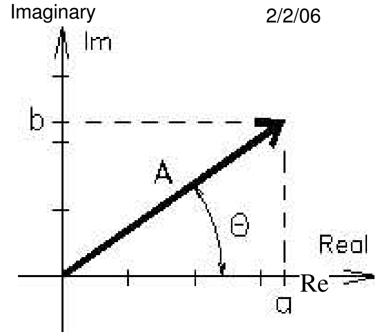
Complex Numbers

ECE 3510

A.Stolp
10/18/00
2/2/06



$j = \sqrt{-1}$ the imaginary number

Rectangular Form $\mathbf{A} = a + b \cdot j$

$\text{Re}(\mathbf{A}) = a \qquad \text{Im}(\mathbf{A}) = b$

Polar Form $\mathbf{A} = A \cdot e^{j \cdot \theta}$

$\text{Re}(\mathbf{A}) = A \cdot \cos(\theta) \qquad \text{Im}(\mathbf{A}) = A \cdot \sin(\theta)$

Conversions $A = |\mathbf{A}| = \sqrt{a^2 + b^2} \qquad \theta = \arg(\mathbf{A}) = \text{atan}\left(\frac{b}{a}\right)$

$a = A \cdot \cos(\theta) \qquad b = A \cdot \sin(\theta)$

$\mathbf{A} = A \cdot e^{j \cdot \theta} = A \cdot \cos(\theta) + A \cdot \sin(\theta) \cdot j \qquad \mathbf{A} = a + b \cdot j = \left(\sqrt{a^2 + b^2}\right) \cdot e^{j \cdot \text{atan}\left(\frac{b}{a}\right)}$

Special Cases $j := \sqrt{-1} = e^{j \cdot 90 \cdot deg} \qquad \frac{1}{j} = -j = e^{-j \cdot 90 \cdot deg} \qquad e^{j \cdot 0 \cdot deg} = 1 \qquad e^{-j \cdot 180 \cdot deg} = e^{-j \cdot 180 \cdot deg} = -1$

$j \cdot e^{j \cdot \theta} = e^{j \cdot (\theta + 90 \cdot deg)}$

Define a 2nd number: rect: $\mathbf{D} = c + d \cdot j$ polar: $\mathbf{D} = D \cdot e^{j \cdot \phi}$

Equality $\mathbf{A} = \mathbf{D}$ if and only if $a = c$ and $b = d$ OR $A = D$ and $\theta = \phi$

Addition and Subtraction

$\mathbf{A} + \mathbf{D} = (a + b \cdot j) + (c + d \cdot j) = (a + c) + (b + d) \cdot j$

$\mathbf{A} - \mathbf{D} = (a + b \cdot j) - (c + d \cdot j) = (a - c) + (b - d) \cdot j$

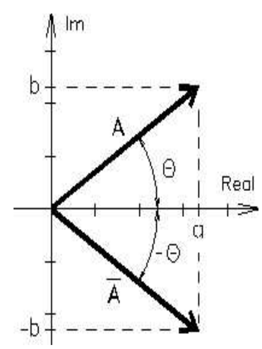
Convert polars to rectangular form first

Multiplication and Division $\mathbf{A} \cdot \mathbf{D} = (a + b \cdot j) \cdot (c + d \cdot j) = (a \cdot c - b \cdot d) + (b \cdot c + a \cdot d) \cdot j$

Rectangular: $\frac{\mathbf{A}}{\mathbf{D}} = \frac{a + b \cdot j}{c + d \cdot j} = \frac{a + b \cdot j}{c + d \cdot j} \cdot \frac{c - d \cdot j}{c - d \cdot j} = \frac{a \cdot c + b \cdot d}{c^2 + d^2} + \frac{b \cdot c - a \cdot d}{c^2 + d^2} \cdot j$

Polar: $\mathbf{A} \cdot \mathbf{D} = A \cdot e^{j \cdot \theta} \cdot D \cdot e^{j \cdot \phi} = A \cdot D \cdot e^{j \cdot (\theta + \phi)}$

$\frac{\mathbf{A}}{\mathbf{D}} = \frac{A \cdot e^{j \cdot \theta}}{D \cdot e^{j \cdot \phi}} = \frac{A}{D} \cdot e^{j \cdot (\theta - \phi)}$



Powers $\mathbf{A}^n = A^n \cdot e^{j \cdot n \cdot \theta} = A^n \cdot \cos(n \cdot \theta) + A^n \cdot \sin(n \cdot \theta) \cdot j$ Convert rectangulars first, usually

Conjugates

complex number	Conjugate
$\mathbf{A} = a + b \cdot j$	$\overline{\mathbf{A}} = a - b \cdot j$
$\mathbf{A} = A \cdot e^{j \cdot \theta}$	$\overline{\mathbf{A}} = A \cdot e^{-j \cdot \theta}$
$\mathbf{F} = \frac{3 + 5 \cdot j}{(2 - 6 \cdot j) \cdot e^{j \cdot 40 \cdot deg}}$	$\overline{\mathbf{F}} = \frac{3 - 5 \cdot j}{(2 + 6 \cdot j) \cdot e^{-j \cdot 40 \cdot deg}}$

$\overline{\overline{\mathbf{A}}} = \mathbf{A}$

Euler's equation $e^{j \cdot \alpha} = \cos(\alpha) + j \cdot \sin(\alpha)$ OR: $\cos(\alpha) = \frac{e^{j \cdot \alpha} + e^{-j \cdot \alpha}}{2} \qquad \sin(\alpha) = \frac{e^{j \cdot \alpha} - e^{-j \cdot \alpha}}{2 \cdot j}$

$e^{j \cdot (\omega \cdot t + \theta)} = \cos(\omega \cdot t + \theta) + j \cdot \sin(\omega \cdot t + \theta)$

$\text{Re}\left[e^{j \cdot (\omega \cdot t + \theta)}\right] = \cos(\omega \cdot t + \theta)$

If we freeze this at time t=0, then we can represent $\cos(\omega \cdot t + \theta)$ by $e^{j \cdot \theta}$

Calculus Remember, when we write $e^{j \cdot \theta}$, we really mean $e^{j \cdot (\omega \cdot t + \theta)}$

$$\frac{d}{dt}\mathbf{A} = \frac{d}{dt}\left(A \cdot e^{j \cdot \theta}\right) = j \cdot \omega \cdot A \cdot e^{j \cdot \theta} = \omega \cdot A \cdot e^{j \cdot (\theta + 90 \cdot deg)}$$

$$\int \mathbf{A}\, dt = \int A \cdot e^{j \cdot \theta}\, dt = \frac{1}{j \cdot \omega} \cdot A \cdot e^{j \cdot \theta} = \frac{1}{\omega} \cdot A \cdot e^{j \cdot (\theta - 90 \cdot deg)}$$

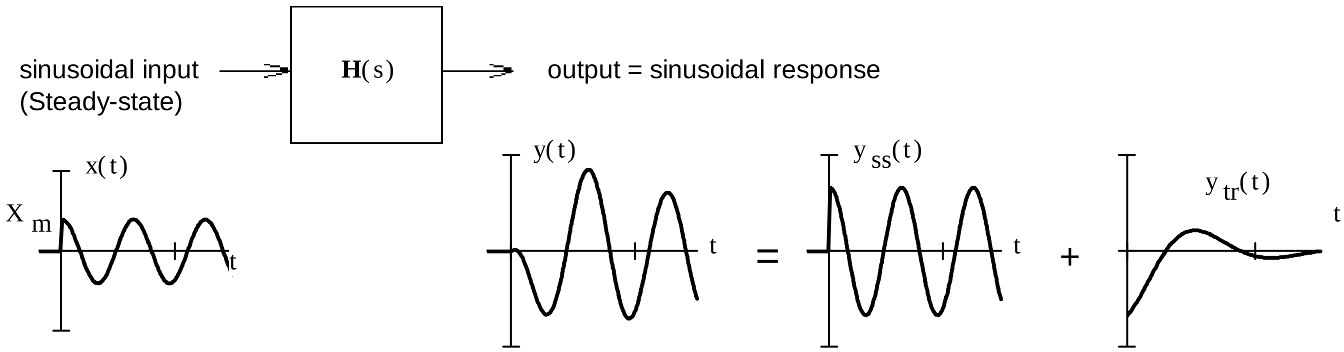
Sinusoidal Response Notes

For BIBO Systems

A.Stolp
2/8/21

The sinusoidal response of a system is the output when the input is a sinusoidal (which starts at time = 0).

System Sinusoidal Response



Complete step response = steady-state response + transient response

$$\mathbf{Y}(s) = \mathbf{X}(s)\cdot\mathbf{H}(s) \quad = \quad X_m\cdot\mathbf{H}(j\omega)\cdot u(t) \quad + \quad \mathbf{Y}_{tr}(s)$$

$\mathbf{H}(j\omega)$ = phasor-type transfer function

Sinusoidal Input

$$\cos(\omega\cdot t)\cdot u(t) \quad \Longleftrightarrow \quad \frac{s}{s^2+\omega^2} \qquad\qquad \sin(\omega\cdot t)\cdot u(t) \quad \Longleftrightarrow \quad \frac{b}{s^2+\omega^2}$$

General sinusoidal input: $\left[X_{mc}\cdot\cos(\omega\cdot t) + X_{ms}\cdot(\sin(\omega\cdot t)\cdot u(t))\right]\cdot u(t)$ $\qquad \mathbf{X}(s) = \dfrac{X_{mc}\cdot s + X_{ms}\cdot\omega}{s^2+\omega^2}$

OR $\quad X_m\cdot\cos(\omega\cdot t+\theta)\cdot u(t) = \left[X_{mc}\cdot\cos(\omega\cdot t) + X_{ms}\cdot(\sin(\omega\cdot t)\cdot u(t))\right]\cdot u(t)$

$$X_{mc} = X_m\cdot\cos(\theta) \qquad X_{ms} = -X_m\cdot\sin(\theta)$$

Steady-State Response & $\mathbf{H}(j\omega)$

$$\mathbf{Y}(s) = \mathbf{X}(s)\cdot\mathbf{H}(s) = \frac{X_{mc}\cdot s + X_{ms}\cdot\omega}{s^2+\omega^2}\cdot\mathbf{H}(s) = \text{Complete sinusoidal response}$$

$$= \frac{A\cdot s + B\cdot\omega}{s^2+\omega^2} + \frac{C}{(\,\mathbf{\cdot}\,)} + \frac{D}{(\,\mathbf{\cdot}\,)} + \frac{E}{(\,\mathbf{\cdot}\,)} + \ldots$$

partial fraction expansion:

$$\mathbf{Y}(s) = \frac{X_{mc}\cdot s + X_{ms}\cdot\omega}{s^2+\omega^2}\cdot\mathbf{H}(s) = \frac{A\cdot s + B\cdot\omega}{s^2+\omega^2} + \left[\frac{C}{(\,\mathbf{\cdot}\,)} + \frac{D}{(\,\mathbf{\cdot}\,)} + \frac{E}{(\,\mathbf{\cdot}\,)}\right]\cdot s$$

steady-state response + transient response

$$\mathbf{Y}_{ss}(s) \quad + \quad \mathbf{Y}_{tr}(s)$$

multiply both sides by: $\left(s^2+\omega^2\right)$

$$\left(X_{mc}\cdot s + X_{ms}\cdot\omega\right)\cdot\mathbf{H}(s) = A\cdot s + B\cdot\omega + \left[\frac{C}{(\,\mathbf{\cdot}\,)} + \frac{D}{(\,\mathbf{\cdot}\,)} + \frac{E}{(\,\mathbf{\cdot}\,)}\right]\cdot\left(s^2+\omega^2\right)$$

set $s := j\omega$

$$\left(X_{mc}\cdot j\omega + X_{ms}\cdot\omega\right)\cdot\mathbf{H}(j\omega) = A\cdot j\omega + B\cdot\omega + \left[\frac{C}{(\,\mathbf{\cdot}\,)} + \frac{D}{(\,\mathbf{\cdot}\,)} + \frac{E}{(\,\mathbf{\cdot}\,)}\right]\cdot 0$$

divide both sides by $j\omega$ $\qquad \mathbf{X}(j\omega)\cdot\mathbf{H}(j\omega) = A - B\cdot j = \mathbf{Y}_{ss}(j\omega)$ = steady-state response in phasor form (real is cosine, imaginary is -sine)

$\mathbf{X}(j\omega)$ = the input expressed in phasor form NOT $\mathbf{X}(s)$ with $s := j\omega$, that would be ∞

$\mathbf{H}(j\omega)$ = the steady-state sinusoidal transfer function
= phasor-type transfer function

The **transient part** would be found by finishing the partial-fraction expansion.

