Complex Numbers

ECE 3510

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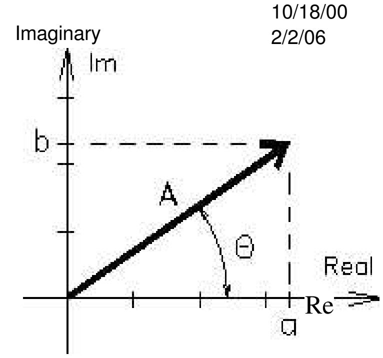
10/18/00
2/2/06



$j = \sqrt{-1}$ the imaginary number

Rectangular Form $\mathbf{A} = a + b \cdot j$

$\mathrm{Re}(\mathbf{A}) = a \qquad \mathrm{Im}(\mathbf{A}) = b$

Polar Form $\mathbf{A} = A \cdot e^{j \cdot \theta}$

$\mathrm{Re}(\mathbf{A}) = A \cdot \cos(\theta) \qquad \mathrm{Im}(\mathbf{A}) = A \cdot \sin(\theta)$

Conversions $A = |\mathbf{A}| = \sqrt{a^2 + b^2} \qquad \theta = \arg(\mathbf{A}) = \mathrm{atan}\left(\frac{b}{a}\right)$

$a = A \cdot \cos(\theta) \qquad b = A \cdot \sin(\theta)$

$\mathbf{A} = A \cdot e^{j \cdot \theta} = A \cdot \cos(\theta) + A \cdot \sin(\theta) \cdot j \qquad \mathbf{A} = a + b \cdot j = \left(\sqrt{a^2 + b^2}\right) \cdot e^{j \cdot \mathrm{atan}\left(\frac{b}{a}\right)}$

Special Cases $j := \sqrt{-1} = e^{j \cdot 90 \cdot deg} \qquad \frac{1}{j} = -j = e^{-j \cdot 90 \cdot deg} \qquad e^{j \cdot 0 \cdot deg} = 1 \qquad e^{-j \cdot 180 \cdot deg} = e^{-j \cdot 180 \cdot deg} = -1$

$j \cdot e^{j \cdot \theta} = e^{j \cdot (\theta + 90 \cdot deg)}$

Define a 2nd number: rect: $\mathbf{D} = c + d \cdot j$ polar: $\mathbf{D} = D \cdot e^{j \cdot \phi}$

Equality $\mathbf{A} = \mathbf{D}$ if and only if $a = c$ and $b = d$ OR $A = D$ and $\theta = \phi$

Addition and Subtraction

$\mathbf{A} + \mathbf{D} = (a + b \cdot j) + (c + d \cdot j) = (a + c) + (b + d) \cdot j$

$\mathbf{A} - \mathbf{D} = (a + b \cdot j) - (c + d \cdot j) = (a - c) + (b - d) \cdot j$

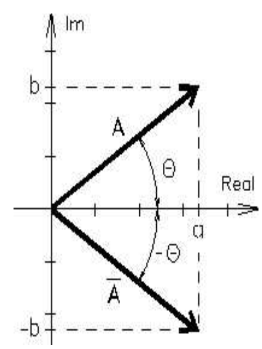
Convert polars to rectangular form first

Multiplication and Division $\mathbf{A} \cdot \mathbf{D} = (a + b \cdot j) \cdot (c + d \cdot j) = (a \cdot c - b \cdot d) + (b \cdot c + a \cdot d) \cdot j$

Rectangular: $\frac{\mathbf{A}}{\mathbf{D}} = \frac{a + b \cdot j}{c + d \cdot j} = \frac{a + b \cdot j}{c + d \cdot j} \cdot \frac{c - d \cdot j}{c - d \cdot j} = \frac{a \cdot c + b \cdot d}{c^2 + d^2} + \frac{b \cdot c - a \cdot d}{c^2 + d^2} \cdot j$

Polar: $\mathbf{A} \cdot \mathbf{D} = A \cdot e^{j \cdot \theta} \cdot D \cdot e^{j \cdot \phi} = A \cdot D \cdot e^{j \cdot (\theta + \phi)}$

$\frac{\mathbf{A}}{\mathbf{D}} = \frac{A \cdot e^{j \cdot \theta}}{D \cdot e^{j \cdot \phi}} = \frac{A}{D} \cdot e^{j \cdot (\theta - \phi)}$



Powers $\mathbf{A}^n = A^n \cdot e^{j \cdot n \cdot \theta} = A^n \cdot \cos(n \cdot \theta) + A^n \cdot \sin(n \cdot \theta) \cdot j$ Convert rectangulars first, usually

Conjugates

complex number	Conjugate
$\mathbf{A} = a + b \cdot j$	$\overline{\mathbf{A}} = a - b \cdot j$
$\mathbf{A} = A \cdot e^{j \cdot \theta}$	$\overline{\mathbf{A}} = A \cdot e^{-j \cdot \theta}$
$\mathbf{F} = \frac{3 + 5 \cdot j}{(2 - 6 \cdot j) \cdot e^{j \cdot 40 \cdot deg}}$	$\overline{\mathbf{F}} = \frac{3 - 5 \cdot j}{(2 + 6 \cdot j) \cdot e^{-j \cdot 40 \cdot deg}}$

$\overline{\overline{\mathbf{A}}} = \mathbf{A}$

Euler's equation $e^{j \cdot \alpha} = \cos(\alpha) + j \cdot \sin(\alpha)$ OR: $\cos(\alpha) = \frac{e^{j \cdot \alpha} + e^{-j \cdot \alpha}}{2} \qquad \sin(\alpha) = \frac{e^{j \cdot \alpha} - e^{-j \cdot \alpha}}{2 \cdot j}$

$e^{j \cdot (\omega \cdot t + \theta)} = \cos(\omega \cdot t + \theta) + j \cdot \sin(\omega \cdot t + \theta)$

$\mathrm{Re}\left[e^{j \cdot (\omega \cdot t + \theta)}\right] = \cos(\omega \cdot t + \theta)$

If we freeze this at time t=0, then we can represent $\cos(\omega \cdot t + \theta)$ by $e^{j \cdot \theta}$

Calculus Remember, when we write $e^{j \cdot \theta}$, we really mean $e^{j \cdot (\omega \cdot t + \theta)}$

$$\frac{d}{dt}\mathbf{A} = \frac{d}{dt}\left(A \cdot e^{j \cdot \theta}\right) = j \cdot \omega \cdot A \cdot e^{j \cdot \theta} = \omega \cdot A \cdot e^{j \cdot (\theta + 90 \cdot deg)}$$

$$\int \mathbf{A}\, dt = \int A \cdot e^{j \cdot \theta}\, dt = \frac{1}{j \cdot \omega} \cdot A \cdot e^{j \cdot \theta} = \frac{1}{\omega} \cdot A \cdot e^{j \cdot (\theta - 90 \cdot deg)}$$

ECE 3510 Complex Numbers Notes

Sinusoidal Response Notes

For BIBO Systems

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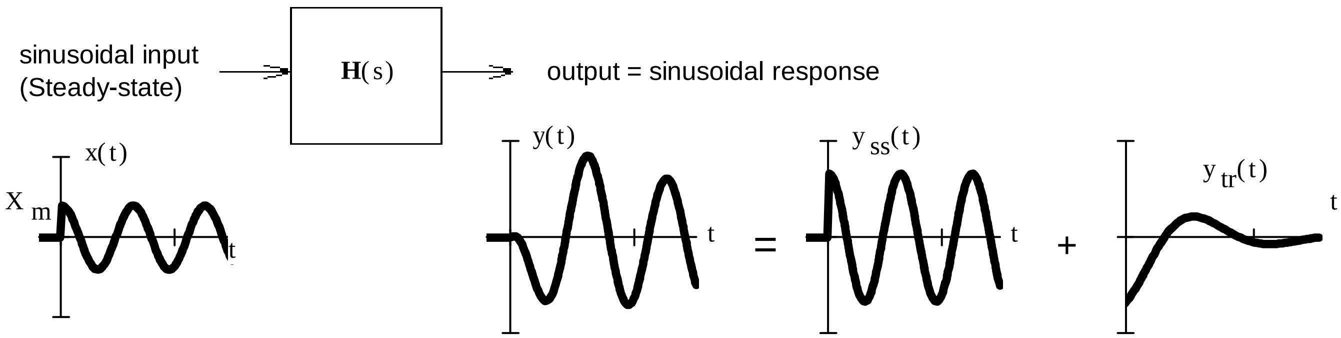
2/8/21,
1/31/23

The sinusoidal response of a system is the output when the input is a sinusoidal (which starts at time = 0).

