$

Fluid Inertia



$$I = \frac{M}{A^2} = \frac{\rho \cdot length}{A}$$

Inductor

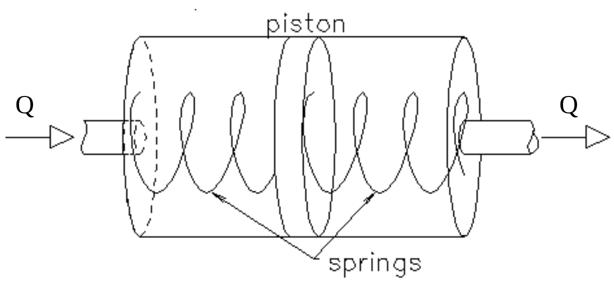


$I \cdot s$

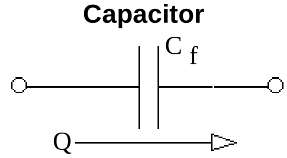
Across variable
energy storage:

$$E = \frac{1}{2} \cdot C_f \cdot P^2$$

Fluid Capacitors

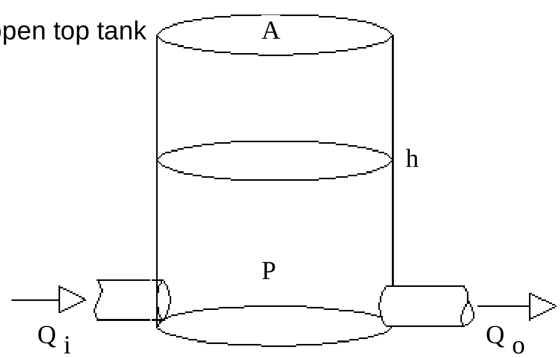


Capacitor

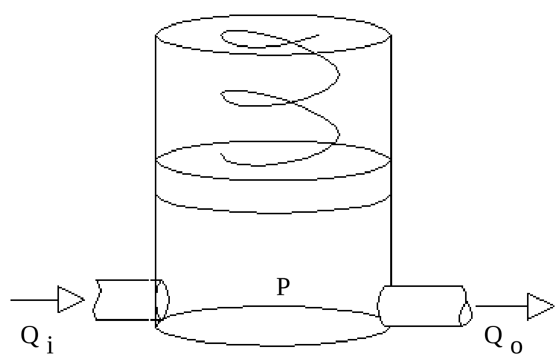


$$\frac{1}{C_f \cdot s}$$

open top tank



OR

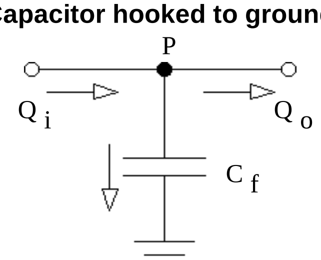


for all capacitors

$$C_f = \frac{\Delta volume}{\Delta pressure} = \frac{\Delta h \cdot A}{\Delta h \cdot \rho \cdot g} = \frac{A}{\rho \cdot g}$$

For open top tank

Capacitor hooked to ground



Gyrators

Pistons and Turbines convert through variables to across variables & vice versa, so there are no good electrical analogies.

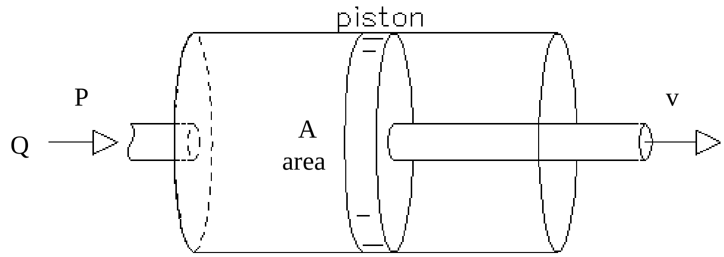
Yet you can still transform an impedance from a mechanical system into the fluid system. You'll find that capacitors become inductors, inductors become capacitors and parallel swaps with series.