Steady-State Response by Phasors

Expression of signals as phasors

T = Period

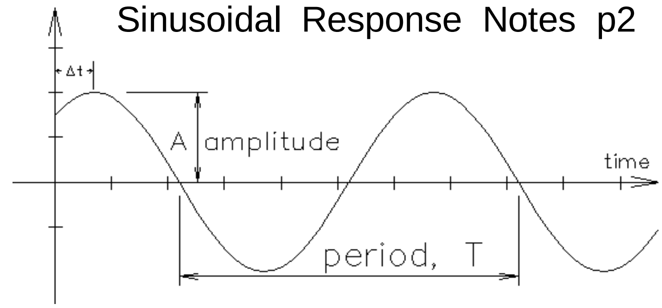
f = frequency, cycles / second $\quad f = \frac{1}{T} = \frac{\omega}{2\cdot\pi}$

ω = radian frequency, radians/sec $\quad \omega = 2\cdot\pi\cdot f$

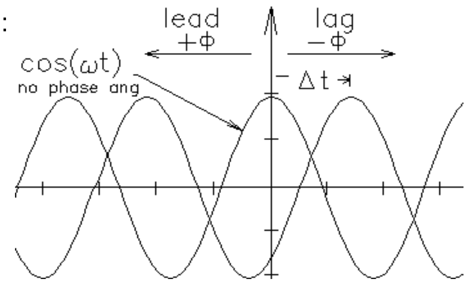
A = amplitude

Phase: $\phi = -\frac{\Delta t}{T}\cdot 360\cdot deg$ or: $\phi = -\frac{\Delta t}{T}\cdot 2\cdot\pi\cdot rad$

$y(t) = A\cdot\cos(\omega\cdot t + \phi)$



Phase:



		Phasor
voltage:	$v(t) = V_p\cdot\cos(\omega\cdot t + \phi)$	$\mathbf{V}(j\omega) = V_p\cdot e^{j\cdot\phi}$
current:	$i(t) = I_p\cdot\cos(\omega\cdot t + \phi)$	$\mathbf{I}(j\omega) = I_p\cdot e^{j\cdot\phi}$

Ex1 Let's assume the input to your system is $v_1(t) = 3.2\cdot V\cdot\cos(\omega\cdot t + 15\cdot deg)$ $\quad \mathbf{V_1}(j\omega) = 3.2\cdot V\cdot e^{j\cdot 15\cdot deg}$

or: $\mathbf{V_1}(j\omega) = 3.2V\ \underline{/15^o}$

In rectangular form:

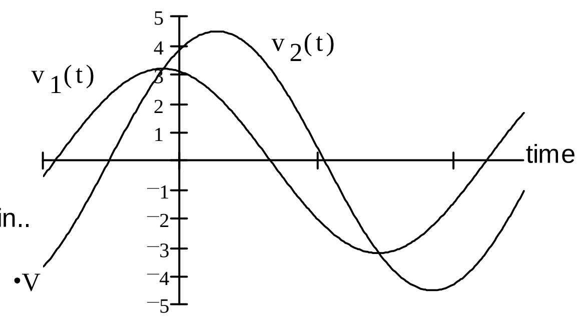
$3.2\cdot V\cdot\cos(15\cdot deg) = 3.091\cdot V$ $\qquad 3.2\cdot V\cdot\sin(15\cdot deg) = 0.828\cdot V$

or: $\mathbf{V_1}(j\omega) = (3.091 + 0.828\cdot j)\cdot V$

Ex2 What if a signal is the sum of two sinusoids.

$v_1(t) = 3.2\cdot V\cdot\cos(\omega\cdot t + 15\cdot deg)$

$v_2(t) = 4.5\cdot V\cdot\sin(\omega\cdot t + 60\cdot deg)$ $\qquad v_3(t) = v_1(t) + v_2(t)$



I'm going to drop the (jω) notation from the phasor notation, it gets cumbersome, but remember that phasors are in the frequency domain..

From Ex1: $\mathbf{V_1} := 3.2\cdot V\cdot e^{j\cdot 15\cdot deg} = 3.2V\ \underline{/15^o}$ $\qquad \mathbf{V_1} = 3.091 + 0.828j\ \cdot V$

Phasors are based on cosines, so express $v_2(t)$ as a cosine. $\qquad$ Remember: $\sin(\omega t) = \cos(\omega\cdot t - 90\cdot deg)$

So: $v_2(t) = 4.5\cdot V\cdot\cos(\omega\cdot t + 60\cdot deg - 90\cdot deg) = 4.5\cdot V\cdot\cos(\omega\cdot t - 30\cdot deg)$

$\mathbf{V_2} = 4.5V\ \underline{/-30^o}$ $\qquad$ or: $\mathbf{V_2} := 4.5\cdot V\cdot e^{-j\cdot 30\cdot deg}$

$4.5\cdot V\cdot\cos(-30\cdot deg) = 3.897\cdot V$ $\qquad 4.5\cdot V\cdot\sin(-30\cdot deg) = -2.25\cdot V$

$\mathbf{V_2} = 3.897 - 2.25j\ \cdot V$ \
$\mathbf{V_1} = 3.091 + 0.828j\ \cdot V$ } add
/

Add real parts: $\quad 3.897 + 3.091 = 6.988$

Add imaginary parts: $\quad -2.25 + 0.828 = -1.422$

$\mathbf{V_3} := \mathbf{V_1} + \mathbf{V_2}$

$\mathbf{V_3} = 6.988 - 1.422j\ \cdot V$ $\qquad$ sum

Change $\mathbf{V_3}$ back to polar coordinates:

$\sqrt{6.988^2 + 1.422^2} = 7.131$ $\qquad \operatorname{atan}\left(\frac{-1.422}{6.988}\right) = -11.502\cdot deg$

OR, in Mathcad notation (you'll see these in future solutions):

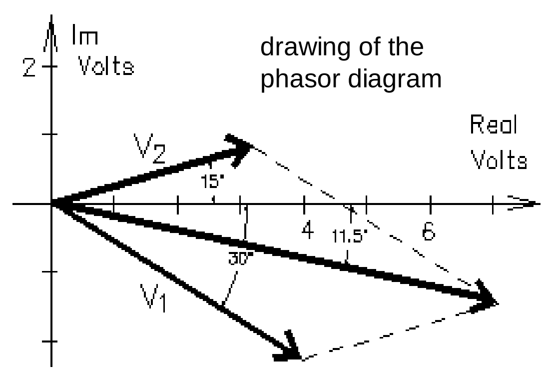
$|\mathbf{V_3}| = 7.131\cdot V$ $\qquad \arg(\mathbf{V_3}) = -11.5\cdot deg$

$\mathbf{V_3}(j\omega) = 7.131V\ \underline{/-11.5^o}$ $\quad$ or: $\mathbf{V_3}(j\omega) = 7.131\cdot V\cdot e^{-j\cdot 11.5\cdot deg}$

$\mathbf{V_3}$ may also be converted back to the time domain:

$v_3(t) = v_1(t) + v_2(t) = 7.13\cdot\cos(\omega\cdot t - 11.5\cdot deg)\cdot V$

drawing of the phasor diagram



Magnitude and Phase of transfer functions With steady-state sinusoidal inputs

Ex3 a) Find the magnitude and phase of the following transfer function at this frequency: $\omega := 2\cdot\frac{rad}{sec}$

$$\mathbf{H}(s) = \frac{2\cdot s^2 + 5\cdot s + 20}{s^2 + 1\cdot s + 10} = \frac{2\cdot s^2 + \frac{5}{sec}\cdot s + \frac{20}{sec^2}}{s^2 + \frac{1}{sec}\cdot s + \frac{10}{sec^2}}$$ Expressed with proper units

$s := j\cdot\omega$

$$\mathbf{H}(j\cdot\omega) = \frac{2\cdot(j\cdot\omega)^2 + 5\cdot(j\cdot\omega) + 20}{(j\cdot\omega)^2 + \frac{1}{sec}\cdot(j\cdot\omega) + \frac{10}{sec^2}} = \frac{\left(20 - 2\cdot\omega^2\right) + (5\cdot\omega)\cdot j}{\left(10 - \omega^2\right) + (1\cdot\omega)\cdot j} = \frac{\left(20 - 2\cdot 2^2\right) + (5\cdot 2)\cdot j}{\left(10 - 2^2\right) + (1\cdot 2)\cdot j} = \frac{12 + 10\cdot j}{6 + 2\cdot j}$$

without units

$$\left|\mathbf{H}(j\cdot\omega)\right| = M = \frac{\sqrt{12^2 + 10^2}}{\sqrt{6^2 + 2^2}} = 2.47$$

$$\underline{/\ H(j\omega)} = \operatorname{atan}\left(\frac{10}{12}\right) - \operatorname{atan}\left(\frac{2}{6}\right) = 21.37\cdot deg$$

b) Find the steady-state sinusoidal output if the input is: $3.2\cdot V\cdot\cos(2\cdot t + 15\cdot deg)$ $\quad \mathbf{V}_{\mathbf{in}} := 3.2\cdot V\cdot e^{j\cdot 15\cdot deg} = 3.2V\ \underline{/15^o}$

$$\mathbf{V}_{\mathbf{outss}}(j\omega) = \mathbf{V}_{\mathbf{in}}(j\omega)\cdot\mathbf{H}(j\cdot\omega) = \left(3.2\cdot V\cdot e^{j\cdot 15\cdot deg}\right)\cdot\left(2.47\cdot e^{j\cdot 21.37\cdot deg}\right) = 3.2\cdot V\cdot 2.47\cdot e^{j\cdot(15\cdot deg + 21.37\cdot deg)}$$

$$= 7.904\cdot V\cdot e^{j\cdot 36.37\cdot deg} = 6.364 + 4.687j\ \cdot V$$

$$v_{outss}(t) = 7.904\cdot V\cdot\cos(2\cdot t + 36.37\cdot deg)$$

Ex4 a) Find the magnitude and phase of the following transfer function at this frequency: $f := 5\cdot Hz$ $\quad \omega := 2\cdot\pi\cdot f$

$\omega = 31.42\cdot\frac{rad}{sec}$

$$\mathbf{H}(s) = \frac{s^2 + \frac{20}{sec}\cdot s + \frac{1000}{sec^2}}{s^2 + \frac{10}{sec}\cdot s + \frac{800}{sec^2}}$$

$$s := j\cdot\omega = j\cdot 31.42\cdot\frac{rad}{sec} \qquad \mathbf{H}(j\cdot\omega) = \frac{(j\cdot\omega)^2 + \frac{20}{sec}\cdot(j\cdot\omega) + \frac{1000}{sec^2}}{(j\cdot\omega)^2 + \frac{10}{sec}\cdot(j\cdot\omega) + \frac{800}{sec^2}} = \frac{(j\cdot 31.42)^2 + 20\cdot(j\cdot 31.42) + 1000}{(j\cdot 31.42)^2 + 10\cdot(j\cdot 31.42) + 800}$$ without units

$$= \frac{13.04 + 628.319\cdot j}{-186.96 + 314.159\cdot j} = 1.459 - 0.91j$$

$$\left|\mathbf{H}(j\cdot\omega)\right| = M = \sqrt{1.459^2 + 0.91^2} = 1.72 \qquad \underline{/\ H(j\omega)} = \operatorname{atan}\left(\frac{-0.91}{1.459}\right) = -31.95\cdot deg$$

b) Find the steady-state sinusoidal output if the input is: $x(t) = 4\cdot\cos(2\cdot\pi\cdot 5\cdot Hz\cdot t)$

$\mathbf{X}(\omega) = 4 + 0j$ and then $\mathbf{Y}(\omega) = 4\cdot(1.459 - 0.91\cdot j) = 5.836 - 3.64j$

Note that you can use the rectangular form of $\mathbf{H}(j\cdot\omega)$

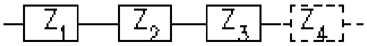
$$y(t) = 5.836\cdot\cos\left(31.42\cdot\frac{rad}{sec}\cdot t\right) + 3.64\cdot\sin\left(31.42\cdot\frac{rad}{sec}\cdot t\right)$$

note that the sine carries the opposite sign as the imaginary part.

$$\sqrt{5.836^2 + 3.64^2} = 6.878 \qquad \operatorname{atan}\left(\frac{-3.64}{5.836}\right) = -31.95\cdot deg \qquad y(t) = 6.88\cdot\cos\left(31.42\cdot\frac{rad}{sec}\cdot t - 32\cdot deg\right)$$

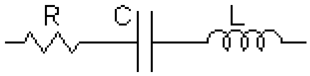
Impedances

series:



$$Z_{eq} = Z_1 + Z_2 + Z_3 \; + \ldots$$

Example:

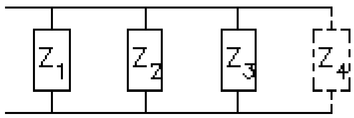


$$Z_{eq} = R + \frac{1}{C\cdot s} + L\cdot s$$

Voltage divider:

$$V_{Zn} = V_{total}\cdot\frac{Z_n}{Z_1 + Z_2 + Z_3 \; + \ldots}$$

parallel:

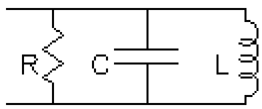


$$Z_{eq} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \; + \ldots}$$

Current divider:

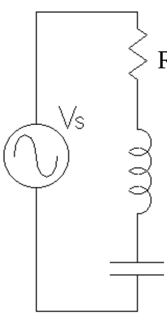
$$I_{Zn} = I_{total}\cdot\frac{\frac{1}{Z_n}}{\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \; + \ldots}$$

Example:



$$Z_{eq} = \frac{1}{\frac{1}{R} + \frac{1}{\frac{1}{C\cdot s}} + \frac{1}{L\cdot s}} = \frac{1}{\frac{1}{R} + C\cdot s + \frac{1}{L\cdot s}}$$

Ex5



$R := 500\cdot\Omega$

$L := 80\cdot mH$

$C := 0.4\cdot\mu F$

a) Find the steady-state $\mathbf{V_R}$ and $v_R(t)$ given $v_S(t)$ is a 12 Vpp cosine wave at: $f := 2\cdot kHz$

$$V_S(j\omega) = 6\cdot V\cdot e^{j0} \qquad \omega := 2\cdot\pi\cdot f \qquad \omega = 12566\cdot\frac{rad}{sec}$$

Transfer function for $\mathbf{V_R}$ as the output: $\mathbf{H}(s) = \frac{\mathbf{V_R}(s)}{\mathbf{V_S}(s)} = \frac{R}{R + L\cdot s + \frac{1}{C\cdot s}}$

$$V_R(j\omega) = 6\cdot V\cdot\frac{R}{R + L\cdot(j\cdot\omega) + \frac{1}{C\cdot(j\cdot\omega)}} = 1.666 - 2.687j \;\cdot V \qquad \mathbf{V_R} = 3.163V\ \underline{/-58.2}^o$$

$$v_R(t) = 3.163\cdot V\cdot\cos\left(12566\cdot\frac{rad}{sec}\cdot t - 58.2\cdot deg\right)$$

b) Find the current: $\frac{\mathbf{I}(s)}{\mathbf{V}(s)} = \frac{1}{\mathbf{Z}(s)} \qquad \mathbf{I}(s) = \frac{\mathbf{V}(s)}{\mathbf{Z}(s)} \qquad s := j\cdot\omega = j\cdot 12566\cdot\frac{rad}{sec}$

$$\mathbf{I} := \frac{6\cdot V\cdot e^{j\cdot 0}}{R + L\cdot(j\cdot\omega) + \frac{1}{C\cdot(j\cdot\omega)}} = \frac{6\cdot V}{500 + 0.080\cdot(j\cdot 12566) + \frac{1}{0.4\cdot 10^{-6}\cdot(j\cdot 12566)}} = \frac{6\cdot V}{500 + 1005.3\cdot j - 198.95\cdot j} = \frac{6\cdot V}{500 + 806.366\cdot j}$$

$$\sqrt{500^2 + 806.366^2} = 948.802$$

$$\operatorname{atan}\left(\frac{806.366}{500}\right) = 58.198\cdot deg$$

magnitude: $\frac{6\cdot V}{948.8\cdot\Omega} = 6.324\cdot mA$

angle: $0\cdot deg - 58.2\cdot deg = -58.2\cdot deg$ $\qquad \mathbf{I} = 6.324mA\ \underline{/-58.2}^o$

c) Draw a phasor diagram of all the voltages.