System Sinusoidal Response



Complete step response = steady-state response + transient response

$\mathbf{Y}(s) = \mathbf{X}(s)\cdot\mathbf{H}(s) \quad = \quad X_m\cdot\mathbf{H}(j\omega)\cdot u(t) \quad + \quad \mathbf{Y}_{tr}(s)$

$\mathbf{H}(j\omega)$ = phasor-type transfer function

Sinusoidal Input

$\cos(\omega\cdot t)\cdot u(t) \quad \Leftrightarrow \quad \dfrac{s}{s^2+\omega^2}$ $\qquad$ $\sin(\omega\cdot t)\cdot u(t) \quad \Leftrightarrow \quad \dfrac{\omega}{s^2+\omega^2}$

General sinusoidal input: $\left[X_{mc}\cdot\cos(\omega\cdot t) + X_{ms}\cdot(\sin(\omega\cdot t)\cdot u(t))\right]\cdot u(t)$ $\qquad$ $\mathbf{X}(s) = \dfrac{X_{mc}\cdot s + X_{ms}\cdot\omega}{s^2+\omega^2}$

OR $\quad X_m\cdot\cos(\omega\cdot t+\theta)\cdot u(t) = \left[X_{mc}\cdot\cos(\omega\cdot t) + X_{ms}\cdot(\sin(\omega\cdot t)\cdot u(t))\right]\cdot u(t)$

$X_{mc} = X_m\cdot\cos(\theta) \qquad X_{ms} = -X_m\cdot\sin(\theta)$ $\quad$ note that the sine carries the opposite sign as you might expect.

Steady-State Response & $\mathbf{H}(j\omega)$

$\mathbf{Y}(s) = \mathbf{X}(s)\cdot\mathbf{H}(s) = \dfrac{X_{mc}\cdot s + X_{ms}\cdot\omega}{s^2+\omega^2}\cdot\mathbf{H}(s)$ = Complete sinusoidal response

$= \dfrac{A\cdot s + B\cdot\omega}{s^2+\omega^2} + \dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)} + \dfrac{E}{(\blacksquare)} + \ldots$

partial fraction expansion: $\quad \mathbf{Y}(s) = \dfrac{X_{mc}\cdot s + X_{ms}\cdot\omega}{s^2+\omega^2}\cdot\mathbf{H}(s) = \dfrac{A\cdot s + B\cdot\omega}{s^2+\omega^2} + \left[\dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)} + \dfrac{E}{(\blacksquare)}\right]\cdot s$

steady-state response + transient response

$\mathbf{Y}_{ss}(s) \quad + \quad \mathbf{Y}_{tr}(s)$

multiply both sides by: $(s^2+\omega^2)$ $\quad (X_{mc}\cdot s + X_{ms}\cdot\omega)\cdot\mathbf{H}(s) = A\cdot s + B\cdot\omega + \left[\dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)} + \dfrac{E}{(\blacksquare)}\right]\cdot(s^2+\omega^2)$

set $s := j\omega$ $\quad (X_{mc}\cdot j\omega + X_{ms}\cdot\omega)\cdot\mathbf{H}(j\omega) = A\cdot j\omega + B\cdot\omega + \left[\dfrac{C}{(\blacksquare)} + \dfrac{D}{(\blacksquare)} + \dfrac{E}{(\blacksquare)}\right]\cdot 0$

divide both sides by $j\omega$ $\quad (X_{mc} - X_{ms}\cdot j)\cdot\mathbf{H}(j\omega)$

$\mathbf{X}(\omega)\cdot\mathbf{H}(j\omega) = A - B\cdot j = \mathbf{Y}_{ss}(\omega)$ = steady-state response in phasor form

(real is cosine, imaginary is -sine)

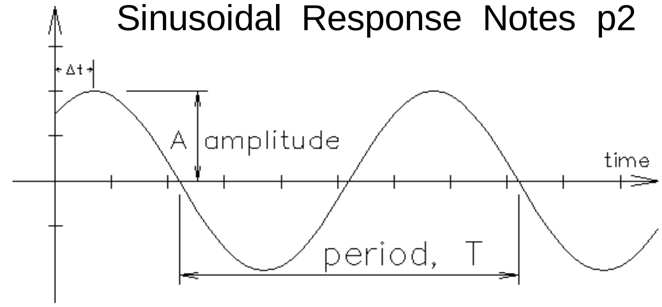
$\mathbf{X}(\omega)$ = the input expressed in phasor form NOT $\mathbf{X}(s)$ with $s := \omega$ or $s := j\omega$ (that would be ∞)

$\mathbf{H}(j\omega)$ = the steady-state sinusoidal transfer function

= phasor-type transfer function

The **transient part** would be found by finishing the partial-fraction expansion.

Steady-State Response by Phasors



Expression of signals as phasors

T = Period

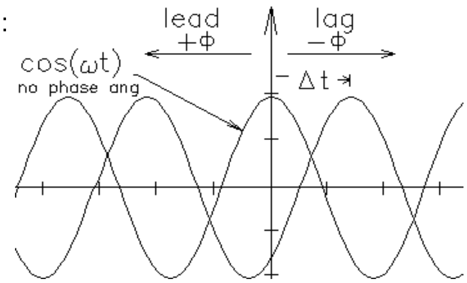
f = frequency, cycles / second $\quad f = \frac{1}{T} = \frac{\omega}{2\cdot\pi}$

ω = radian frequency, radians/sec $\quad \omega = 2\cdot\pi\cdot f$

A = amplitude

Phase: $\phi = -\frac{\Delta t}{T}\cdot 360\cdot deg$ $\quad$ or: $\phi = -\frac{\Delta t}{T}\cdot 2\cdot\pi\cdot rad$

Phase:



$y(t) = A\cdot\cos(\omega\cdot t + \phi)$

		Phasor
voltage:	$v(t) = V_p\cdot\cos(\omega\cdot t + \phi)$	$\mathbf{V}(j\omega) = V_p\cdot e^{j\cdot\phi}$
current:	$i(t) = I_p\cdot\cos(\omega\cdot t + \phi)$	$\mathbf{I}(j\omega) = I_p\cdot e^{j\cdot\phi}$

Ex1 Let's assume the input to your system is $v_1(t) = 3.2\cdot V\cdot\cos(\omega\cdot t + 15\cdot deg)$ $\quad \mathbf{V_1}(j\omega) = 3.2\cdot V\cdot e^{j\cdot 15\cdot deg}$

or: $\mathbf{V_1}(j\omega) = 3.2V\ \underline{/15^o}$

In rectangular form:

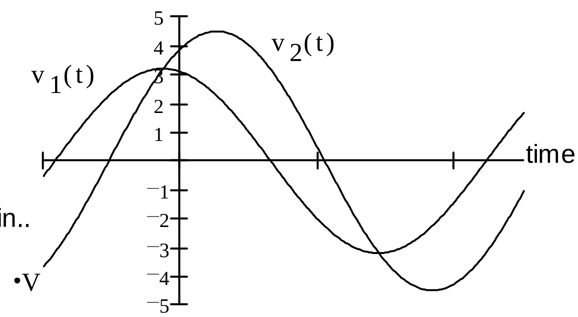
$3.2\cdot V\cdot\cos(15\cdot deg) = 3.091\cdot V$ $\qquad 3.2\cdot V\cdot\sin(15\cdot deg) = 0.828\cdot V$

or: $\mathbf{V_1}(j\omega) = (3.091 + 0.828\cdot j)\cdot V$

Ex2 What if a signal is the sum of two sinusoids.

$v_1(t) = 3.2\cdot V\cdot\cos(\omega\cdot t + 15\cdot deg)$

$v_2(t) = 4.5\cdot V\cdot\sin(\omega\cdot t + 60\cdot deg)$ $\qquad v_3(t) = v_1(t) + v_2(t)$



I'm going to drop the (jω) notation from the phasor notation, it gets cumbersome, but remember that phasors are in the frequency domain..