Piston & Cylinder

Fluid power input

$P = \frac{F}{A}$

$Q = A\cdot v$



$v = \frac{Q}{A}$

$F = A\cdot P$

$\mathbf{Z}_{\mathbf{tran}} = \frac{v}{F} =$ an impedance representing the mechanical system connected here.

$$\text{fluid}\quad \mathbf{Z}_{\mathbf{eq}} = \frac{P}{Q} = \frac{\left(\frac{F}{A}\right)}{A\cdot v} = \frac{1}{A^2}\cdot\frac{F}{v} = \frac{1}{A^2}\cdot\frac{1}{\mathbf{Z}_{\mathbf{tran}}} = \frac{1}{A^2\cdot\mathbf{Z}_{\mathbf{tran}}}$$

If the input is mechanical linear motion power and the output is fluid power:

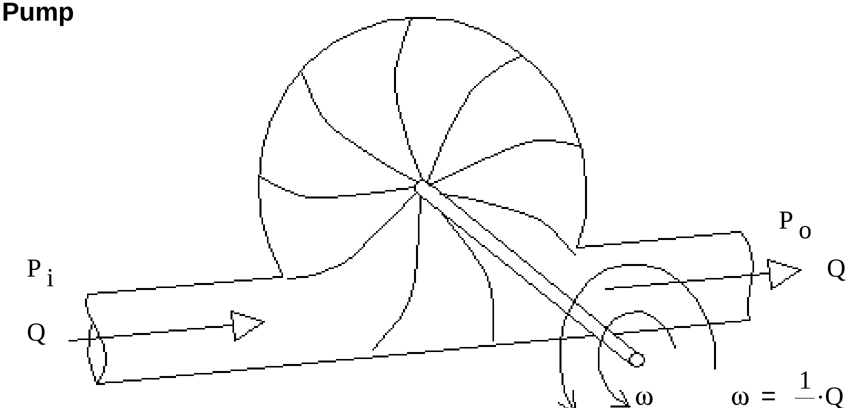
$$\text{translational mechcanical}\quad \mathbf{Z}_{\mathbf{eq}} = \frac{v}{F} = \frac{\left(\frac{Q}{A}\right)}{A\cdot P} = \frac{1}{A^2}\cdot\frac{Q}{P} = \frac{1}{A^2}\cdot\frac{1}{\mathbf{Z}_{\mathbf{fluid}}} = \frac{1}{A^2\cdot\mathbf{Z}_{\mathbf{fluid}}}$$

Do Example 5

Turbine or Pump

Fluid power input

(turbine)



$\omega = \frac{1}{K}\cdot Q$

$\Delta\cdot P = P_i - P_o$

T

$T = K\left(P_i - P_o\right)$

$= K\cdot\Delta P$

$\mathbf{Z}_{\mathbf{rot}} = \frac{\omega}{T} =$ an impedance representing the mechanical system connected to the rotating shaft.

$$\text{fluid}\quad \mathbf{Z}_{\mathbf{eq}} = \frac{\Delta\cdot P}{Q} = \frac{\left(\frac{T}{K}\right)}{K\cdot\omega} = \frac{1}{K^2}\cdot\frac{T}{\omega} = \frac{1}{K^2}\cdot\frac{1}{\mathbf{Z}_{\mathbf{rot}}} = \frac{1}{K^2\cdot\mathbf{Z}_{\mathbf{rot}}} \quad \text{(turbine)}$$

If the input is mechanical rotational power and the output is fluid power (pump):