$\mathbf{V_L} = \mathbf{I}\cdot\mathbf{Z_L}$ $\quad 6.324\cdot mA\cdot 1005\cdot\Omega = 6.356\cdot V$

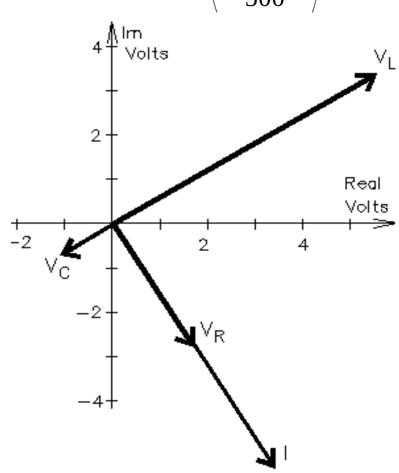
$-58.2\cdot deg + 90\cdot deg = 31.8\cdot deg$

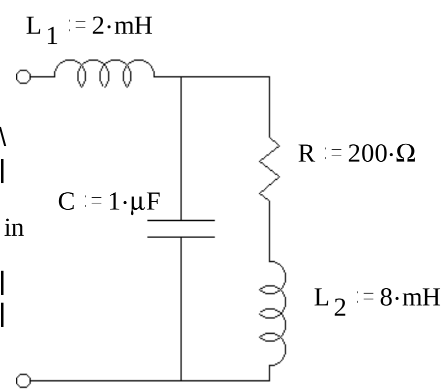
$\mathbf{V_L} = 6.356V\ \underline{/\,31.8}^o$

$\mathbf{V_C} = \mathbf{I}\cdot\mathbf{Z_C}$ $\quad 6.324\cdot mA\cdot(-199)\cdot\Omega = -1.258\cdot V$

$-58.2\cdot deg + (90)\cdot deg = 31.8\cdot deg$

$\mathbf{V_C} = -1.258V\ \underline{/\,31.8}^o = 1.258V\ \underline{/-148.2}^o$





Ex6 a) Find the steady-state $\mathbf{V_C}$ and $v_C(t)$ given $v_{in}(t)$ is a 12 Vp cosine wave at: $f := 2.5\cdot kHz$ with a 20º leading phase angle.

Transfer function for $\mathbf{V_C}$ as the output: $\mathbf{H}(s) = \dfrac{\mathbf{V_C}(s)}{\mathbf{V_{in}}(s)}$

$$\mathbf{H}(s) = \frac{\dfrac{1}{\dfrac{1}{R+L_2\cdot s}+C\cdot s}}{L_1\cdot s+\dfrac{1}{\dfrac{1}{R+L_2\cdot s}+C\cdot s}} \qquad \frac{\left(\dfrac{1}{R+L_2\cdot s}+C\cdot s\right)}{\left(\dfrac{1}{R+L_2\cdot s}+C\cdot s\right)}$$

$$= \frac{1}{L_1\cdot s\cdot\left(\dfrac{1}{R+L_2\cdot s}+C\cdot s\right)+1}$$

$\mathbf{V_{in}} := 12\cdot V\cdot e^{j\cdot 20\cdot deg}$ $\qquad \omega := 2\cdot\pi\cdot f$ $\qquad \omega = 15708\cdot\dfrac{rad}{sec}$

$$\mathbf{H}(j\omega) = \frac{1}{L_1\cdot(j\cdot\omega)\cdot\left[\dfrac{1}{R+L_2\cdot(j\cdot\omega)}+C\cdot(j\cdot\omega)\right]+1} = \frac{1}{0.002\cdot(j\cdot 15708)\cdot\left[\dfrac{1}{200+0.008\cdot(j\cdot 15708)}+10^{-6}\cdot(j\cdot 15708)\right]+1}$$

$$= \frac{1}{31.416\cdot j\ \cdot\left[3.585\cdot 10^{-3}-2.252\cdot 10^{-3}\cdot j\ +(j\cdot 0.015708)\right]+1} = \frac{1}{1-0.423+0.113\cdot j} = \frac{1}{0.588\cdot e^{j\cdot 11.039\cdot deg}}$$

$$= 1.7\cdot e^{-j\cdot 11.039\cdot deg}$$

$$\mathbf{V_C} = \mathbf{V_{in}}(j\omega)\cdot\mathbf{H}(j\omega) = 12\cdot V\cdot e^{j\cdot 20\cdot deg}\cdot 1.7\cdot e^{-j\cdot 11.039\cdot deg} = 12\cdot V\cdot 1.7\ \underline{/\ 20 - 11.039^{o}} = 20.4V\ \underline{/\ 8.96^{o}}$$

$$v_c(t) = 20.4\cdot V\cdot\cos\left(15708\cdot\frac{rad}{sec}\cdot t+8.96\cdot deg\right)$$

a) Find the steady-state $\mathbf{I_{L2}}$ and $i_{L2}(t)$.

Transfer function for $\mathbf{I_{L2}}$ as the output: $\mathbf{H}(s) = \dfrac{\mathbf{I_{L2}}(s)}{\mathbf{V_{in}}(s)} = \dfrac{\left(\dfrac{\mathbf{V_C}(s)}{R+L_2\cdot s}\right)}{\mathbf{V_{in}}(s)} = \dfrac{\mathbf{V_C}(s)}{\mathbf{V_{in}}(s)}\cdot\dfrac{1}{R+L_2\cdot s}$

$$\mathbf{H}(s) = \frac{1}{L_1\cdot s\cdot\left(\dfrac{1}{R+L_2\cdot s}+C\cdot s\right)+1}\cdot\frac{1}{(R+L_2\cdot s)} = \frac{1}{L_1\cdot s+L_1\cdot s\cdot C\cdot s\cdot(R+L_2\cdot s)+(R+L_2\cdot s)}$$

$$\mathbf{H}(j\omega) = \frac{1}{L_1\cdot(j\cdot\omega)+L_1\cdot C\cdot(j\cdot\omega)^2\cdot\left[R+L_2\cdot(j\cdot\omega)\right]+\left[R+L_2\cdot(j\cdot\omega)\right]} = 5.249\cdot 10^{-3} - 4.926\cdot 10^{-3} j \quad \cdot\frac{1}{\Omega} = \frac{7.198}{k\Omega}\cdot e^{-j\cdot 43.181\cdot deg}$$

$$\mathbf{I_{L2}} = \mathbf{V_{in}}(j\omega)\cdot\mathbf{H}(j\omega) = 12\cdot V\cdot e^{j\cdot 20\cdot deg}\cdot\left(\frac{7.198}{k\Omega}\cdot e^{-j\cdot 43.181\cdot deg}\right) = 12\cdot V\cdot\frac{7.198}{k\Omega}\ \underline{/\ 20 - 43.181^{o}} = 86.38mA\ \underline{/\ -23.18^{o}}$$

$$i_{L2}(t) = 86.38\cdot mA\cdot\cos\left(15708\cdot\frac{rad}{sec}\cdot t-23.18\cdot deg\right)$$

Ex7 This system: $\mathbf{H}(s) = \frac{s+20}{s+5}$ Has this input: $x(t) = 4\cdot\sin(12\cdot t - 40\cdot deg)\cdot u(t)$

a) Use steady-state AC analysis to find the steady-state output. $y_{ss}(t) = ?$

$$\text{AC steady-state } \mathbf{H}(j\omega) = \mathbf{H}(j\cdot 12) = \frac{j\cdot 12+20}{j\cdot 12+5} = \frac{\sqrt{12^2+20^2}\cdot e^{j\cdot atan\left(\frac{12}{20}\right)}}{\sqrt{12^2+5^2}\cdot e^{j\cdot atan\left(\frac{12}{5}\right)}} = \frac{23.324\cdot e^{j\cdot 30.964\cdot deg}}{13\cdot e^{j\cdot 67.38\cdot deg}}$$

$$= 1.794 \ \angle -36.416\cdot deg$$

$\mathbf{X}(j\omega) = 4\cdot e^{-j\cdot 130\cdot deg}$ Note 90º phase-lag because it's given as a sine wave

$$\mathbf{Y}_{ss}(j\omega) = 4\cdot 1.794 = 7.176 \ \angle\ -130 - 36.416 = -166.416$$

$$= 7.176 \ \angle -166.416\cdot deg$$

$$y_{ss}(t) = 7.176\cdot\cos(12\cdot t - 166.416\cdot deg)$$

b) Express the output, and separate into 3 partial fractions that you can find in the laplace transform table without using complex numbers. Show what they are, but don't find the coefficients.

Find the input as a sum of a pure sine and cosine $-4\cdot\sin(-130\cdot deg) = 3.064$ $4\cdot\cos(-130\cdot deg) = -2.571$

so: $4\cdot\sin(12\cdot t - 40\cdot deg)\cdot u(t) = (3.064\cdot\sin(12\cdot t) - 2.571\cdot\cos(12\cdot t))\cdot u(t)$

$$\mathbf{Y}(s) = \frac{3.064\cdot 12 - 2.571\cdot s}{s^2+144}\cdot\frac{(s+20)}{(s+5)} = \frac{A}{s+5} + \frac{B\cdot s}{\left(s^2+144\right)} + \frac{C\cdot 12}{\left(s^2+144\right)}$$

c) Continue with the partial fraction expansion just far enough to find the **transient** coefficient as a number.

$$(3.064\cdot 12 - 2.571\cdot s)\cdot(s+20) = A\cdot\left(s^2+144\right) + B\cdot s\cdot(s+5) + C\cdot 12\cdot(s+5)$$

let $s := -5$

$$(3.064\cdot 12 - 2.571\cdot(-5))\cdot(-5+20) = A\cdot\left(5^2+144\right) + 0 + 0$$

$$A := \frac{(s+20)\cdot(3.064\cdot 12 - 2.571\cdot s)}{25+144} \qquad A = 4.404$$

d) Express the complete (both transient and steady-state) output as a function of time. $y(t) = ?$

$$y(t) = \left(4.404\cdot e^{-5\cdot t} + 7.176\cdot\cos(12\cdot t - 166.416\cdot deg)\right)\cdot u(t)$$

$7.176\cdot\cos(-166.416\cdot deg) = -6.975$

$-7.176\cdot\sin(-166.416\cdot deg) = 1.685$

Either answer

$$y(t) = \left(4.404\cdot e^{-5\cdot t} - 6.975\cdot\cos(12\cdot t) + 1.685\cdot\sin(12\cdot t)\right)\cdot u(t)$$

e) What is the time constant of the transient part this expression? $\tau = ?$ $= \frac{1}{5}$

ECE 3510 Electrical Analogies of Mechanical Systems

A.Stolp
2/14/06,
rev,2/11/09

This method is based on the method of Linear Graphs. In Linear Graphs, all systems are reduced to a universal symbology peculiar to Linear Graphs (system graph) and then analyzed by methods very similar to those used with electric circuits. Since we, as electrical engineers, are already well versed in the symbology of electric circuits, my variation of linear graphs reduces mechanical systems to electric circuits and leaves them in that symbology for ready analysis by methods, tools and computer programs that we are already familiar with.