From Ex1: $\mathbf{V_1} := 3.2\cdot V\cdot e^{j\cdot 15\cdot deg} = 3.2V\ \underline{/15^o}$ $\qquad \mathbf{V_1} = 3.091 + 0.828j\ \cdot V$

Phasors are based on cosines, so express $v_2(t)$ as a cosine. $\qquad$ Remember: $\sin(\omega t) = \cos(\omega\cdot t - 90\cdot deg)$

So: $v_2(t) = 4.5\cdot V\cdot\cos(\omega\cdot t + 60\cdot deg - 90\cdot deg) = 4.5\cdot V\cdot\cos(\omega\cdot t - 30\cdot deg)$

$\mathbf{V_2} = 4.5V\ \underline{/-30^o}$ $\qquad$ or: $\mathbf{V_2} := 4.5\cdot V\cdot e^{-j\cdot 30\cdot deg}$

$4.5\cdot V\cdot\cos(-30\cdot deg) = 3.897\cdot V$ $\qquad 4.5\cdot V\cdot\sin(-30\cdot deg) = -2.25\cdot V$

$\mathbf{V_2} = 3.897 - 2.25j\ \cdot V$ \
$\mathbf{V_1} = 3.091 + 0.828j\ \cdot V$ } add
/

Add real parts: $\quad 3.897 + 3.091 = 6.988$

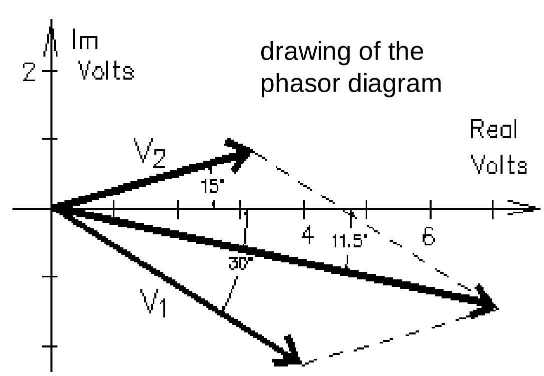
Add imaginary parts: $\quad -2.25 + 0.828 = -1.422$

$\mathbf{V_3} := \mathbf{V_1} + \mathbf{V_2}$

$\mathbf{V_3} = 6.988 - 1.422j\ \cdot V$ $\qquad$ sum

Change $\mathbf{V_3}$ back to polar coordinates:

$\sqrt{6.988^2 + 1.422^2} = 7.131$ $\qquad \operatorname{atan}\left(\frac{-1.422}{6.988}\right) = -11.502\cdot deg$



OR, in Mathcad notation (you'll see these in future solutions):

$\left|\mathbf{V_3}\right| = 7.131\cdot V$ $\qquad \arg(\mathbf{V_3}) = -11.5\cdot deg$

$\mathbf{V_3}(j\omega) = 7.131V\ \underline{/-11.5^o}$ $\quad$ or: $\mathbf{V_3}(j\omega) = 7.131\cdot V\cdot e^{-j\cdot 11.5\cdot deg}$

$\mathbf{V_3}$ may also be converted back to the time domain:

$v_3(t) = v_1(t) + v_2(t) = 7.13\cdot\cos(\omega\cdot t - 11.5\cdot deg)\cdot V$

Magnitude and Phase of transfer functions With steady-state sinusoidal inputs

Ex3 a) Find the magnitude and phase of the following transfer function at this frequency: $\omega := 2 \cdot \frac{rad}{sec}$

$$\mathbf{H}(s) = \frac{2 \cdot s^2 + 5 \cdot s + 20}{s^2 + 1 \cdot s + 10} = \frac{2 \cdot s^2 + \frac{5}{sec} \cdot s + \frac{20}{sec^2}}{s^2 + \frac{1}{sec} \cdot s + \frac{10}{sec^2}}$$ Expressed with proper units

$s := j \cdot \omega$

$$\mathbf{H}(j \cdot \omega) = \frac{2 \cdot (j \cdot \omega)^2 + 5 \cdot (j \cdot \omega) + 20}{(j \cdot \omega)^2 + \frac{1}{sec} \cdot (j \cdot \omega) + \frac{10}{sec^2}} = \frac{\left(20 - 2 \cdot \omega^2\right) + (5 \cdot \omega) \cdot j}{\left(10 - \omega^2\right) + (1 \cdot \omega) \cdot j} = \frac{\left(20 - 2 \cdot 2^2\right) + (5 \cdot 2) \cdot j}{\left(10 - 2^2\right) + (1 \cdot 2) \cdot j} = \frac{12 + 10 \cdot j}{6 + 2 \cdot j}$$

without units

$$|\mathbf{H}(j \cdot \omega)| = M = \frac{\sqrt{12^2 + 10^2}}{\sqrt{6^2 + 2^2}} = 2.47 \qquad \angle H(j\omega) = \operatorname{atan}\left(\frac{10}{12}\right) - \operatorname{atan}\left(\frac{2}{6}\right) = 21.37 \cdot deg$$

b) Find the steady-state sinusoidal output if the input is: $3.2 \cdot V \cdot \cos(2 \cdot t + 15 \cdot deg)$ $\quad \mathbf{V}_{in} := 3.2 \cdot V \cdot e^{j \cdot 15 \cdot deg} = 3.2V \angle 15^o$

$$\mathbf{V}_{outss}(j\omega) = \mathbf{V}_{in}(j\omega) \cdot \mathbf{H}(j \cdot \omega) = \left(3.2 \cdot V \cdot e^{j \cdot 15 \cdot deg}\right) \cdot \left(2.47 \cdot e^{j \cdot 21.37 \cdot deg}\right) = 3.2 \cdot V \cdot 2.47 \cdot e^{j \cdot (15 \cdot deg + 21.37 \cdot deg)}$$

$$= 7.904 \cdot V \cdot e^{j \cdot 36.37 \cdot deg} = 6.364 + 4.687j \cdot V$$

$$v_{outss}(t) = 7.904 \cdot V \cdot \cos(2 \cdot t + 36.37 \cdot deg)$$

Ex4 a) Find the magnitude and phase of the following transfer function at this frequency: $f := 5 \cdot Hz$ $\quad \omega := 2 \cdot \pi \cdot f$

$\omega = 31.42 \cdot \frac{rad}{sec}$

$$\mathbf{H}(s) = \frac{s^2 + \frac{20}{sec} \cdot s + \frac{1000}{sec^2}}{s^2 + \frac{10}{sec} \cdot s + \frac{800}{sec^2}}$$

$$s := j \cdot \omega = j \cdot 31.42 \cdot \frac{rad}{sec} \qquad \mathbf{H}(j \cdot \omega) = \frac{(j \cdot \omega)^2 + \frac{20}{sec} \cdot (j \cdot \omega) + \frac{1000}{sec^2}}{(j \cdot \omega)^2 + \frac{10}{sec} \cdot (j \cdot \omega) + \frac{800}{sec^2}} = \frac{(j \cdot 31.42)^2 + 20 \cdot (j \cdot 31.42) + 1000}{(j \cdot 31.42)^2 + 10 \cdot (j \cdot 31.42) + 800}$$ without units

$$= \frac{13.04 + 628.319 \cdot j}{-186.96 + 314.159 \cdot j} = 1.459 - 0.91j$$

$$|\mathbf{H}(j \cdot \omega)| = M = \sqrt{1.459^2 + 0.91^2} = 1.72 \qquad \angle H(j\omega) = \operatorname{atan}\left(\frac{-0.91}{1.459}\right) = -31.95 \cdot deg$$

b) Find the steady-state sinusoidal output if the input is: $x(t) = 4 \cdot \cos(2 \cdot \pi \cdot 5 \cdot Hz \cdot t)$

$\mathbf{X}(\omega) = 4 + 0j$ and then $\mathbf{Y}(\omega) = 4 \cdot (1.459 - 0.91 \cdot j) = 5.836 - 3.64j$

Note that you can use the rectangular form of $\mathbf{H}(j \cdot \omega)$

$$y(t) = 5.836 \cdot \cos\left(31.42 \cdot \frac{rad}{sec} \cdot t\right) + 3.64 \cdot \sin\left(31.42 \cdot \frac{rad}{sec} \cdot t\right)$$

note that the sine carries the opposite sign as the imaginary part.