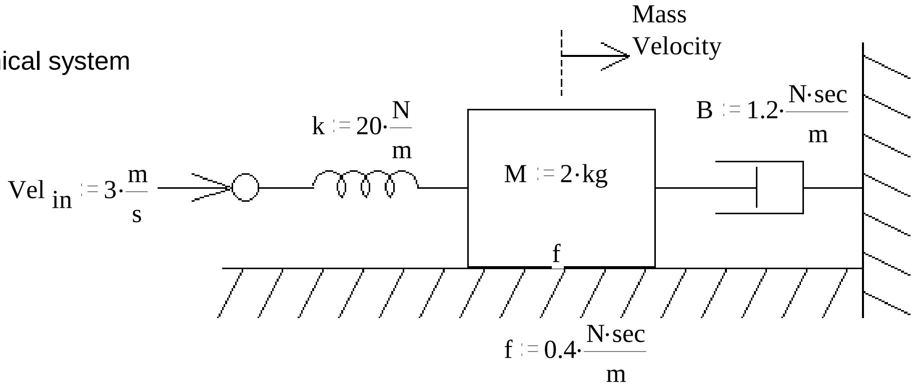
$$\text{rotational mechanical}\quad \mathbf{Z}_{\mathbf{eq}} = \frac{\omega}{T} = \frac{\left(\frac{Q}{K}\right)}{K\cdot\Delta P} = \frac{1}{K^2}\cdot\frac{Q}{\Delta P} = \frac{1}{K^2}\cdot\frac{1}{\mathbf{Z}_{\mathbf{fluid}}} = \frac{1}{K^2\cdot\mathbf{Z}_{\mathbf{fluid}}} \quad \text{(pump)}$$

Note: $\Delta P = P_o - P_i$ across the pump

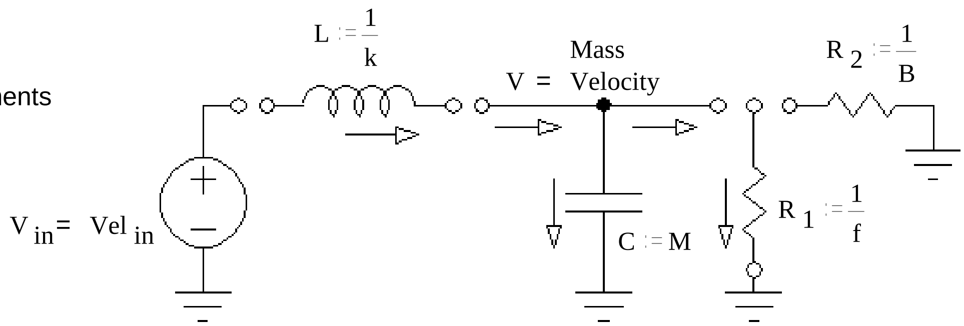
A.Stolp
2/11/09,
rev,1/30/15

Example 1

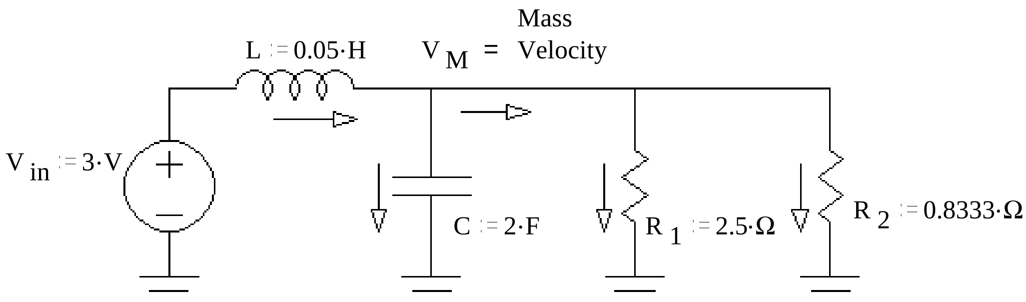
Mechanical system



Elements



Circuit



$$R_{eq} := \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}} \qquad R_{eq} = 0.625\cdot\Omega$$