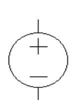
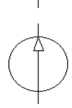
Across and Through Variables

	Across Variable	**Through Variable**
Electrical	V = voltage (volts) or (V)	I = current (Amps) or (A)
Mechanical translational	v = velocity $\left(\frac{m}{sec}\right)$	F = force (newtons) or (N) or $\left(Kg\cdot\frac{m}{sec^2}\right)$
Mechanical rotational	ω = angular velocity $\left(\frac{rad}{sec}\right)$	T = torque $(N\cdot m)$
Fluid	P = pressure $\left(\frac{N}{m^2}\right)$ or (Pa)	Q = flow $\left(\frac{m^3}{sec}\right)$

Elements

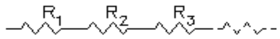
	Dissipation	**Across Variable Energy Storage**	**Through Variable Energy Storage**
Electrical	R = resistance $\left(\frac{V}{A}\right)$ or (Ω)	C = capacitance $\left(\frac{A\cdot sec}{V}\right)$ or (F)	L = inductor $\left(\frac{V\cdot sec}{A}\right)$ or (H)
Mechanical translational	B = damping $\left(\frac{N\cdot sec}{m}\right)$	M = mass (Kg) or $\left(\frac{N\cdot sec^2}{m}\right)$	k = Spring constant $\left(\frac{N}{m}\right)$
Mechanical rotational	B = damping $(N\cdot m\cdot sec)$ or $\left[\frac{N\cdot m}{\left(\frac{rad}{sec}\right)}\right]$	J = moment of inertia $(Kg\cdot m^2)$ or $\left(\frac{N\cdot m^3}{sec^2}\right)$	k = Spring constant $\left(\frac{N\cdot m}{rad}\right)$
Fluid	R_f = fluid resistance $\left(\frac{N\cdot sec}{m^5}\right)$	C_f = fluid capacitance $\left(\frac{m^5}{N}\right)$	I = fluid inertia $\left(\frac{Kg}{m^4}\right)$

Basic Electric Circuit Analysis

Element	Parts like resistors, capacitors, inductors & transformers
Wires and connections	Direct the current, but do not affect voltage
Circuit	Wires and elements connected to form loops
Voltage	Measured as a difference **across** an element
Current	Flows **through** a wire or element
Kirchhoff's Current Law (KCL)	Current in = current out of all elements, wires & connections
Kirchhoff's Voltage Law (KVL)	Voltage gains = voltage "losses" around any circuit loop
Node	Connected wires and connections which all have the same voltage
Ground	Zero-reference node for all other nodal voltages
Branch	Connected wires and elements which all have the same current
Power $P = V\cdot I$	Power = Across variable x Through variable
Voltage Source 	Constant voltage regardless of current in or out
Current Source 	Constant current regardless of voltage + or -

Passive Electrical Elements

Resistors

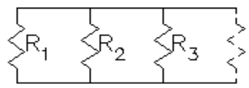


series: $R_{eq} = R_1 + R_2 + R_3 + \ldots$

Exactly the **same current** through each resistor

voltage divider: $V_{Rn} = V_{total} \cdot \dfrac{R_n}{R_1 + R_2 + R_3 + \ldots}$

parallel: $R_{eq} = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \ldots}$



Exactly the **same voltage** across each resistor

current divider: $I_{Rn} = I_{total} \cdot \dfrac{\dfrac{1}{R_n}}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \ldots}$

Resistors dissipate power $P = V \cdot I = I^2 \cdot R = \dfrac{V^2}{R}$

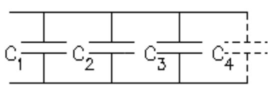
Capacitors

$C = \dfrac{Q}{V}$ $\quad$ $farad = \dfrac{coul}{volt} = \dfrac{amp \cdot sec}{volt}$ $\quad$ $v_C = \dfrac{1}{C} \cdot \displaystyle\int_{-\infty}^{t} i_C \, dt = \dfrac{1}{C} \cdot \int_0^t i_C \, dt + v_C(0)$ (initial voltage) $\quad$ $i_C = C \cdot \dfrac{d}{dt} v_C$

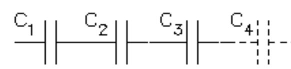
Energy stored in electric field: $E_C = \dfrac{1}{2} \cdot C \cdot V_C^{\,2}$

Capacitor voltage **cannot** change instantaneously

parallel: $C_{eq} = C_1 + C_2 + C_3 + \ldots$



series: $C_{eq} = \dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2} + \dfrac{1}{C_3} + \ldots}$



Steady-state sinusoids:
Impedance: $\mathbf{Z_C} = \dfrac{1}{j \cdot \omega \cdot C} = \dfrac{-j}{\omega \cdot C}$

Current leads voltage by 90 deg

Laplace:
Impedance: $\mathbf{Z_C} = \dfrac{1}{C \cdot s}$

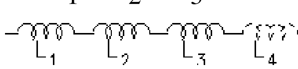
Inductors

$henry = \dfrac{volt \cdot sec}{amp}$ $\quad$ $i_L = \dfrac{1}{L} \cdot \displaystyle\int_{-\infty}^{t} v_L \, dt = \dfrac{1}{L} \cdot \int_0^t v_L \, dt + i_L(0)$ (initial current) $\quad$ $v_L = L \cdot \dfrac{d}{dt} i_L$

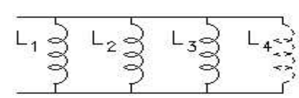
Energy stored in magnetic field: $E_L = \dfrac{1}{2} \cdot L \cdot I_L^{\,2}$

Inductor current **cannot** change instantaneously

series: $L_{eq} = L_1 + L_2 + L_3 + \ldots$



parallel: $L_{eq} = \dfrac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2} + \dfrac{1}{L_3} + \ldots}$



Steady-state sinusoids:
Impedance: $\mathbf{Z_L} = j \cdot \omega \cdot L$

Current lags voltage by 90 deg

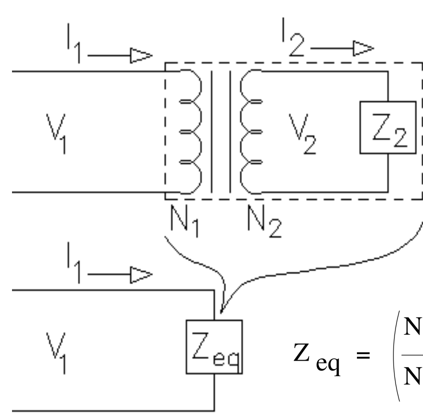
Laplace:
Impedance: $\mathbf{Z_L} = L \cdot s$

Transformers (ideal)

Ideal: $P_1 = P_2$ $\quad$ power in = power out

Turns ratio $= N = \dfrac{N_1}{N_2} = \dfrac{V_1}{V_2} = \dfrac{I_2}{I_1}$

Note: some books define the turns ratio as N_2/N_1

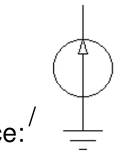
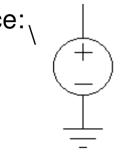
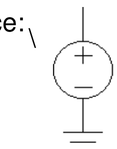


$Z_{eq} = \left(\dfrac{N_1}{N_2}\right)^2 \cdot Z_2$

Equivalent impedance in primary: $\mathbf{Z_{eq}} = N^2 \cdot \mathbf{Z_2} = \left(\dfrac{N_1}{N_2}\right)^2 \cdot \mathbf{Z_2}$

You can replace the entire transformer and load with ($\mathbf{Z_{eq}}$).
This "impedance transformation" can work across systems.

Mechanical system with linear motion (translational)

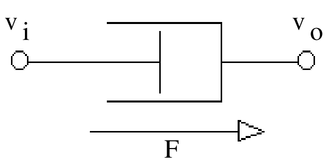
	Mechanical translational	**Electrical**	
Through Variable:	F = Force (N)	I = current (A)	Source: 
Across Variable:	v = velocity $\left(\frac{m}{sec}\right)$	V = voltage (V)	Source: 
$\int v\,dt$	x = displacement (m)	Source:  $v = \frac{d}{dt}x$ or $s \cdot X(s)$	
$\frac{V(s)}{s}$	$X(s)$ = displacement $(m \cdot sec)$ (in freq domain)		

Dissipation element:

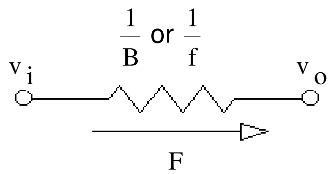
power

$$P = v \cdot F = \frac{F^2}{B}$$
$$= v^2 \cdot B$$

Damper or friction



Resistor



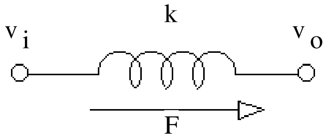
Impedance

$\frac{1}{B}$ or $\frac{1}{f}$

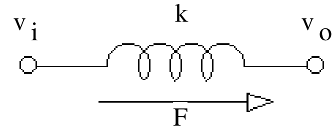
Through variable energy storage:

$$E = \frac{1}{2} \cdot \frac{1}{k} \cdot F^2 = \frac{1}{2} \cdot k \cdot x^2$$
(F=kx)

Spring

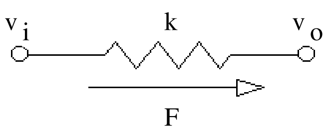


Inductor



$\frac{s}{k}$

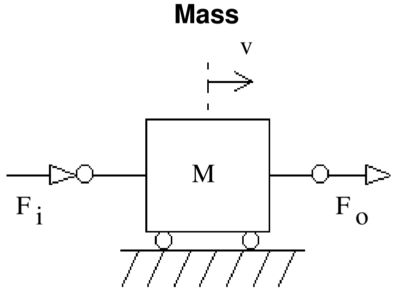
Springs are sometimes shown like this:



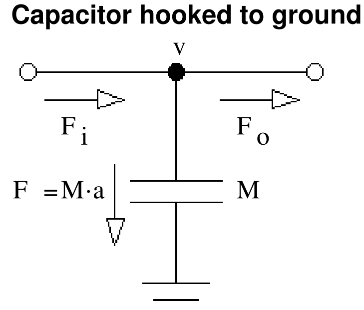
Through variable energy storage:

$$E = \frac{1}{2} \cdot M \cdot v^2$$

Mass

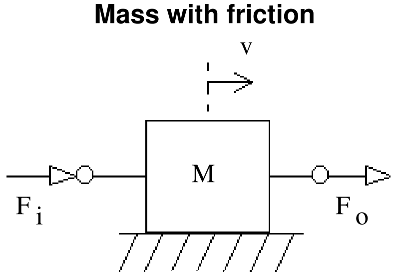


Capacitor hooked to ground



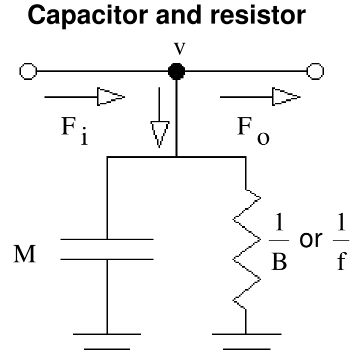
$\frac{1}{M \cdot s}$

Mass with friction



B or f
sliding friction

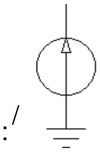
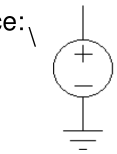
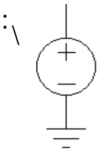
Capacitor and resistor



$$\frac{1}{\frac{1}{\left(\frac{1}{M \cdot s}\right)} + \frac{1}{\left(\frac{1}{B}\right)}}$$

$$\frac{1}{M \cdot s + B}$$

Mechanical system with circular motion (rotational)

	Mechanical rotational	**Electrical**	
Through Variable:	T = Torque $(N\cdot m)$	I = current (A)	Source: 
Across Variable:	ω = angular velocity $\left(\frac{rad}{sec}\right)$	V = voltage (V)	Source: 
$\int \omega \, dt$	θ = angular displacement (rad)	Source:  $\omega = \frac{d}{dt}\theta$ or $s\cdot\theta(s)$	
$\frac{\omega(s)}{s}$	$\theta(s)$ = angular displacement $(rad\cdot sec)$ (in freq domain)		

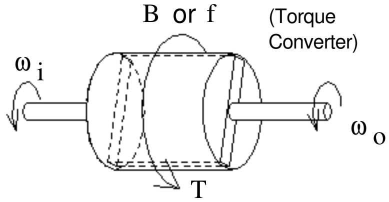
Dissipation element:

power

$$P = v\cdot T = \frac{T^2}{B} = \omega^2\cdot B$$

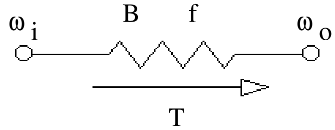
Damper or friction

B or f (Torque Converter)



Resistor

$\frac{1}{B}$ or $\frac{1}{f}$



Impedance

$\frac{1}{B}$ or $\frac{1}{f}$

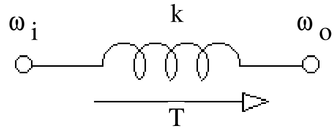
Through variable energy storage:

$$E = \frac{1}{2}\cdot\frac{1}{k}\cdot T^2$$

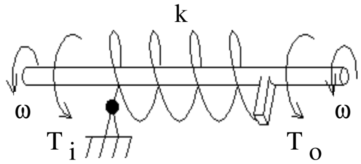
Springs

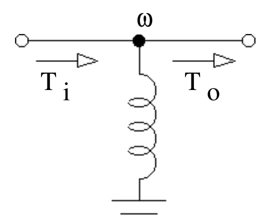
Inductor

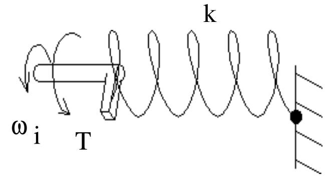
$\frac{1}{k}$

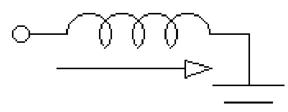


$\frac{s}{k}$





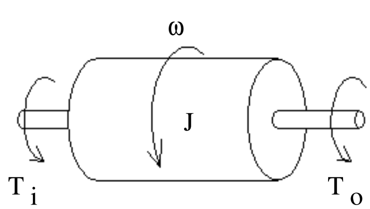




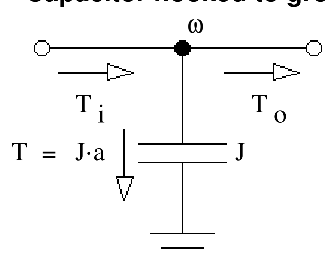
Through variable energy storage:

$$E = \frac{1}{2}\cdot J\cdot\omega^2$$

Moment of Inertia, J

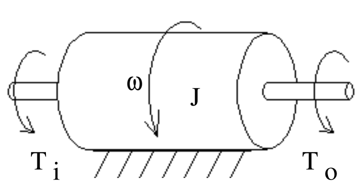


Capacitor hooked to ground



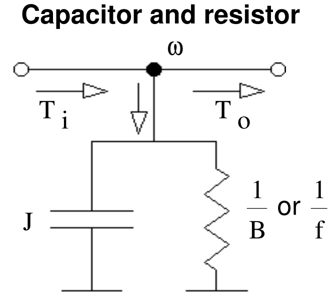
$\frac{1}{J\cdot s}$

J with friction



B or f
sliding friction

Capacitor and resistor



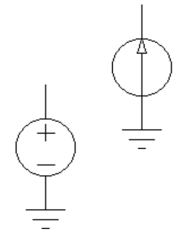
$\frac{1}{B}$ or $\frac{1}{f}$

$$\frac{1}{\frac{1}{\left(\frac{1}{J\cdot s}\right)} + \frac{1}{\left(\frac{1}{B}\right)}}$$

$$\frac{1}{J\cdot s + B}$$

Fluid (hydraulic) system

Sources

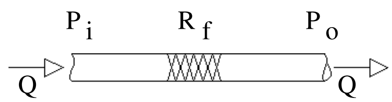


	Fluid	**Electrical**
Through Variable:	Q = volumetric flow rate $\left(\frac{m^3}{sec}\right)$	I = current (A)
Across Variable:	P = Pressure $\left(\frac{N}{m^2}\right)$ or (Pa)	V = voltage (V)