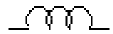
$$\sqrt{5.836^2 + 3.64^2} = 6.878 \qquad \operatorname{atan}\left(\frac{-3.64}{5.836}\right) = -31.95 \cdot deg \qquad y(t) = 6.88 \cdot \cos\left(31.42 \cdot \frac{rad}{sec} \cdot t - 32 \cdot deg\right)$$

Impedances

Resistor

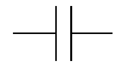
$$v_R = i_R \cdot R$$
$$\mathbf{V}_R(s) = R\cdot \mathbf{I}(s)$$
$$\mathbf{Z}_R = R$$

Inductor



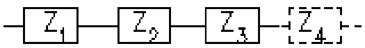
$$v_L(t) = L\cdot\frac{d}{dt} i_L(t)$$
$$\mathbf{V}_L(s) = s\cdot L\cdot \mathbf{I}_L(s)$$
$$\mathbf{Z}_L = L\cdot s$$

Capacitor



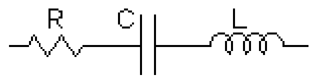
$$i_C(t) = C\cdot\frac{d}{dt} v_C(t)$$
$$\mathbf{I}_C(s) = C\cdot s\cdot \mathbf{V}_C(s)$$
$$\mathbf{V}_C(\omega) = \frac{1}{C\cdot s}\cdot \mathbf{I}(\omega)$$
$$\mathbf{Z}_C = \frac{1}{C\cdot s}$$

series:



$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 \ + \ldots$$

Example:

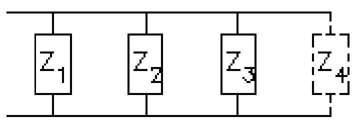


$$\mathbf{Z}_{eq} = R + \frac{1}{C\cdot s} + L\cdot s$$

Voltage divider:

$$\mathbf{V}_{Zn} = \mathbf{V}_{total}\cdot\frac{\mathbf{Z}_n}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \ldots}$$

parallel:

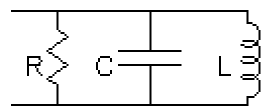


$$\mathbf{Z}_{eq} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} \ + \ldots}$$

Current divider:

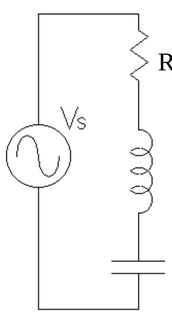
$$\mathbf{I}_{Zn} = \mathbf{I}_{total}\cdot\frac{\frac{1}{\mathbf{Z}_n}}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \ldots}$$

Example:



$$\mathbf{Z}_{eq} = \frac{1}{\frac{1}{R} + \frac{1}{\frac{1}{C\cdot s}} + \frac{1}{L\cdot s}} = \frac{1}{\frac{1}{R} + C\cdot s + \frac{1}{L\cdot s}}$$

Ex5



$R := 500\cdot\Omega$

$L := 80\cdot mH$

$C := 0.4\cdot\mu F$

a) Find the steady-state $\mathbf{V}_R$ and $v_R(t)$ given $v_S(t)$ is a 12 Vpp cosine wave at: $f := 2\cdot kHz$

$$\mathbf{V}_S(\omega) = 6\cdot V\cdot e^{j0} \qquad \omega := 2\cdot\pi\cdot f \qquad \omega = 12566\cdot\frac{rad}{sec}$$

Transfer function for $\mathbf{V}_R$ as the output: $\mathbf{H}(s) = \dfrac{\mathbf{V}_R(s)}{\mathbf{V}_S(s)} = \dfrac{R}{R + L\cdot s + \frac{1}{C\cdot s}}$

$$\mathbf{V}_R(\omega) = 6\cdot V\cdot\frac{R}{R + L\cdot(j\cdot\omega) + \frac{1}{C\cdot(j\cdot\omega)}} = 1.666 - 2.687j \ \cdot V \qquad \mathbf{V}_R = 3.163V \ \underline{/-58.2}^o$$

$$v_R(t) = 3.163\cdot V\cdot\cos\left(12566\cdot\frac{rad}{sec}\cdot t - 58.2\cdot deg\right)$$

b) Find the current: $\dfrac{\mathbf{I}(s)}{\mathbf{V}(s)} = \dfrac{1}{\mathbf{Z}(s)} \qquad \mathbf{I}(s) = \dfrac{\mathbf{V}(s)}{\mathbf{Z}(s)} \qquad s := j\cdot\omega = j\cdot 12566\cdot\dfrac{rad}{sec}$

$$\mathbf{I} := \frac{6\cdot V\cdot e^{j\cdot 0}}{R + L\cdot(j\cdot\omega) + \frac{1}{C\cdot(j\cdot\omega)}} = \frac{6\cdot V}{500 + 0.080\cdot(j\cdot 12566) + \frac{1}{0.4\cdot 10^{-6}\cdot(j\cdot 12566)}} = \frac{6\cdot V}{500 + 1005.3\cdot j \ - 198.95\cdot j} = \frac{6\cdot V}{500 + 806.366\cdot j}$$

$$\sqrt{500^2 + 806.366^2} = 948.802$$

$$\mathrm{atan}\left(\frac{806.366}{500}\right) = 58.198\cdot deg$$

magnitude: $\dfrac{6\cdot V}{948.8\cdot\Omega} = 6.324\cdot mA$

angle: $0\cdot deg - 58.2\cdot deg = -58.2\cdot deg \qquad \mathbf{I} = 6.324mA \ \underline{/-58.2}^o$

c) Draw a phasor diagram of all the voltages.

$\mathbf{V}_L = \mathbf{I}\cdot\mathbf{Z}_L \qquad 6.324\cdot mA\cdot 1005\cdot\Omega = 6.356\cdot V$

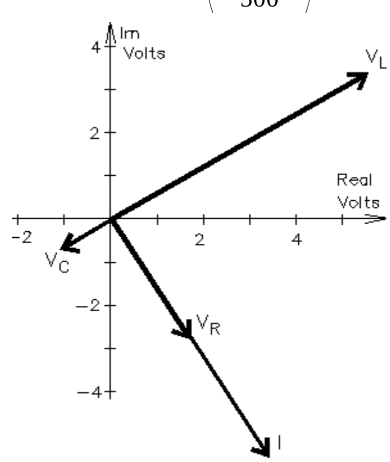
$-58.2\cdot deg + 90\cdot deg = 31.8\cdot deg$

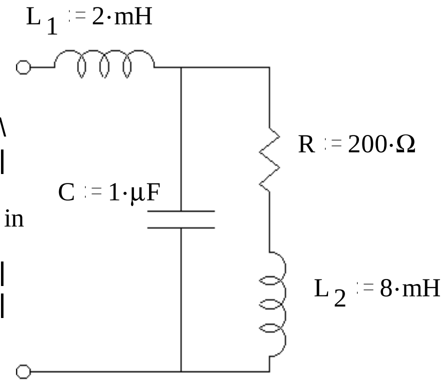
$\mathbf{V}_L = 6.356V \ \underline{/ 31.8}^o$

$\mathbf{V}_C = \mathbf{I}\cdot\mathbf{Z}_C \qquad 6.324\cdot mA\cdot(-199)\cdot\Omega = -1.258\cdot V$

$-58.2\cdot deg + (90)\cdot deg = 31.8\cdot deg$

$\mathbf{V}_C = -1.258V \ \underline{/ 31.8}^o = 1.258V \ \underline{/-148.2}^o$





Ex6 a) Find the steady-state $\mathbf{V_C}$ and $v_C(t)$ given $v_{in}(t)$ is a 12 Vp cosine wave at: $f := 2.5\cdot kHz$ with a 20º leading phase angle.

Transfer function for $\mathbf{V_C}$ as the output: $\mathbf{H}(s) = \dfrac{\mathbf{V_C}(s)}{\mathbf{V_{in}}(s)}$

$$\mathbf{H}(s) = \frac{\dfrac{1}{\dfrac{1}{R+L_2\cdot s}+C\cdot s}}{L_1\cdot s+\dfrac{1}{\dfrac{1}{R+L_2\cdot s}+C\cdot s}} \qquad \frac{\left(\dfrac{1}{R+L_2\cdot s}+C\cdot s\right)}{\left(\dfrac{1}{R+L_2\cdot s}+C\cdot s\right)}$$

$$= \frac{1}{L_1\cdot s\cdot\left(\dfrac{1}{R+L_2\cdot s}+C\cdot s\right)+1}$$

$\mathbf{V_{in}} := 12\cdot V\cdot e^{j\cdot 20\cdot deg}$ $\qquad \omega := 2\cdot\pi\cdot f$ $\qquad \omega = 15708\cdot\dfrac{rad}{sec}$

$$\mathbf{H}(j\omega) = \frac{1}{L_1\cdot(j\cdot\omega)\cdot\left[\dfrac{1}{R+L_2\cdot(j\cdot\omega)}+C\cdot(j\cdot\omega)\right]+1} = \frac{1}{0.002\cdot(j\cdot 15708)\cdot\left[\dfrac{1}{200+0.008\cdot(j\cdot 15708)}+10^{-6}\cdot(j\cdot 15708)\right]+1}$$

$$= \frac{1}{31.416\cdot j\ \cdot\left[3.585\cdot 10^{-3}-2.252\cdot 10^{-3}\cdot j\ +(j\cdot 0.015708)\right]+1} = \frac{1}{1-0.423+0.113\cdot j} = \frac{1}{0.588\cdot e^{j\cdot 11.039\cdot deg}}$$

$$= 1.7\cdot e^{-j\cdot 11.039\cdot deg}$$

$$\mathbf{V_C} = \mathbf{V_{in}}(j\omega)\cdot\mathbf{H}(j\omega) = 12\cdot V\cdot e^{j\cdot 20\cdot deg}\cdot 1.7\cdot e^{-j\cdot 11.039\cdot deg} = 12\cdot V\cdot 1.7\ \underline{/\ 20 - 11.039^o} = 20.4V\ \underline{/\ 8.96^o}$$

$$v_c(t) = 20.4\cdot V\cdot\cos\left(15708\cdot\frac{rad}{sec}\cdot t+8.96\cdot deg\right)$$

a) Find the steady-state $\mathbf{I_{L2}}$ and $i_{L2}(t)$.