$$B_{eq} := \frac{1}{R_{eq}}$$

Transfer function

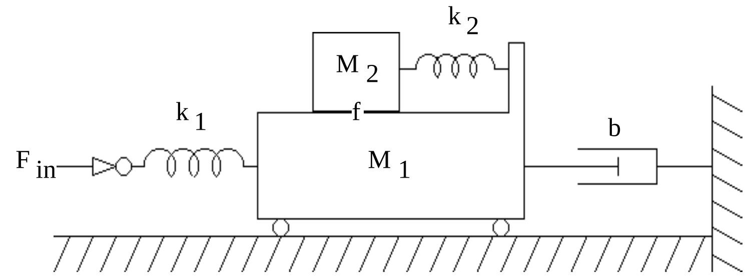
$$\frac{V_M(s)}{V_{in}(s)} = \frac{\frac{1}{C\cdot s+\frac{1}{R_{eq}}}}{L\cdot s+\frac{1}{C\cdot s+\frac{1}{R_{eq}}}}\cdot\frac{\left(C\cdot s+\frac{1}{R_{eq}}\right)}{\left(C\cdot s+\frac{1}{R_{eq}}\right)} = \frac{1}{L\cdot C\cdot s^2+\frac{L}{R_{eq}}\cdot s+1}\cdot\frac{\left(\frac{1}{L\cdot C}\right)}{\left(\frac{1}{L\cdot C}\right)} = \frac{\frac{1}{L\cdot C}}{s^2+\frac{1}{C\cdot R_{eq}}\cdot s+\frac{1}{L\cdot C}}$$

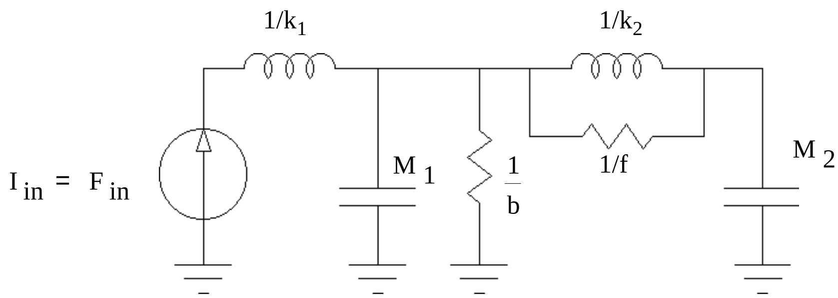
$$\frac{V_M(s)}{Vel_{in}(s)} = \frac{\frac{k}{M}}{s^2+\frac{B_{eq}}{M}\cdot s+\frac{k}{M}} = \frac{\frac{20\cdot N}{2\cdot kg\cdot m}}{s^2+\frac{1.6\cdot N\cdot sec}{2\cdot kg\cdot m}\cdot s+\left(\frac{20\cdot N}{2\cdot kg\cdot m}\right)} = \frac{10}{s^2+0.8\cdot s+10}$$