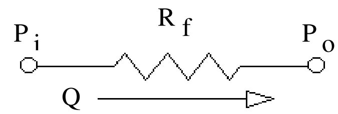
Dissipation element: power

$$P = P \cdot Q = \frac{Q^2}{R_f} = P^2 \cdot R_f$$

Fluid resistance



Resistor



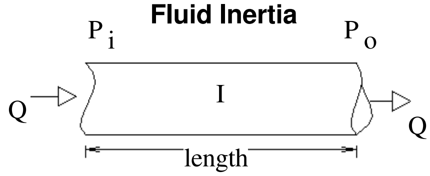
Impedance

R_f

Through variable energy storage:

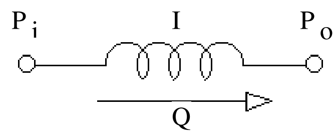
$$E = \frac{1}{2} \cdot I \cdot Q^2$$

Fluid Inertia



$$I = \frac{M}{A^2} = \frac{\rho \cdot length}{A}$$

Inductor

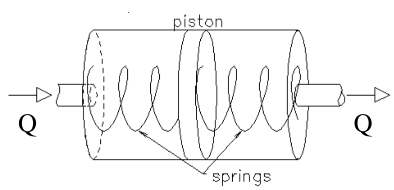


$I \cdot s$

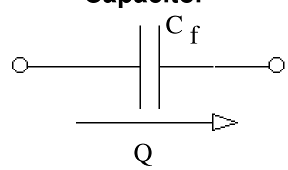
Through variable energy storage:

$$E = \frac{1}{2} \cdot C_f \cdot P^2$$

Fluid Capacitors

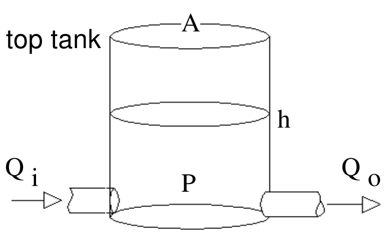


Capacitor

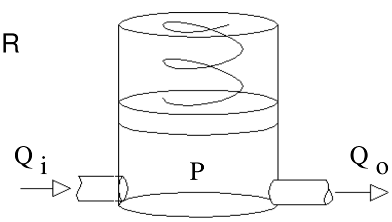


$\frac{1}{C_f s}$

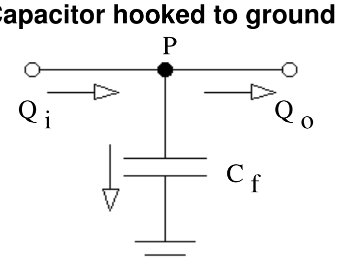
open top tank



OR



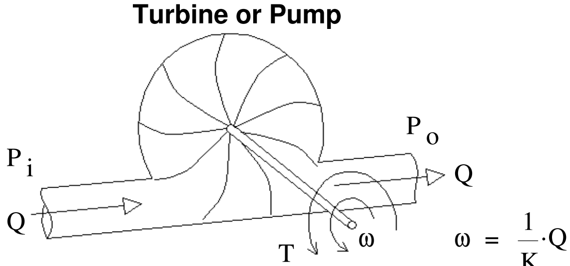
Capacitor hooked to ground



$$C_f = \frac{\Delta volume}{\Delta pressure} = \frac{\Delta h \cdot A}{\Delta h \cdot \rho \cdot g} = \frac{A}{\rho \cdot g}$$

for all capacitors — For open top tank

Turbine or Pump



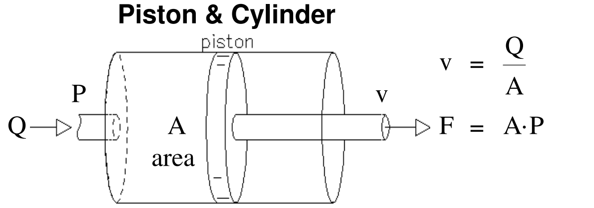
$$\omega = \frac{1}{K} \cdot Q$$

$$T = K\left(P_o - P_i\right)$$

Piston & Cylinder

$$P = \frac{F}{A}$$

$$Q = A \cdot v$$



$$v = \frac{Q}{A}$$

$$F = A \cdot P$$

Turbines & pistons convert through variables to across variables & vice versa, so there are no good electrical analogies.

Yet you can still transform an impedance from a mechanical system into the fluid system. You,ll find that capacitors become inductors, inductors become capacitors and parallel swaps with series.

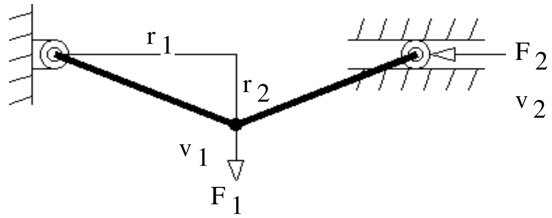
$$\mathbf{Z}_{eq} = \frac{\Delta \cdot P}{Q} = \frac{\left(\frac{T}{K}\right)}{K \cdot \omega} = \frac{1}{K^2} \cdot \frac{T}{\omega} = \frac{1}{K^2} \cdot \frac{1}{\mathbf{Z}_2} = \frac{1}{K^2 \cdot \mathbf{Z}_2}$$

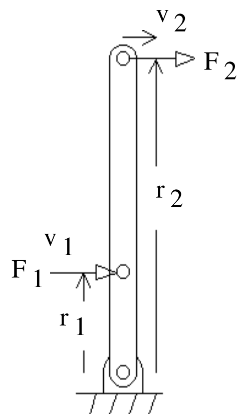
$$\mathbf{Z}_{eq} = \frac{P}{Q} = \frac{\left(\frac{F}{A}\right)}{A \cdot v} = \frac{1}{A^2} \cdot \frac{F}{v} = \frac{1}{A^2} \cdot \frac{1}{\mathbf{Z}_2} = \frac{1}{A^2 \cdot \mathbf{Z}_2}$$

Transducers and Transformers

A transducer converts power from one type to another. We can model many of them with transformers.
Transformers increase the through variable and correspondingly decrease the across variable or vice-versa.

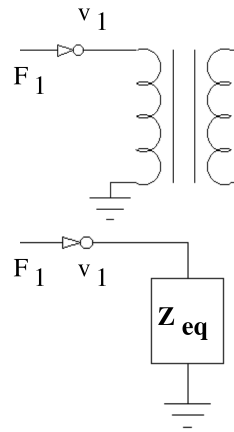
Levers





$$N = \frac{r_1}{r_2} = \frac{v_1}{v_2} = \frac{F_2}{F_1}$$

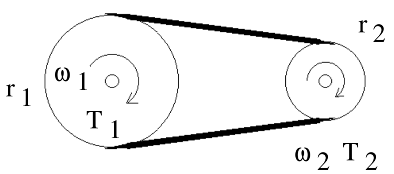
(not really this simple)

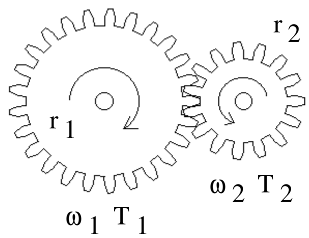


$$Z_{eq} = N^2 \cdot Z_2 = \left(\frac{r_1}{r_2}\right)^2 \cdot Z_2$$

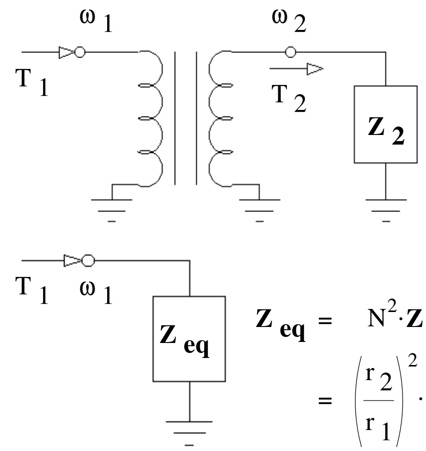
Belts, chains, & gears

r = radius of pulley or pitch radius of gears





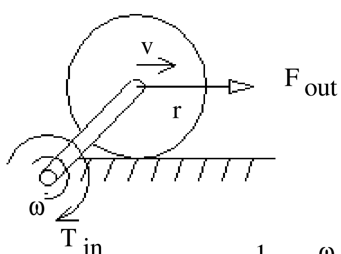
$$N = \frac{r_2}{r_1} = \frac{\omega_1}{\omega_2} = \frac{T_2}{T_1} = \text{gear tooth ratio} \left(\frac{N_2}{N_1}\right)$$

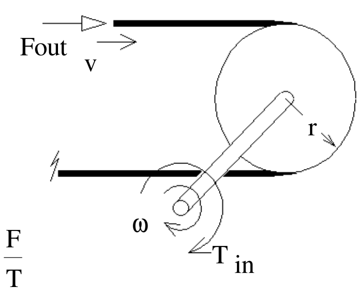


$$Z_{eq} = N^2 \cdot Z_2 = \left(\frac{r_2}{r_1}\right)^2 \cdot Z_2$$

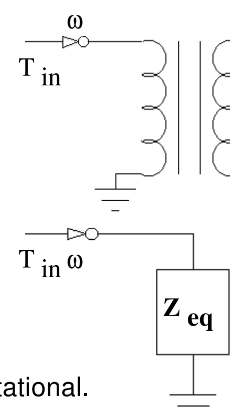
Tires, racks, & conveyors

r = radius of wheel or pitch radius of pinion gear





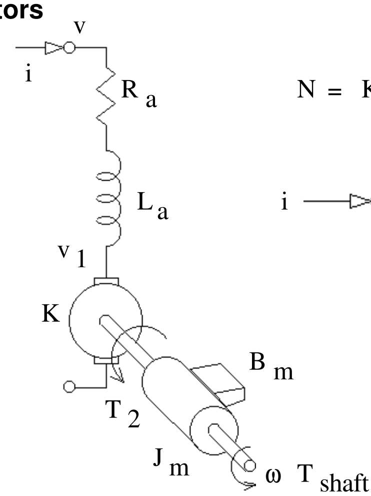
$$N = \frac{1}{r} = \frac{\omega}{v} = \frac{F}{T}$$



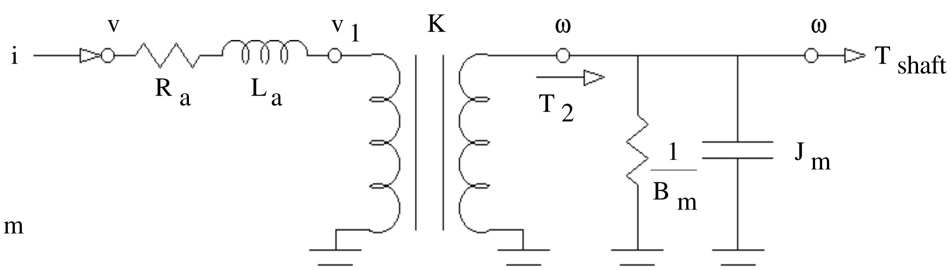
$$Z_{eq} = N^2 \cdot Z_2 = \left(\frac{1}{r}\right)^2 \cdot Z_2$$

Note: $N = r$ if the input is linear motion and output is rotational.