Transfer function for $\mathbf{I_{L2}}$ as the output: $\mathbf{H}(s) = \dfrac{\mathbf{I_{L2}}(s)}{\mathbf{V_{in}}(s)} = \dfrac{\left(\dfrac{\mathbf{V_C}(s)}{R+L_2\cdot s}\right)}{\mathbf{V_{in}}(s)} = \dfrac{\mathbf{V_C}(s)}{\mathbf{V_{in}}(s)}\cdot\dfrac{1}{R+L_2\cdot s}$

$$\mathbf{H}(s) = \frac{1}{L_1\cdot s\cdot\left(\dfrac{1}{R+L_2\cdot s}+C\cdot s\right)+1}\cdot\frac{1}{\left(R+L_2\cdot s\right)} = \frac{1}{L_1\cdot s+L_1\cdot s\cdot C\cdot s\cdot\left(R+L_2\cdot s\right)+\left(R+L_2\cdot s\right)}$$

$$\mathbf{H}(j\omega) = \frac{1}{L_1\cdot(j\cdot\omega)+L_1\cdot C\cdot(j\cdot\omega)^2\cdot\left[R+L_2\cdot(j\cdot\omega)\right]+\left[R+L_2\cdot(j\cdot\omega)\right]} = 5.249\cdot 10^{-3}-4.926\cdot 10^{-3}j \quad \cdot\frac{1}{\Omega} = \frac{7.198}{k\Omega}\cdot e^{-j\cdot 43.181\cdot deg}$$

$$\mathbf{I_{L2}} = \mathbf{V_{in}}(j\omega)\cdot\mathbf{H}(j\omega) = 12\cdot V\cdot e^{j\cdot 20\cdot deg}\cdot\left(\frac{7.198}{k\Omega}\cdot e^{-j\cdot 43.181\cdot deg}\right) = 12\cdot V\cdot\frac{7.198}{k\Omega}\ \underline{/\ 20 - 43.181^o} = 86.38mA\ \underline{/\ -23.18^o}$$

$$i_{L2}(t) = 86.38\cdot mA\cdot\cos\left(15708\cdot\frac{rad}{sec}\cdot t-23.18\cdot deg\right)$$

Ex7 This system: $\mathbf{H}(s) = \dfrac{s+20}{s+5}$ Has this input: $x(t) = 4\cdot\sin(12\cdot t - 40\cdot deg)\cdot u(t)$

a) Use steady-state AC analysis to find the steady-state output. $y_{ss}(t) = ?$

$$\text{AC steady-state } \mathbf{H}(j\omega) = \mathbf{H}(j\cdot 12) = \frac{j\cdot 12+20}{j\cdot 12+5} = \frac{\sqrt{12^2+20^2}\cdot e^{j\cdot \operatorname{atan}\left(\frac{12}{20}\right)}}{\sqrt{12^2+5^2}\cdot e^{j\cdot \operatorname{atan}\left(\frac{12}{5}\right)}} = \frac{23.324\cdot e^{j\cdot 30.964\cdot deg}}{13\cdot e^{j\cdot 67.38\cdot deg}}$$

$$= 1.794 \;/\!\_\_\, -36.416\cdot deg$$

$\mathbf{X}(j\omega) = 4\cdot e^{-j\cdot 130\cdot deg}$ Note 90º phase-lag because it's given as a sine wave

$$\mathbf{Y}_{ss}(j\omega) = 4\cdot 1.794 = 7.176 \;/\!\_\_\; -130 - 36.416 = -166.416$$

$$= 7.176 \;/\!\_\_\, -166.416\cdot deg$$

$$y_{ss}(t) = 7.176\cdot\cos(12\cdot t - 166.416\cdot deg)$$

b) Express the output, and separate into 3 partial fractions that you can find in the laplace transform table without using complex numbers. Show what they are, but don't find the coefficients.

Find the input as a sum of a pure sine and cosine $-4\cdot\sin(-130\cdot deg) = 3.064$ $4\cdot\cos(-130\cdot deg) = -2.571$

so: $4\cdot\sin(12\cdot t - 40\cdot deg)\cdot u(t) = (3.064\cdot\sin(12\cdot t) - 2.571\cdot\cos(12\cdot t))\cdot u(t)$

$$\mathbf{Y}(s) = \frac{3.064\cdot 12 - 2.571\cdot s}{s^2+144}\cdot\frac{(s+20)}{(s+5)} = \frac{A}{s+5} + \frac{B\cdot s}{\left(s^2+144\right)} + \frac{C\cdot 12}{\left(s^2+144\right)}$$

c) Continue with the partial fraction expansion just far enough to find the **transient** coefficient as a number.

$$(3.064\cdot 12 - 2.571\cdot s)\cdot(s+20) = A\cdot\left(s^2+144\right) + B\cdot s\cdot(s+5) + C\cdot 12\cdot(s+5)$$

let $s := -5$

$$(3.064\cdot 12 - 2.571\cdot(-5))\cdot(-5+20) = A\cdot\left(5^2+144\right) + 0 + 0$$

$$A := \frac{(s+20)\cdot(3.064\cdot 12 - 2.571\cdot s)}{25+144} \qquad A = 4.404$$

d) Express the complete (both transient and steady-state) output as a function of time. $y(t) = ?$

$$y(t) = \left(4.404\cdot e^{-5\cdot t} + 7.176\cdot\cos(12\cdot t - 166.416\cdot deg)\right)\cdot u(t)$$

$7.176\cdot\cos(-166.416\cdot deg) = -6.975$

$-7.176\cdot\sin(-166.416\cdot deg) = 1.685$

Either answer

$$y(t) = \left(4.404\cdot e^{-5\cdot t} - 6.975\cdot\cos(12\cdot t) + 1.685\cdot\sin(12\cdot t)\right)\cdot u(t)$$

e) What is the time constant of the transient part this expression? $\tau = ? \quad = \dfrac{1}{5}$

ECE 3510 Effect of initial conditions

See Bodson text, section 3.5

Best explained by example, IF: $\mathbf{H}(s) = \dfrac{b_2 \cdot s^2 + b_1 \cdot s + b_0}{s^2 + a_1 \cdot s + a_0} \cdot \mathbf{X}(s)$

We would normally say: $\mathbf{Y}(s) = \dfrac{b_2 \cdot s^2 + b_1 \cdot s + b_0}{s^2 + a_1 \cdot s + a_0} \cdot \mathbf{X}(s)$

But that ignores initial conditions. So let's deconstruct and then reconstruct with initial conditions included.

$$\mathbf{Y}(s) \cdot \left(s^2 + a_1 \cdot s + a_0\right) = \left(b_2 \cdot s^2 + b_1 \cdot s + b_0\right) \cdot \mathbf{X}(s)$$

$$s^2 \cdot \mathbf{Y}(s) + a_1 \cdot s \cdot \mathbf{Y}(s) + a_0 \cdot \mathbf{Y}(s) = b_2 \cdot s^2 \cdot \mathbf{X}(s) + b_1 \cdot s \cdot \mathbf{X}(s) + b_0 \cdot \mathbf{X}(s)$$

$$\frac{d^2}{dt^2} y(t) + a_1 \cdot \frac{d}{dt} y(t) + a_0 \cdot y(t) = b_2 \cdot \frac{d^2}{dt^2} x(t) + b_1 \cdot \frac{d}{dt} x(t) + b_0 \cdot x(t)$$