without units

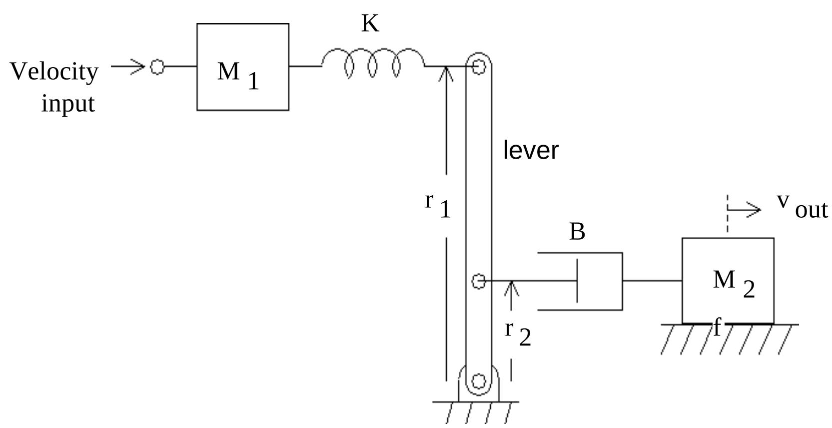
same, either way

Example 2

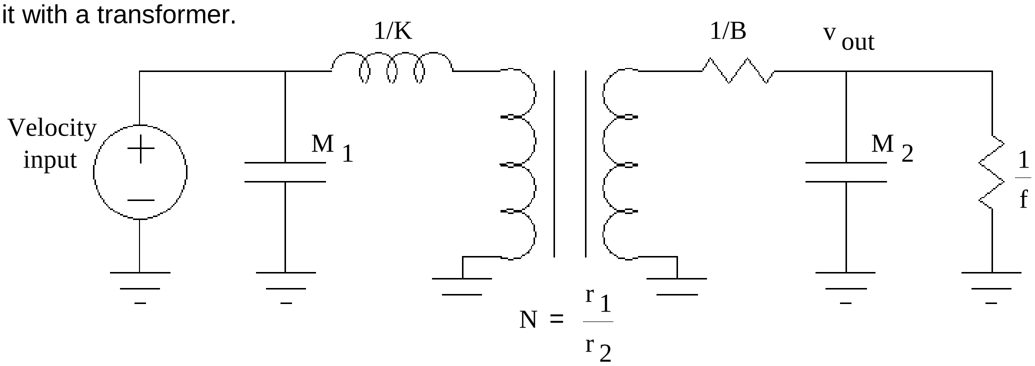




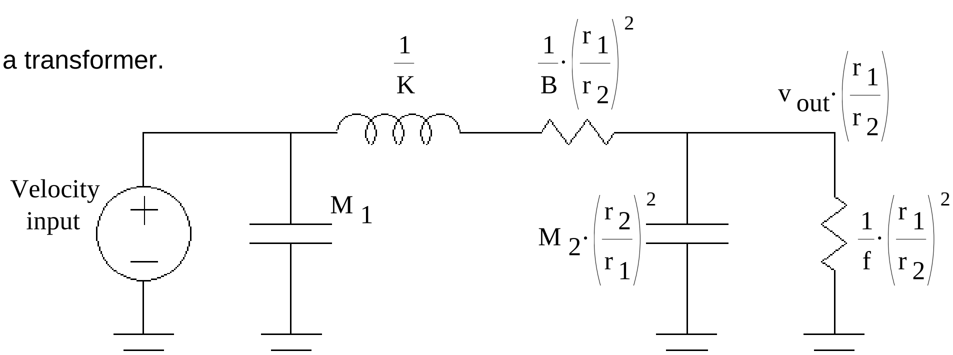
Example 3



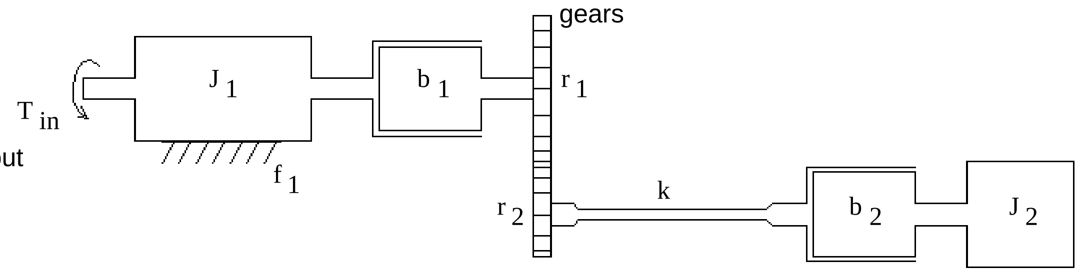
Circuit with a transformer.



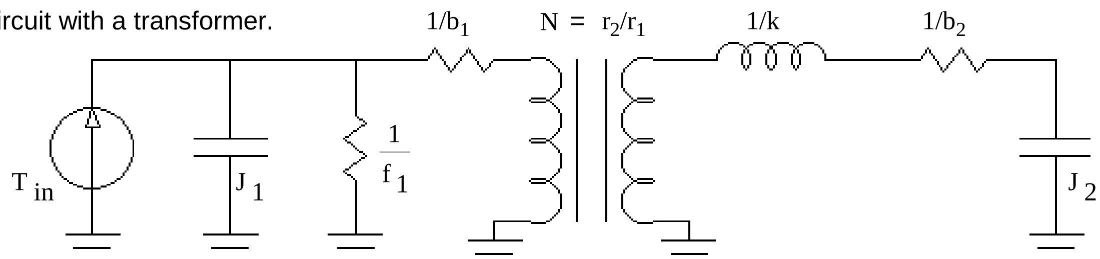
Circuit without a transformer.