DC Motors



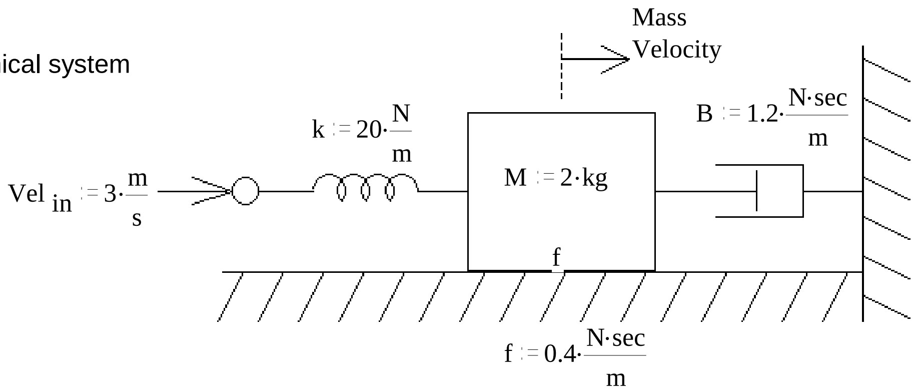
$$N = K = \frac{v_1}{\omega} = \frac{T_2}{i}$$



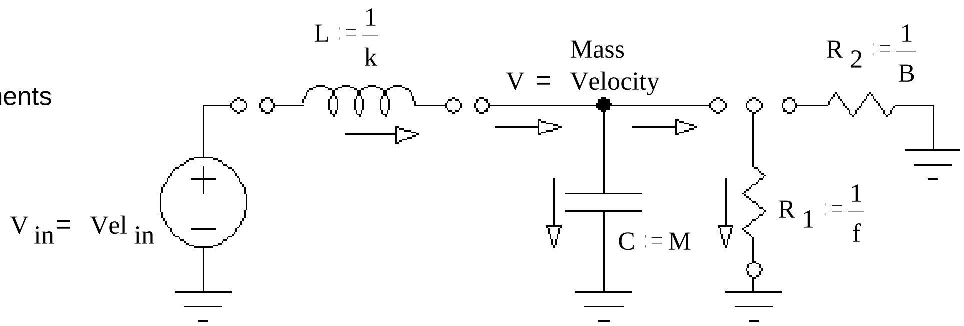
A.Stolp
2/11/09,
rev,1/30/15

Example 1

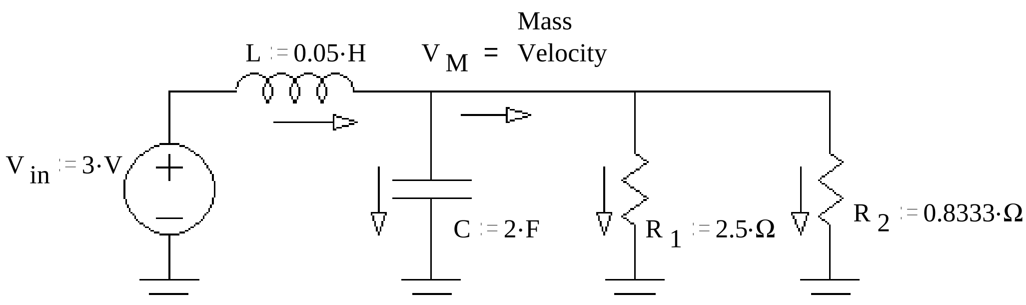
Mechanical system



Elements



Circuit



$$R_{eq} := \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} \qquad R_{eq} = 0.625 \cdot \Omega$$

$$B_{eq} := \frac{1}{R_{eq}}$$

Transfer function

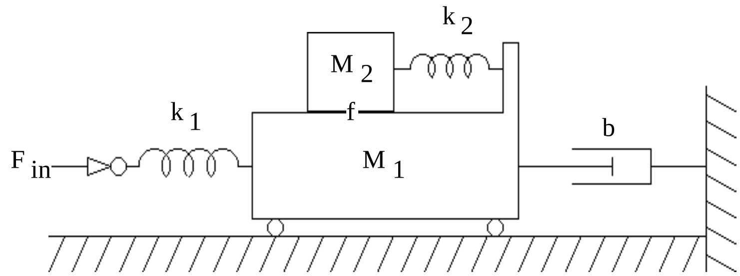
$$\frac{V_M(s)}{V_{in}(s)} = \frac{\dfrac{1}{C\cdot s + \dfrac{1}{R_{eq}}}}{L\cdot s + \dfrac{1}{C\cdot s + \dfrac{1}{R_{eq}}}} \cdot \frac{\left(C\cdot s + \dfrac{1}{R_{eq}}\right)}{\left(C\cdot s + \dfrac{1}{R_{eq}}\right)} = \frac{1}{L\cdot C\cdot s^2 + \dfrac{L}{R_{eq}}\cdot s + 1} \cdot \frac{\left(\dfrac{1}{L\cdot C}\right)}{\left(\dfrac{1}{L\cdot C}\right)} = \frac{\dfrac{1}{L\cdot C}}{s^2 + \dfrac{1}{C\cdot R_{eq}}\cdot s + \dfrac{1}{L\cdot C}}$$

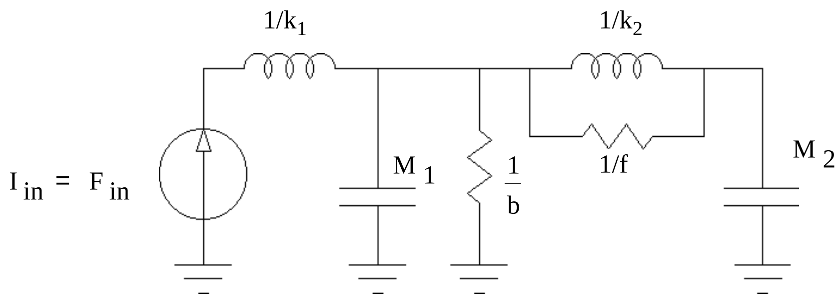
$$\frac{V_M(s)}{Vel_{in}(s)} = \frac{\dfrac{k}{M}}{s^2 + \dfrac{B_{eq}}{M}\cdot s + \dfrac{k}{M}} = \frac{\dfrac{20\cdot N}{2\cdot kg\cdot m}}{s^2 + \dfrac{1.6\cdot N\cdot sec}{2\cdot kg\cdot m}\cdot s + \left(\dfrac{20\cdot N}{2\cdot kg\cdot m}\right)} = \frac{10}{s^2 + 0.8\cdot s + 10}$$

without units

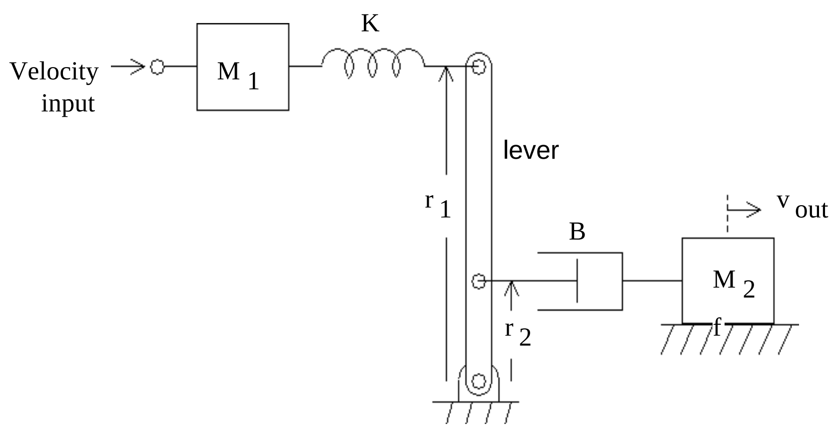
same, either way

Example 2

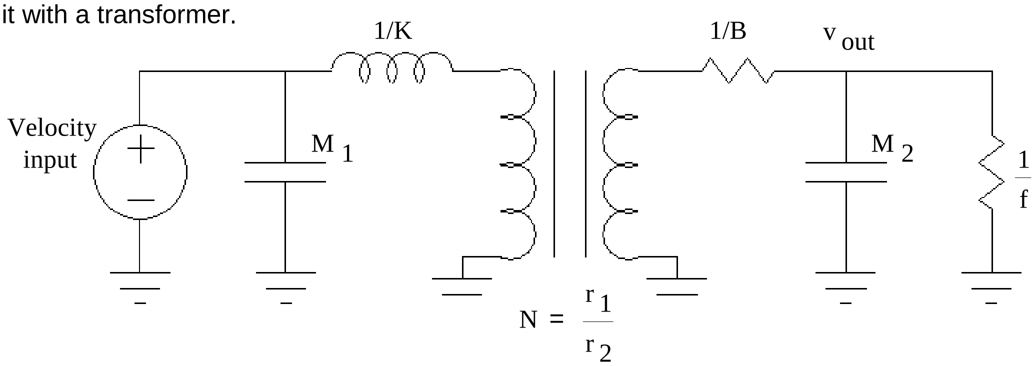




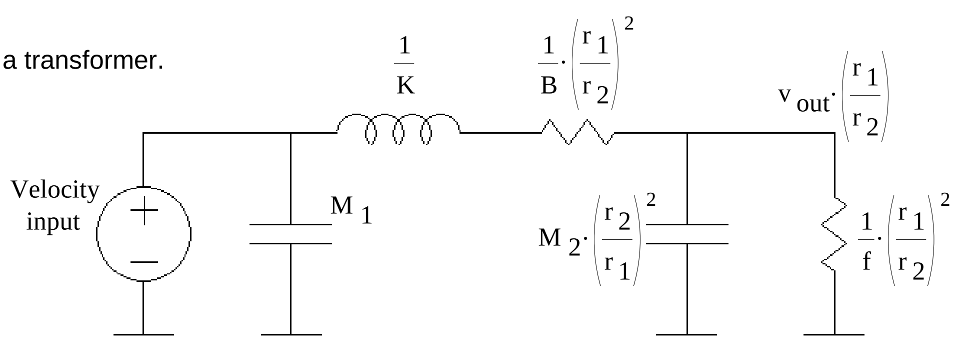
Example 3



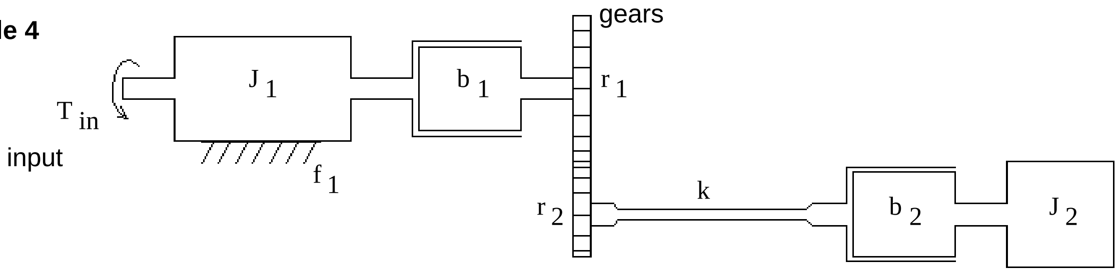
Circuit with a transformer.



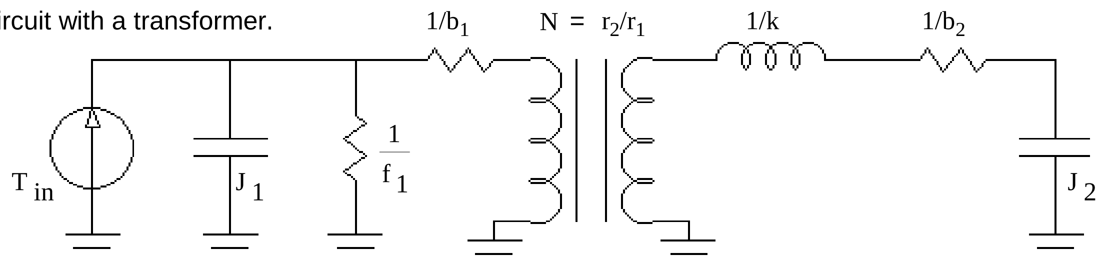
Circuit without a transformer.



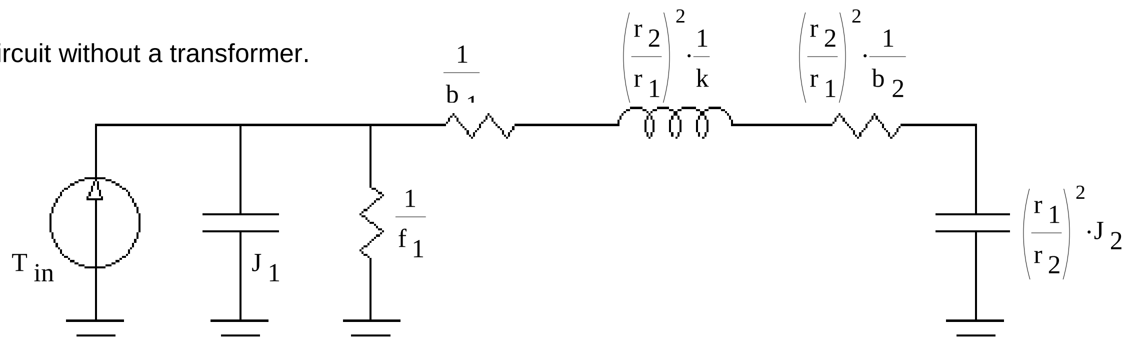
Example 4



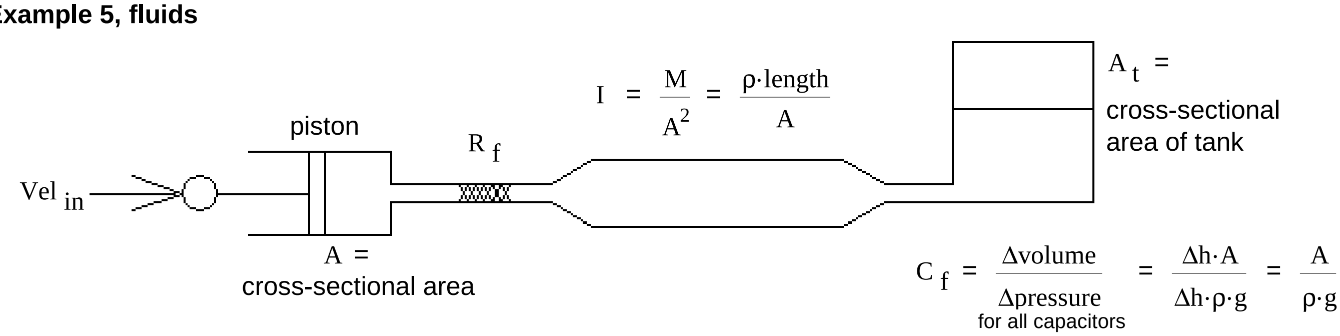
Circuit with a transformer.

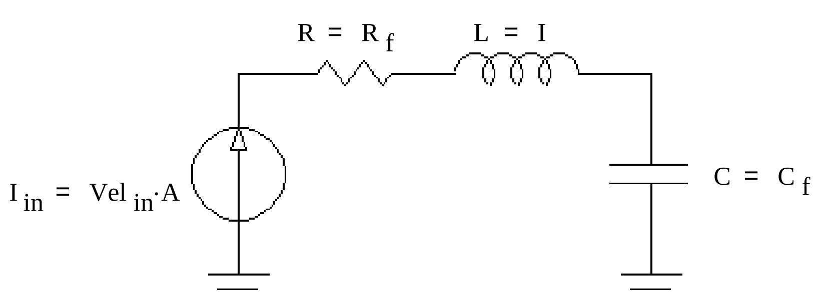


Circuit without a transformer.