Laplace Properties

Operation	f(t)	F(s)
Time differentiation	$\frac{d}{dt} f(t)$	$s \cdot \mathbf{F}(s) - f(0-)$
	$\frac{d^2}{dt^2} f(t)$	$s^2 \cdot \mathbf{F}(s) - s \cdot f(0-) - \frac{d}{dt} f(0-)$

initial slope

$$\frac{d^2}{dt^2} y(t) + a_1 \cdot \frac{d}{dt} y(t) + a_0 \cdot y(t) = b_2 \cdot \frac{d^2}{dt^2} x(t) + b_1 \cdot \frac{d}{dt} x(t) + b_0 \cdot x(t)$$

$$\left(s^2 \cdot \mathbf{Y}(s) - s \cdot y(0-) - \frac{d}{dt} y(0-)\right) + a_1 \cdot \left(s \cdot \mathbf{Y}(s) - y(0-)\right) + a_0 \cdot \mathbf{Y}(s) =$$

$$b_2 \cdot \left(s^2 \cdot \mathbf{X}(s) - s \cdot x(0-) - \frac{d}{dt} x(0-)\right) + b_1 \cdot \left(s \cdot \mathbf{X}(s) - x(0-)\right) + b_0 \cdot \mathbf{X}(s)$$

$$\left(s^2 \cdot \mathbf{Y}(s) + a_1 \cdot s \cdot \mathbf{Y}(s) + a_0 \cdot \mathbf{Y}(s)\right) - s \cdot y(0-) - \frac{d}{dt} y(0-) - a_1 \cdot y(0-) =$$

$$\left(b_2 \cdot s^2 \cdot \mathbf{X}(s) + b_1 \cdot s \cdot \mathbf{X}(s) + b_0 \cdot \mathbf{X}(s)\right) - b_2 \cdot s \cdot x(0-) - b_2 \cdot \frac{d}{dt} x(0-) - b_1 \cdot x(0-)$$

$$\mathbf{Y}(s) \cdot \left(s^2 + a_1 \cdot s + a_0\right) = \left(b_2 \cdot s^2 + b_1 \cdot s + b_0\right) \cdot \mathbf{X}(s) + \left(s \cdot y(0-) + \frac{d}{dt} y(0-) + a_1 \cdot y(0-) - b_2 \cdot s \cdot x(0-) - b_2 \cdot \frac{d}{dt} x(0-) - b_1 \cdot x(0-)\right)$$

Response to input — Response to initial conditions

$$\mathbf{Y}(s) = \underbrace{\frac{b_2 \cdot s^2 + b_1 \cdot s + b_0}{s^2 + a_1 \cdot s + a_0}}_{\mathbf{H}(s)} \cdot \mathbf{X}(s) + \frac{s \cdot y(0-) + \frac{d}{dt} y(0-) + a_1 \cdot y(0-) - b_2 \cdot s \cdot x(0-) - b_2 \cdot \frac{d}{dt} x(0-) - b_1 \cdot x(0-)}{s^2 + a_1 \cdot s + a_0}$$

Forced response — Natural response

Zero-state response — Zero-input response

$$\mathbf{Y}(s) = \overbrace{\frac{b_2 \cdot s^2 + b_1 \cdot s + b_0}{s^2 + a_1 \cdot s + a_0}}^{\mathbf{H}(s)} \cdot \mathbf{X}(s) \quad + \quad \overbrace{\frac{s \cdot y(0-) + \frac{d}{dt} y(0-) + a_1 \cdot y(0-) - b_2 \cdot s \cdot x(0-) - b_2 \cdot \frac{d}{dt} x(0-) - b_1 \cdot x(0-)}{s^2 + a_1 \cdot s + a_0}}^{\text{Initial conditions}}$$

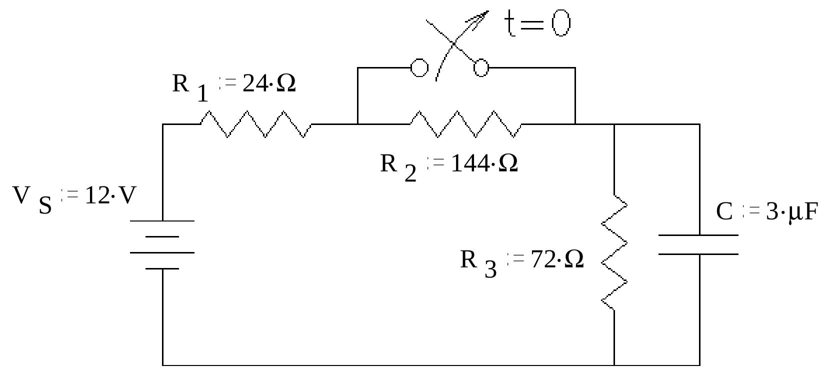
Observations

1. The total response is the sum of two independent components.
2. These values together fully describe the *state* of the 2nd-order system at time $t = 0^-$ (the initial state): $\quad y(0-) \quad \frac{d}{dt} y(0-) \quad x(0-) \quad \frac{d}{dt} x(0-)$
3. Similar denominator for both parts = Share poles = Similar responses
4. Response to Initial conditions always go to zero if system is BIBO.
5. Pole-zero cancellations in right-half plane can cause major problems with internal states of the system.

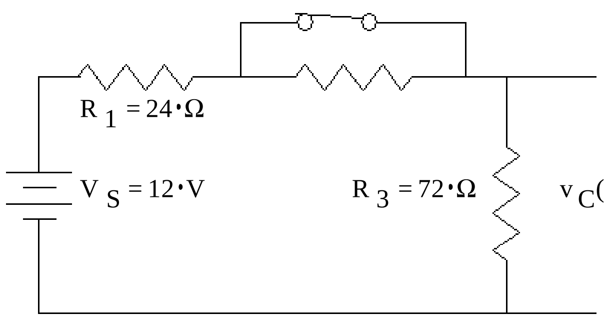
A simple first-order example

The switch has been closed for a long time and is opened at time $t=0$.

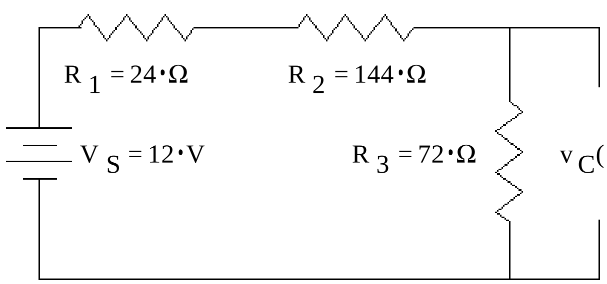
Find the complete expression for $v_C(t)$.



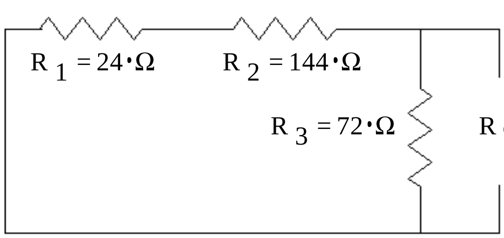
The way you've solved it before:



$$v_C(0) \;=\; V_S \cdot \frac{R_3}{R_1 + R_3} = 9 \cdot V$$



$$v_C(\infty) \;=\; V_S \cdot \frac{R_3}{R_1 + R_2 + R_3} = 3.6 \cdot V$$



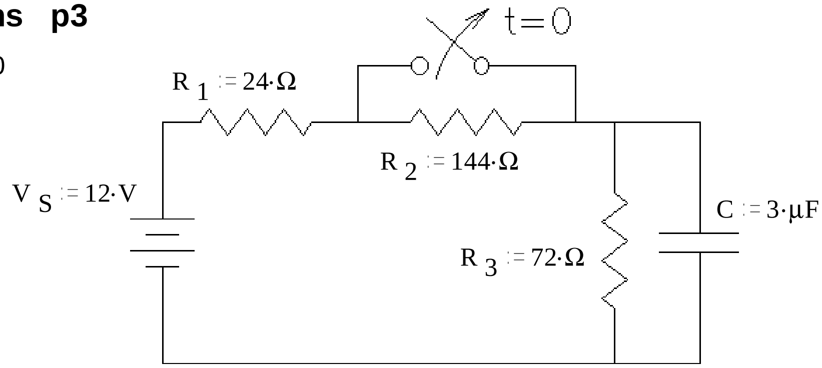
$$R_{Th} := \frac{1}{\left(\frac{1}{R_3} + \frac{1}{R_1 + R_2}\right)} \qquad R_{Th} = 50.4 \cdot \Omega$$

$$\tau := R_{Th} \cdot C \qquad \tau = 151.2 \cdot \mu s$$

$$v_C(t) = v_C(\infty) + \left(v_C(0) - v_C(\infty)\right) \cdot e^{-\frac{t}{\tau}} \quad = \quad 3.6 \cdot V + (9 \cdot V - 3.6 \cdot V) \cdot e^{-\frac{t}{151 \cdot \mu s}} \quad = \quad 3.6 \cdot V + 5.4 \cdot V \cdot e^{-\frac{t}{151 \cdot \mu s}}$$