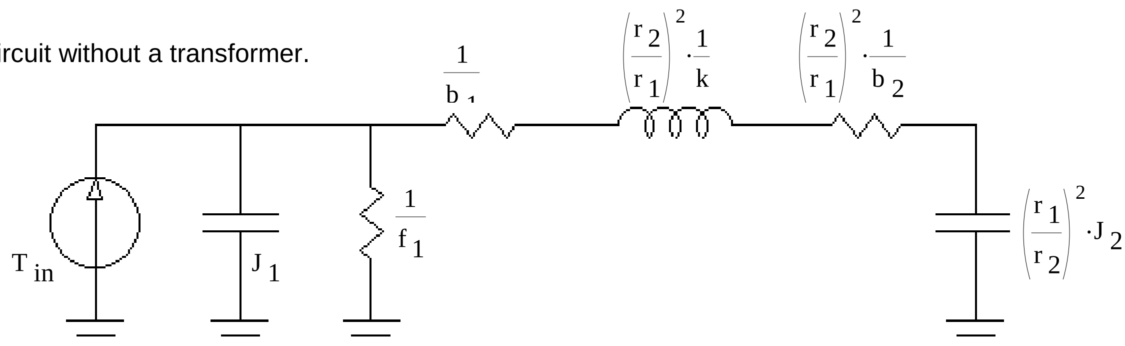
Example 4



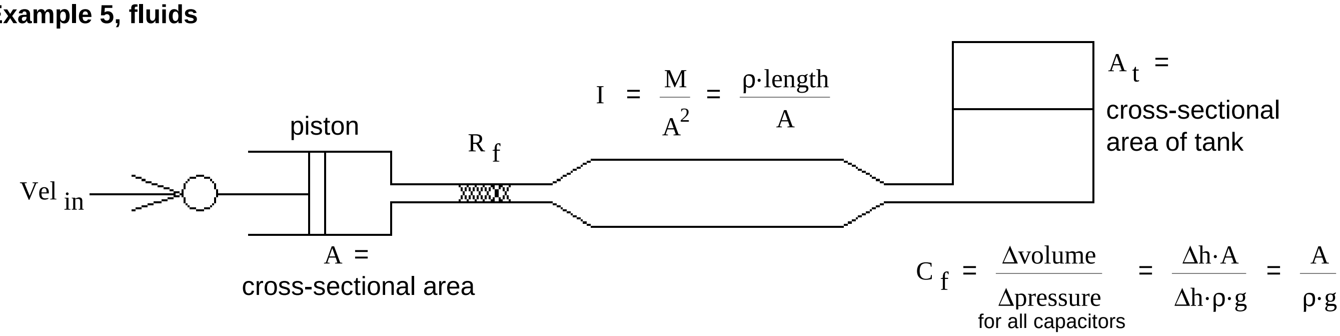
Circuit with a transformer.

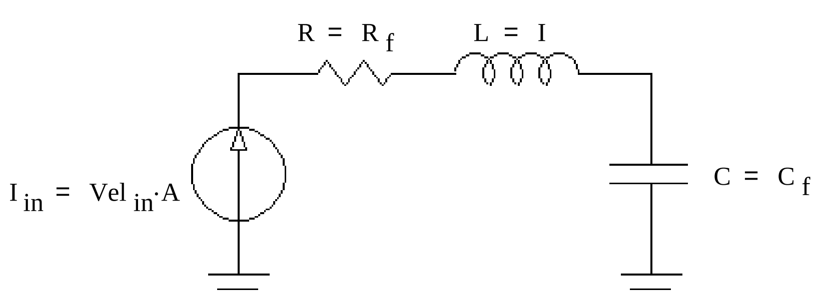


Circuit without a transformer.