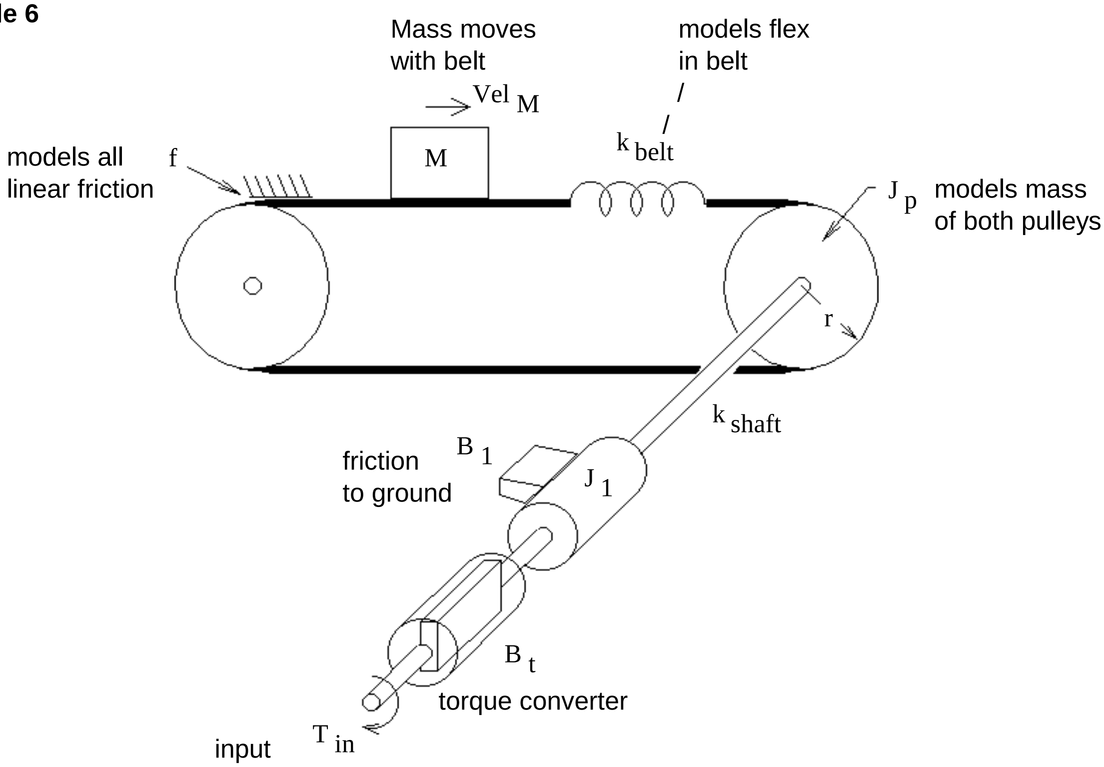
Example 5, fluids



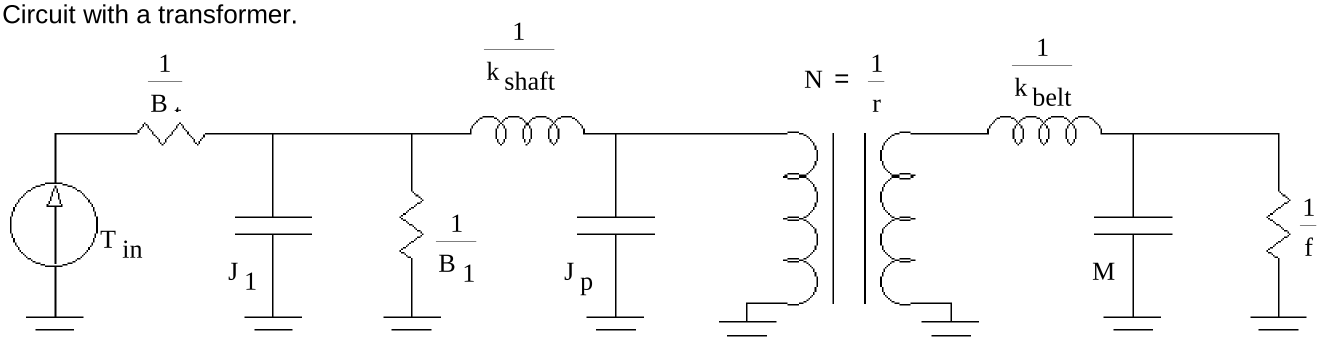


ECE 3510 Mechanical to Electrical Examples p.3

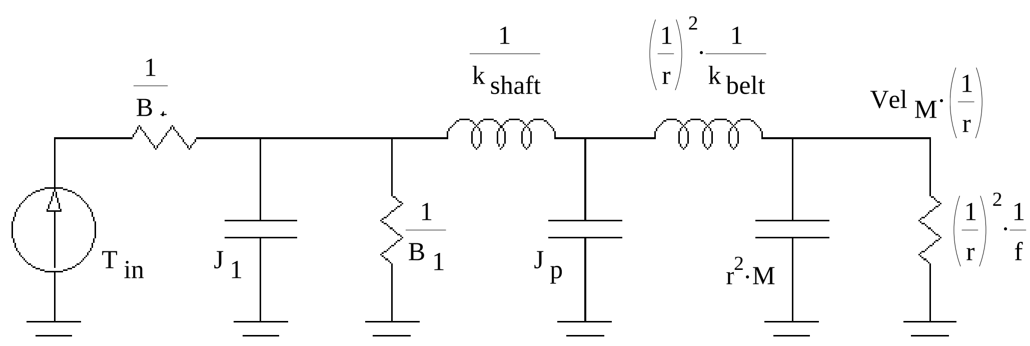
Example 6



Circuit with a transformer.



Circuit without a transformer.



ECE 3510 homework # 6 Review of Steady-State AC Due: Fri, 2/12/21 b

Same as in Week 03 handouts

1. Convert the following complex numbers to polar form ($m\underline{/\theta}$ or $me^{j\theta}$). a) $2.6 + 8.7j$ b) $3 + 4j$ c) $-3 - 4j$

2. Convert the following complex numbers to rectangular form (a + bj). a) $10 \cdot e^{j \cdot 60 \cdot deg}$ b) $10 \cdot e^{-j \cdot 45 \cdot deg}$ c) $20 \cdot e^{j \cdot 120 \cdot deg}$

3. Add or subtract the complex numbers. a) $(3 + 2j) + (6 + 9j)$ b) $(9 - 10j) - (9 + 10j)$

4. Multiply the complex numbers. a) $\left(20 \cdot e^{j \cdot 40 \cdot deg}\right) \cdot \left(10 \cdot e^{j \cdot 60 \cdot deg}\right)$ b) $(-2 - j) \cdot (-6 - 9j)$

5. Divide the complex numbers. a) $\dfrac{20 \cdot e^{j \cdot 40 \cdot deg}}{10 \cdot e^{j \cdot 60 \cdot deg}}$ b) $\dfrac{12 + 10j}{6 + 9j}$

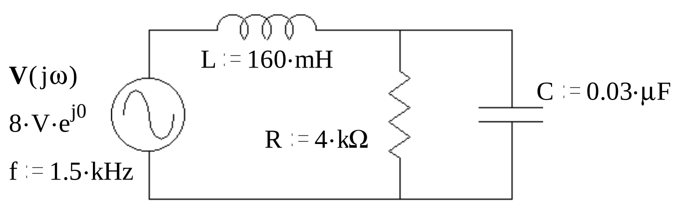
6. Add and subtract the sinusoidal voltages using phasors. Draw a phasor diagram which shows all 4 phasors, and give your final answer in time domain form.

   $v_1(t) = 1.5 \cdot V \cdot \cos(\omega \cdot t + 10 \cdot deg)$ $v_2(t) = 3.2 \cdot V \cdot \cos(\omega \cdot t + 25 \cdot deg)$

   a) Find $v_3(t) = v_1(t) + v_2(t)$ b) Find $v_4(t) = v_1(t) - v_2(t)$

7. a) Find $\mathbf{Z_{eq}}$.

   b) Find the current $\mathbf{I_L}(j\omega)$.



8. Find the steady-state magnitude and phase of each of the following transfer functions. $|H(j \cdot \omega)|$ = ? $\underline{/H(j\omega)}$ = ?

   a) $\omega := 10 \cdot \dfrac{rad}{sec}$ $s = j \cdot \omega$ $H(s) = \dfrac{\dfrac{40}{sec} \cdot s}{s^2 + \dfrac{10}{sec} \cdot s + \dfrac{200}{sec^2}}$

   b) $f := 50 \cdot Hz$ $H(s) = \dfrac{s^2 + \dfrac{1000}{sec} \cdot s}{s^2 + \dfrac{300}{sec} \cdot s + \dfrac{10000}{sec^2}}$

9. Find the following outputs. Express them in the time domain, first as a cosine with a phase angle and then as a sum of cosine and sine with no phase angles:

   a) The input $x(t) = 3 \cdot \cos(10 \cdot t)$ is the input for the transfer function of 8a), above.

   b) The input $x(t) = 5 \cdot \sin(2 \cdot \pi \cdot 50 \cdot t)$ is the input for the transfer function of 8b), above.

   remember, sine is -j

Answers

1. a) $9.08 \cdot e^{j \cdot 73.4 \cdot deg}$ b) $5 \cdot e^{j \cdot 53.1 \cdot deg}$ c) $5 \cdot e^{-j \cdot 126.9 \cdot deg}$

2. a) $5 + 8.66 \cdot j$ b) $7.071 - 7.071 \cdot j$ c) $-10 + 17.321 \cdot j$

3. a) $9 + 11 \cdot j$ b) $-20 \cdot j$

4. a) $200 \cdot e^{j \cdot 100 \cdot deg}$ b) $24.2 \cdot e^{j \cdot 82.9 \cdot deg}$

5. a) $2 \cdot e^{-j \cdot 20 \cdot deg}$ b) $1.385 - 0.41 \cdot j$

6. a) $v_1(t) + v_2(t) = 4.67 \cdot \cos(\omega \cdot t + 20.2 \cdot deg) \cdot V$ b) $v_1(t) - v_2(t) = 1.794 \cdot \cos(\omega \cdot t - 142.5 \cdot deg) \cdot V$

7. a) $1.82 \cdot k\Omega$ $-15.2 \cdot deg$ b) $4.4 \cdot mA$ $15.2 \cdot deg$

8. a) M = 2.828 $45 \cdot deg$ b) M = 2.544 $-25.8 \cdot deg$

9. a) $y(t) = 8.484 \cdot \cos\left(10 \cdot \dfrac{rad}{sec} \cdot t + 45 \cdot deg\right) = 6 \cdot \cos\left(10 \cdot \dfrac{rad}{sec} \cdot t\right) - 6 \cdot \sin\left(10 \cdot \dfrac{rad}{sec} \cdot t\right)$

   b) $y(t) = 12.72 \cdot \cos(2 \cdot \pi \cdot 50 \cdot t - 115.82 \cdot deg) = -5.54 \cdot \cos(2 \cdot \pi \cdot 50 \cdot t) + 11.45 \cdot \sin(2 \cdot \pi \cdot 50 \cdot t)$

1. Given the conditions in example 3.4.3, p.57,
   a) Show all the steps needed to find eq. 3.60.
   b) Use the Laplace transform table to find the results in eq. 3.61 and 3.62 ($y_{ss}(t)$ part).
   c) Show that equations 3.63 & 3.64 can be found from equations 3.62.
   d) Show that equations 3.63 & 3.64 can be found from steady-state analysis of H(s) (see eq. 3.56).

2. Still referring to the system in example 3.4.3, p.57, the input is: $x(t) = x_m \cdot \sin(\omega_o \cdot t)$
   a) Confirm eq. 3.66.
   b) Use any method you want to find M and ϕ_2 in: $y_{ss}(t) = M \cdot x_m \cdot \cos(\omega_o \cdot t + \phi_2)$

      Hint:, you may want to recall that: $\sin(\omega_o \cdot t) = \cos(\omega_o \cdot t - 90 \cdot deg)$

3. This system: $H(s) = \dfrac{3}{s+8}$ Has a cosine input: $x(t) = 4 \cdot \cos(10 \cdot t) \cdot u(t)$
   a) Express the output, $Y(s)$
   b) This separates into 3 partial fractions that you can find in the laplace transform table without using complex numbers. Show what they are, but don't find the coefficients.
   c) Continue with the partial fraction expansion just far enough to find the **transient** coefficient as a number.
   d) Express the transient part as a function of time. $y_{tr}(t)$ = ?
   e) What is the time constant of this expression? τ = ?
   f) Use steady-state AC analysis to find the steady-state output in the form of a cosine with a magnitude and phase angle.

      $y_{ss}(t)$ = ?

4. This system: $H(s) = \dfrac{4}{s+12}$ Has this Cosine input: $x(t) = 5 \cdot \cos(8 \cdot t + 40 \cdot deg) \cdot u(t)$
   a) Use steady-state AC analysis to find the steady-state response ($y_{ss}(t)$) of the system. $y_{ss}(t)$ = ?
   b) Separate the input x(t) into a pure cosine part and a pure sine part.
   c) Use the results of 1b) and 2a), above to find the transient responses to cosine and sine inputs and then add them together to find the total transient response.