ECE 3510 Effect of initial conditions p3

The way we would do the same thing in 3510



$$\mathbf{H}(s) = \frac{\mathbf{V_C}(s)}{\mathbf{V_S}(s)} = \frac{\dfrac{1}{\dfrac{1}{R_3}+C\cdot s}}{R_1+R_2+\dfrac{1}{\dfrac{1}{R_3}+C\cdot s}}\cdot\frac{\left(\dfrac{1}{R_3}+C\cdot s\right)}{\left(\dfrac{1}{R_3}+C\cdot s\right)} = \frac{1}{\dfrac{R_1+R_2}{R_3}+\left(R_1+R_2\right)\cdot C\cdot s+1}\cdot\frac{\left[\dfrac{1}{\left(R_1+R_2\right)\cdot C}\right]}{\left[\dfrac{1}{\left(R_1+R_2\right)\cdot C}\right]}$$

$$= \frac{1}{\dfrac{R_1+R_2}{R_3}+\left(R_1+R_2\right)\cdot C\cdot s+1}\cdot\frac{\left[\dfrac{1}{\left(R_1+R_2\right)\cdot C}\right]}{\left[\dfrac{1}{\left(R_1+R_2\right)\cdot C}\right]} = \frac{\dfrac{1}{\left(R_1+R_2\right)\cdot C}}{s+\left[\dfrac{1}{R_3\cdot C}+\dfrac{1}{\left(R_1+R_2\right)\cdot C}\right]}$$

First-order version of $\mathbf{Y}(s)$ with initial conditions

$$\mathbf{Y}(s) = \underbrace{\frac{b_1\cdot s+b_0}{s+a_0}}_{\mathbf{H}(s)}\cdot\mathbf{X}(s) + \underbrace{\frac{y(0-)-b_1\cdot x(0-)}{s+a_0}}_{\text{Initial conditions}}$$

$b_1 = 0$ $\qquad b_0 := \dfrac{1}{\left(R_1+R_2\right)\cdot C}$ $\qquad b_0 = 1984\cdot sec^{-1}$

$a_0 := \dfrac{1}{R_3\cdot C}+\dfrac{1}{\left(R_1+R_2\right)\cdot C}$ $\qquad a_0 = 6614\cdot sec^{-1}$

$$\mathbf{X}(s) = \frac{12\cdot V}{s}$$

$$\mathbf{Y}(s) = \frac{b_0}{s+a_0}\cdot\frac{12\cdot V}{s} + \frac{y(0-)}{s+a_0}$$

$y(0-) = 9\cdot V$ from above

$$\frac{b_0}{s+a_0}\cdot\frac{12\cdot V}{s} = \frac{A}{s}+\frac{B}{s+a_0}$$

$$A = \mathbf{H}(0)\cdot 12\cdot V = \frac{b_0}{a_0}\cdot 12\cdot V = 3.6\cdot V$$

$$B = b_0\cdot\frac{12\cdot V}{s}\Bigg|_{s:=-a_0} = \frac{b_0}{-a_0}\cdot 12\cdot V = -3.6\cdot V$$

$$= \frac{3.6\cdot V}{s} + \frac{-3.6\cdot V}{s+a_0} + \frac{y(0-)}{s+a_0}$$

$$y(t) = v_C(t) = \left(3.6\cdot V - 3.6\cdot V\cdot e^{-a_0\cdot t} + 9\cdot V\cdot e^{-a_0\cdot t}\right)\cdot u(t)$$

Same as above $\quad v_C(t) = 3.6\cdot V + (9\cdot V - 3.6\cdot V)\cdot e^{-\frac{t}{151\cdot\mu s}}$ $\qquad$ where $\tau = \dfrac{1}{a_0} = \dfrac{sec}{6614} = 151.2\cdot\mu s$

ECE 3510 Effect of initial conditions p3

ECE 3510 State Space

A completely different method where all math is done in the time-domain using linear algebra. See Bodson, section 3.6.

$x(t)$ = The state vector (n x 1 matrix)

n = order of the system

$\frac{d}{dt}x(t)$ = Time derivative of the state vector (n x 1 matrix)

$u(t)$ = The input vector (n_u x 1 matrix)

n_u = number of inputs

$y(t)$ = The state vector (n_y x 1 matrix)

n_y = order of the system

A = The system matrix (n x n matrix)

B = The input matrix (n x n_u matrix)

C = The output matrix (n_y x n matrix)

D = The feed-forward matrix (n_y x n_u matrix)

State Equation: $\frac{d}{dt}x(t) = A\cdot x(t) + B\cdot u(t)$

A third-order, 2-input, 2-output system

$$\begin{pmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{pmatrix} = \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \cdot \begin{pmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{pmatrix} + \begin{pmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{pmatrix} \cdot \begin{pmatrix} \blacksquare \\ \blacksquare \end{pmatrix}$$

Output Equation: $y(t) = C\cdot x(t) + D\cdot u(t)$

$$\begin{pmatrix} \blacksquare \\ \blacksquare \end{pmatrix} = \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \cdot \begin{pmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{pmatrix} + \begin{pmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{pmatrix} \cdot \begin{pmatrix} \blacksquare \\ \blacksquare \end{pmatrix}$$

State Equation: $\frac{d}{dt}x(t) = A\cdot x(t) + B\cdot u(t)$

A third-order, single-input, single-output system

$$\begin{pmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{pmatrix} = \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \cdot \begin{pmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{pmatrix} + \begin{pmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{pmatrix} \cdot u(t)$$

Output Equation: $y(t) = C\cdot x(t) + D\cdot u(t)$

$$y(t) = \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \cdot \begin{pmatrix} \blacksquare \\ \blacksquare \\ \blacksquare \end{pmatrix} + D\cdot u(t)$$

Advantages of the state-space method

- Easily handles multiple inputs, multiple outputs and initial conditions
- Can be used with nonlinear systems
- Can be used with time-varying systems
- Reveals unstable systems that have stable transfer functions (pole-zero cancellations). You can determine:
  - Controllability: State variables can all be affected by the input
  - Observability: State variables are all "observable" from the output
- Basis of Optimal control methods

Advantages and disadvantages of the classical frequency-domain method used in this class

- Simpler to understand and model interconnected systems.
- Rapidly provide stability and transient response information.
- Easy to see the effects of varying system parameters to get a good design.
- Limited to linear, time-invariant systems or systems that can be approximated as such.

ECE 3510 homework 6 Review of Steady-State AC Due: Sat, 2/4/23 b

1. Convert the following complex numbers to polar form ($m\underline{/\theta}$ or $me^{j\theta}$). a) $2.6 + 8.7j$ b) $3 + 4j$ c) $-3 - 4j$

2. Convert the following complex numbers to rectangular form (a + bj). a) $10 \cdot e^{j \cdot 60 \cdot deg}$ b) $10 \cdot e^{-j \cdot 45 \cdot deg}$ c) $20 \cdot e^{j \cdot 120 \cdot deg}$

3. Add or subtract the complex numbers. a) $(3 + 2j) + (6 + 9j)$ b) $(9 - 10j) - (9 + 10j)$

4. Multiply the complex numbers. a) $\left(20 \cdot e^{j \cdot 40 \cdot deg}\right) \cdot \left(10 \cdot e^{j \cdot 60 \cdot deg}\right)$ b) $(-2 - j) \cdot (-6 - 9j)$

5. Divide the complex numbers. a) $\dfrac{20 \cdot e^{j \cdot 40 \cdot deg}}{10 \cdot e^{j \cdot 60 \cdot deg}}$ b) $\dfrac{12 + 10j}{6 + 9j}$

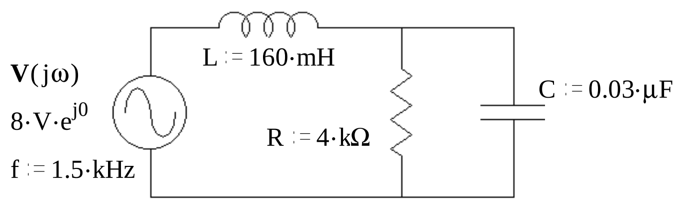
6. Add and subtract the sinusoidal voltages using phasors. Draw a phasor diagram which shows all 4 phasors, and give your final answer in time domain form.