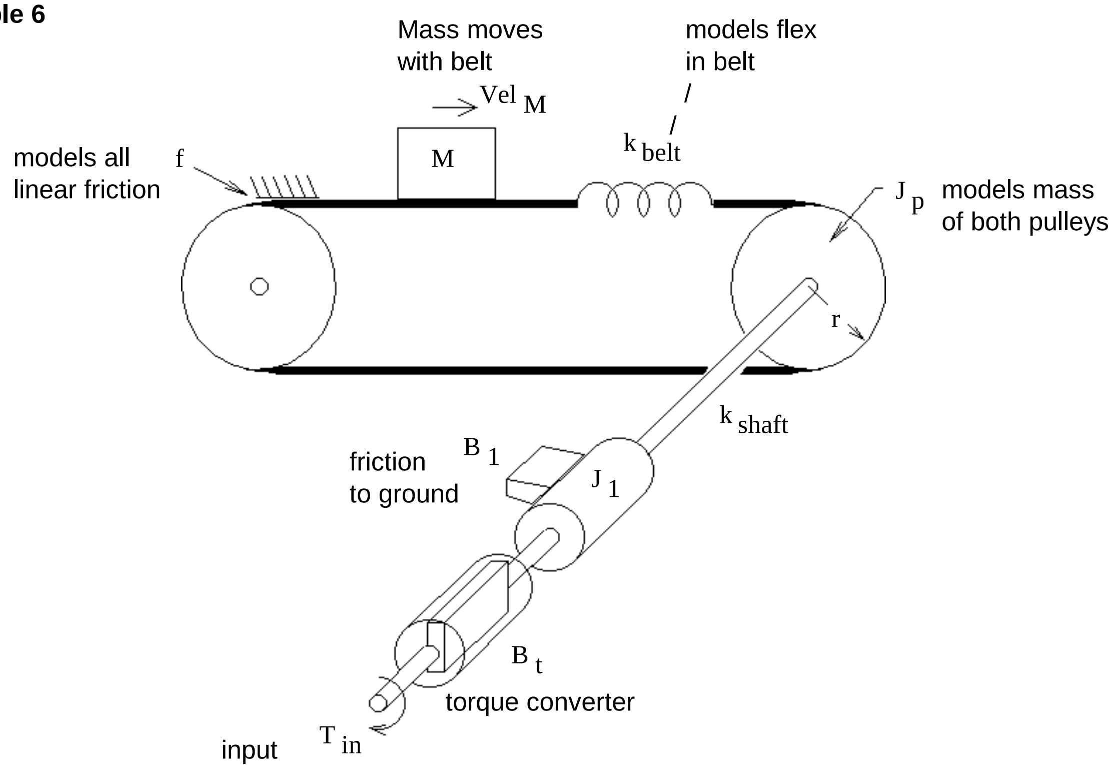


Example 5, fluids

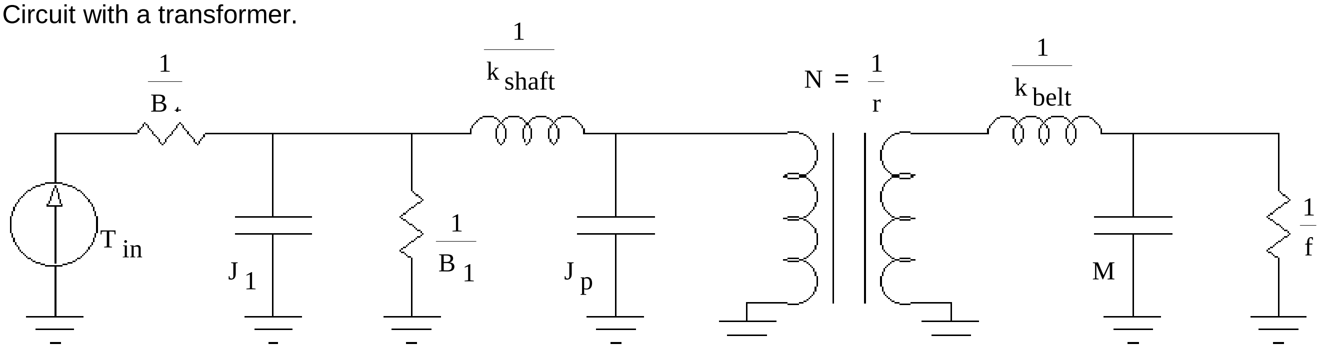




Example 6



Circuit with a transformer.



Circuit without a transformer.