5. Find the steady-state (sinusoidal) magnitude and phase of the following transfer function.

   $|H(j \cdot \omega)|$ = ? $\angle H(j\omega)$ = ? $\omega := 20 \cdot \dfrac{rad}{sec}$ $H(s) = \dfrac{\dfrac{80}{sec} \cdot s - \dfrac{300}{sec^2}}{s^2 + \dfrac{90}{sec^2}}$

6. Express the following signal in the time domain, as a sum of cosine and sine with no phase angles:

   $\omega := 44 \cdot \dfrac{rad}{sec}$ $Y(j\omega) = 3 + 0.5 \cdot j$

7. The following questions refer to the general system whose output is given by eq. 3.70, p.42 in our text.

a) Can a system's response to initial conditions be calculated separately from its response to the input signal? Why or why not?

b) Can you expect a system's response to initial conditions to be similar to its response to a simple input signal? Why or why not?

c) To fully describe the state of the system, how many things do you need to know? List them.

d) If a system is BIBO stable, then what is its final response to initial conditions?

e) The output of a system with nonrepeated poles on the jω-axis which is otherwise BIBO stable can be unbounded for some input signals. Is this also true for initial conditions alone when there is no input signal?
If no, why are the conditions for bounded output not as restrictive if there are only initial conditions and no input?

8. a) List 4 advantages of the state-space method over the frequency domain method we are using in this class.

b) List 2 advantages of the frequency domain method we are using in this class over the state-space method.

Answers

1 & 2a) Answers are right in the book

2.b) $\dfrac{k}{\sqrt{\omega_o^2 + a^2}}$ $\quad -\tan^{-1}\left(\dfrac{\omega_o}{a}\right) - 90\cdot deg$

3. a) $\dfrac{3}{s+8}\cdot\dfrac{4\cdot s}{s^2+100}$ b) $\dfrac{A}{s+8} + \dfrac{B\cdot s}{\left(s^2+100\right)} + \dfrac{C\cdot 10}{\left(s^2+100\right)}$ c) -0.585 d) $-0.585\cdot e^{-8\cdot t}$ e) $125\cdot ms$
f) $0.936\cdot\cos(10\cdot t - 51.34\cdot deg)$

4. a) $1.385\cdot\cos(8\cdot t + 6.3\cdot deg)$ b) $x(t) = (3.83\cdot\cos(8\cdot t) - 3.214\cdot\sin(8\cdot t))\cdot u(t)$ c) $-1.378\cdot e^{-12\cdot t}$

5. $5.251 \quad -79.38\cdot deg$

6. $3\cdot\cos\left(44\cdot\dfrac{rad}{sec}\cdot t\right) - 0.5\cdot\sin\left(44\cdot\dfrac{rad}{sec}\cdot t\right)$

7. c) 4 $\quad y(0) \quad \dfrac{d}{dt}y(0) \quad x(0) \quad \dfrac{d}{dt}x(0)$ d) 0

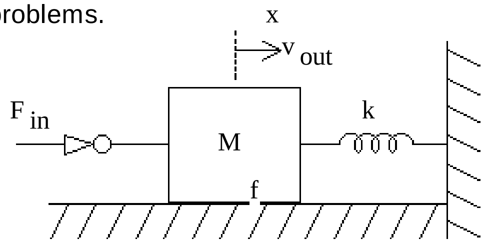
8. See section 3.1 in the Nise textbook.

ECE 3510 homework # 7 p.2

ECE 3510 homework # 8 Due: Fri, 2/19/21 a2

Use the current-force analogy discussed in class for the following problems.

1. a) Find the equivalent electric circuit for the mechanical system shown. F_{in} is an input.



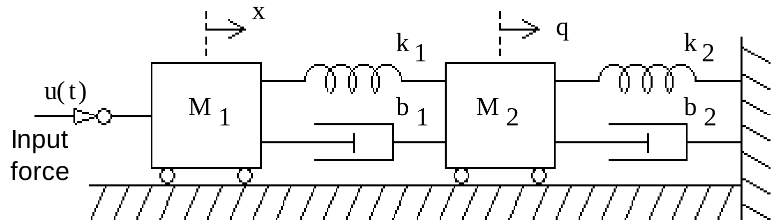
b) Find the transfer function for the system. Put it in the standard form. $\frac{v_{out}(s)}{F_{in}(s)}$

c) Check the units of all coefficients of the transfer function to make sure they agree and work out to the units of velocity over force. Recall that the units of $s = \frac{1}{sec}$

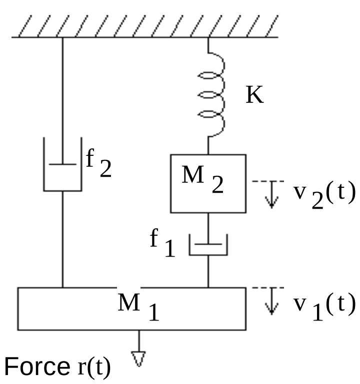
d) The resonant frequency of an electrical circuit can be found from $\frac{1}{\sqrt{L \cdot C}}$. What is it for this system?

e) Find the transfer function for the system. Put it in the standard form. $\frac{x_{out}(s)}{F_{in}(s)}$ Where x is the displacement of the mass rather than its velocity.

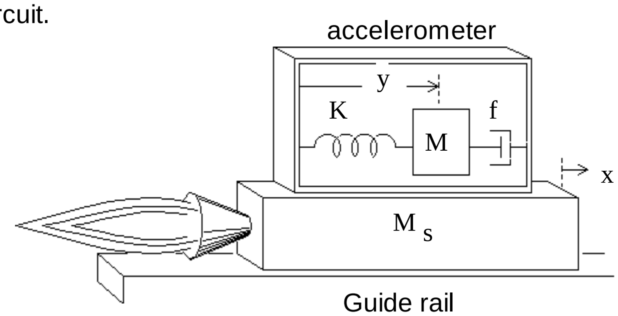
2. Find the equivalent electric circuit for the mechanical system shown. u(t) is an input. Show x-velocity and q-velocity on the circuit.



3. Find the equivalent electric circuit for the mechanical system shown. r(t) is an input. Show v_1 & v_2 on the circuit.

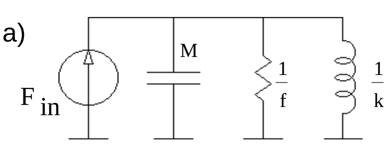


4. Find the equivalent electric circuit for the levitated rocket sled shown. The rocket is a force input. There is no friction between the sled and guide rail, but there is air resistance (which can be modeled in exactly the same way as friction between the sled and guide rail) The accelerometer is firmly mounted onto the sled. Show x-velocity and y-velocity on the circuit.



Answers

1. a)



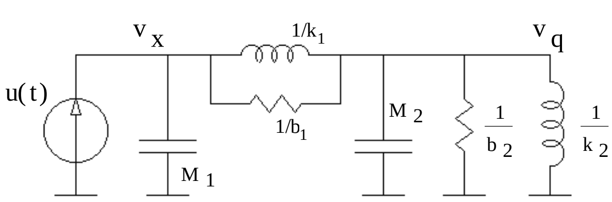
b) $$\frac{\frac{1}{M} \cdot s}{s^2 + \frac{f}{M} \cdot s + \frac{k}{M}}$$

c) Yes, units do all check, show your work.

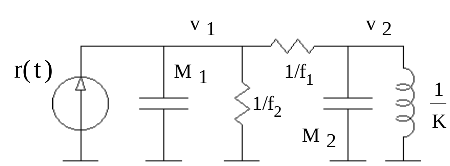
d) $\sqrt{\frac{k}{M}}$

e) $$\frac{\frac{1}{M}}{s^2 + \frac{f}{M} \cdot s + \frac{k}{M}}$$

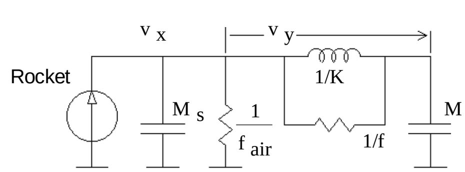
2.



3.



4.

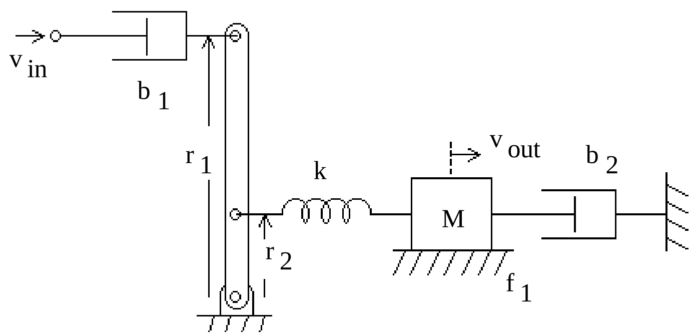


ECE 3510 homework # 9 Due: Tue, 2/23/21

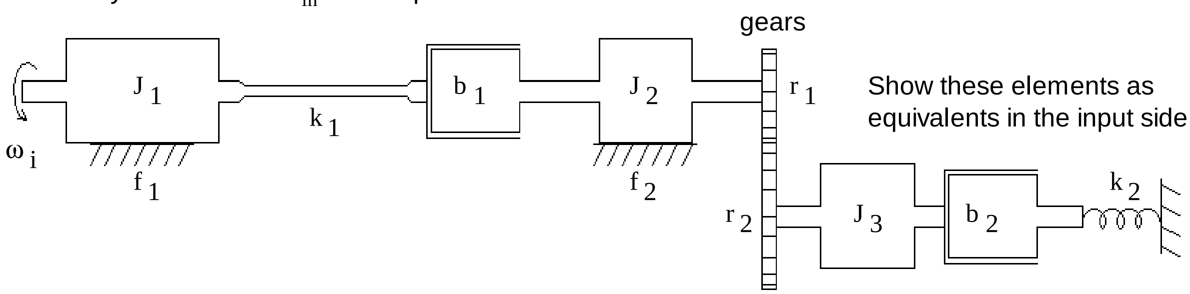
a

Use the current-force analogy discussed in class for the following problems.

1. Find the equivalent electric circuit for the mechanical system shown. v_{in} is the input.

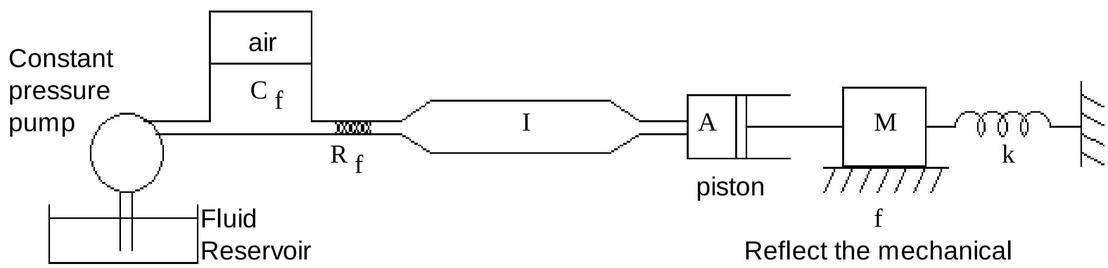


2. Find the equivalent electric circuit for the mechanical system shown. ω_{in} is the input.



Show these elements as equivalents in the input side.

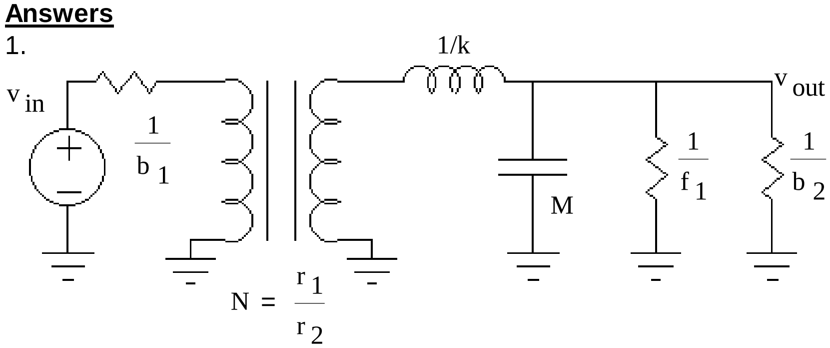
3. Find the equivalent electric circuit for the fluid system shown.



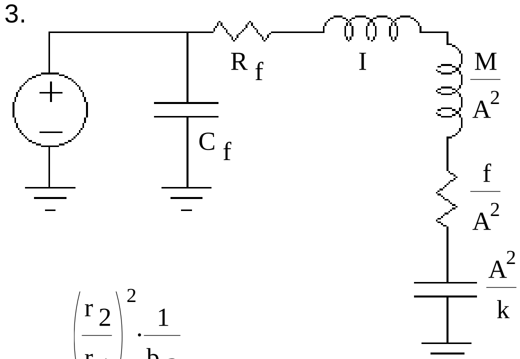
Reflect the mechanical elements into the fluid system.

Answers

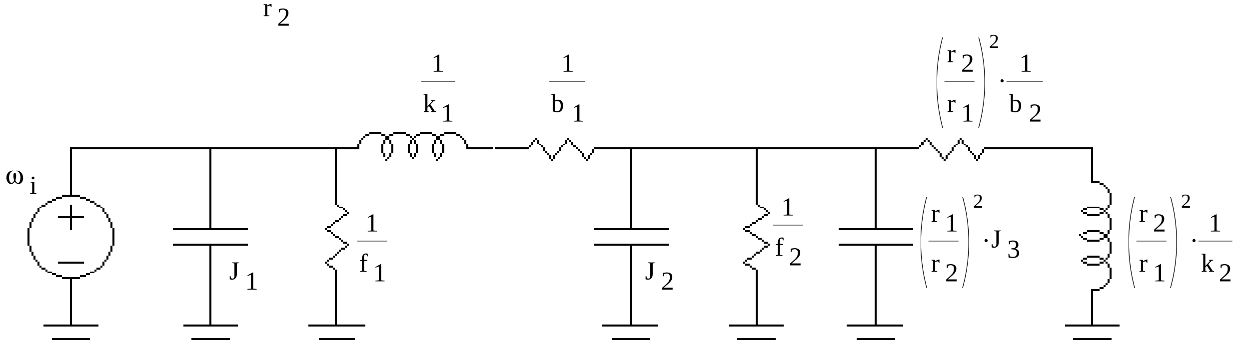
1.



3.



2.



ECE 3510 homework # 9