   $v_1(t) = 1.5 \cdot V \cdot \cos(\omega \cdot t + 10 \cdot deg)$ $v_2(t) = 3.2 \cdot V \cdot \cos(\omega \cdot t + 25 \cdot deg)$

   a) Find $v_3(t) = v_1(t) + v_2(t)$ b) Find $v_4(t) = v_1(t) - v_2(t)$

7. a) Find $\mathbf{Z_{eq}}$.

   b) Find the current $\mathbf{I_L}(j\omega)$.



8. Find the steady-state magnitude and phase of each of the following transfer functions. $|H(j \cdot \omega)|$ = ? $\underline{/H(j\omega)}$ = ?

   a) $\omega := 10 \cdot \dfrac{rad}{sec}$ $s = j \cdot \omega$ $H(s) = \dfrac{\dfrac{40}{sec} \cdot s}{s^2 + \dfrac{10}{sec} \cdot s + \dfrac{200}{sec^2}}$

   b) $f := 50 \cdot Hz$ $H(s) = \dfrac{s^2 + \dfrac{1000}{sec} \cdot s}{s^2 + \dfrac{300}{sec} \cdot s + \dfrac{10000}{sec^2}}$

9. Find the following outputs. Express them in the time domain, first as a cosine with a phase angle and then as a sum of cosine and sine with no phase angles:

   a) The input $x(t) = 3 \cdot \cos(10 \cdot t)$ is the input for the transfer function of 8a), above.

   b) The input $x(t) = 5 \cdot \sin(2 \cdot \pi \cdot 50 \cdot t)$ is the input for the transfer function of 8b), above.

   remember, sine is -j

Answers

1. a) $9.08 \cdot e^{j \cdot 73.4 \cdot deg}$ b) $5 \cdot e^{j \cdot 53.1 \cdot deg}$ c) $5 \cdot e^{-j \cdot 126.9 \cdot deg}$

2. a) $5 + 8.66 \cdot j$ b) $7.071 - 7.071 \cdot j$ c) $-10 + 17.321 \cdot j$ 3. a) $9 + 11 \cdot j$ b) $-20 \cdot j$

4. a) $200 \cdot e^{j \cdot 100 \cdot deg}$ b) $24.2 \cdot e^{j \cdot 82.9 \cdot deg}$ 5. a) $2 \cdot e^{-j \cdot 20 \cdot deg}$ b) $1.385 - 0.41 \cdot j$

6. a) $v_1(t) + v_2(t) = 4.67 \cdot \cos(\omega \cdot t + 20.2 \cdot deg) \cdot V$ b) $v_1(t) - v_2(t) = 1.794 \cdot \cos(\omega \cdot t - 142.5 \cdot deg) \cdot V$

7. a) $1.82 \cdot k\Omega$ $-15.2 \cdot deg$ b) $4.4 \cdot mA$ $15.2 \cdot deg$

8. a) M = 2.828 $45 \cdot deg$ b) M = 2.544 $-25.8 \cdot deg$

9. a) $y(t) = 8.484 \cdot \cos\left(10 \cdot \dfrac{rad}{sec} \cdot t + 45 \cdot deg\right) = 6 \cdot \cos\left(10 \cdot \dfrac{rad}{sec} \cdot t\right) - 6 \cdot \sin\left(10 \cdot \dfrac{rad}{sec} \cdot t\right)$

   b) $y(t) = 12.72 \cdot \cos(2 \cdot \pi \cdot 50 \cdot t - 115.82 \cdot deg) = -5.54 \cdot \cos(2 \cdot \pi \cdot 50 \cdot t) + 11.45 \cdot \sin(2 \cdot \pi \cdot 50 \cdot t)$

1. Given the conditions in Bodson, example 3.4.3, p.43,
   a) Show all the steps needed to find eq. 3.64.
   b) Use the Laplace transform table to find the results in eq. 3.65 and 3.66 ($y_{ss}(t)$ part).
   c) Show that equations 3.67 & 3.68 can be found from equations 3.66.
   d) Show that equations 3.67 & 3.68 can be found from steady-state analysis of $H(s)$ (see eq. 3.56).

2. Still referring to the system in example 3.4.3, p.43, the input is: $x(t) = x_m \cdot \sin(\omega_o \cdot t)$
   a) Confirm eq. 3.70.
   b) Use any method you want to find M and ϕ_2 in: $y_{ss}(t) = M \cdot x_m \cdot \cos(\omega_o \cdot t + \phi_2)$

      Hint:, you may want to recall that: $\sin(\omega_o \cdot t) = \cos(\omega_o \cdot t - 90 \cdot deg)$

3. This system: $H(s) = \dfrac{3}{s+8}$ Has a cosine input: $x(t) = 4 \cdot \cos(10 \cdot t) \cdot u(t)$
   a) Express the output, $Y(s)$
   b) This separates into 3 partial fractions that you can find in the laplace transform table without using complex numbers. Show what they are, but don't find the coefficients.
   c) Continue with the partial fraction expansion just far enough to find the **transient** coefficient as a number.
   d) Express the transient part as a function of time. $y_{tr}(t) = ?$
   e) What is the time constant of this expression? $\tau = ?$
   f) Use steady-state AC analysis to find the steady-state output in the form of a cosine with a magnitude and phase angle.

      $y_{ss}(t) = ?$

4. This system: $H(s) = \dfrac{4}{s+12}$ Has this Cosine input: $x(t) = 5 \cdot \cos(8 \cdot t + 40 \cdot deg) \cdot u(t)$
   a) Use steady-state AC analysis to find the steady-state response ($y_{ss}(t)$) of the system. $y_{ss}(t) = ?$
   b) Separate the input $x(t)$ into a pure cosine part and a pure sine part.
   c) Use the results of 1b) and 2a), above to find the transient responses to cosine and sine inputs and then add them together to find the total transient response.

5. Find the steady-state (sinusoidal) magnitude and phase of the following transfer function.

   $|H(j \cdot \omega)| = ?$ $\quad \underline{/H(j\omega)} = ?$ $\quad \omega := 20 \cdot \dfrac{rad}{sec}$ $\quad H(s) = \dfrac{\dfrac{80}{sec} \cdot s - \dfrac{300}{sec^2}}{s^2 + \dfrac{90}{sec^2}}$

6. Express the following signal in the time domain, as a sum of cosine and sine with no phase angles:

   $\omega := 44 \cdot \dfrac{rad}{sec}$ $\quad Y(j\omega) = 3 + 0.5 \cdot j$

7. The following questions refer to the general system whose output is given by eq. 3.81, p.45 in our text.

a) Can a system's response to initial conditions be calculated separately from its response to the input signal? Why or why not?

b) Can you expect a system's response to initial conditions to be similar to its response to a simple input signal? Why or why not?

c) To fully describe the state of the system, how many things do you need to know? List them.

d) If a system is BIBO stable, then what is its final response to initial conditions?

e) The output of a system with nonrepeated poles on the jω-axis which is otherwise BIBO stable can be unbounded for some input signals. Is this also true for initial conditions alone when there is no input signal?
If no, why are the conditions for bounded output not as restrictive if there are only initial conditions and no input?

8. a) List 4 advantages of the state-space method over the frequency domain method we are using in this class.

b) List 2 advantages of the frequency domain method we are using in this class over the state-space method.

Answers

1 & 2a) Answers are right in the book

2.b) $\frac{k}{\sqrt{\omega_o^2 + a^2}}$ $\qquad -\tan^{-1}\left(\frac{\omega_o}{a}\right) - 90\cdot deg$

3. a) $\frac{3}{s+8}\cdot\frac{4\cdot s}{s^2+100}$ b) $\frac{A}{s+8} + \frac{B\cdot s}{(s^2+100)} + \frac{C\cdot 10}{(s^2+100)}$ c) -0.585 d) $-0.585\cdot e^{-8\cdot t}$ e) $125\cdot ms$

f) $0.936\cdot\cos(10\cdot t - 51.34\cdot deg)$

4. a) $1.385\cdot\cos(8\cdot t + 6.3\cdot deg)$ b) $x(t) = (3.83\cdot\cos(8\cdot t) - 3.214\cdot\sin(8\cdot t))\cdot u(t)$ c) $-1.378\cdot e^{-12\cdot t}$

5. $5.251 \qquad -79.38\cdot deg$

6. $3\cdot\cos\left(44\cdot\frac{rad}{sec}\cdot t\right) - 0.5\cdot\sin\left(44\cdot\frac{rad}{sec}\cdot t\right)$

7. c) 4 $\quad y(0) \quad \frac{d}{dt}y(0) \quad x(0) \quad \frac{d}{dt}x(0)$ d) 0

8. See section 3.1 in the Nise textbook.