ECE 3510 Finish Ch 2

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1/25/09
1/19/16

Non-strictly-proper transforms

section 2.2.5, p.16 in Bodson text

What if the order of the numerator is equal to or even greater than the order of the denominator? $m \geq n$?

Example: $F(s) = \dfrac{2 \cdot s^2 + 100}{s^2 + 8 \cdot s + 41}$ $\quad m := 2 \quad n := 2$

First divide, before partial fraction expansion $s^2 + 8 \cdot s + 41 \,\overline{\big)\; 2 \cdot s^2 + 0 \cdot s + 100}$

"remainder"

$$F(s) = \frac{2 \cdot s^2 + 100}{s^2 + 8 \cdot s + 41} =$$

f(t) =

Delta functions are not very common in real life.

Non-strictly-proper transforms are just as common.

Properties of Signals

Can you tell what f(t) must be just by looking at F(s)? YES, somewhat...

$$\frac{s + 5}{s \cdot \left(s^2 + 4 \cdot s + 13\right) \cdot (s - 10)}$$

$$\frac{s + 5}{s \cdot \left(s^2 + 64\right) \cdot (s + 10)}$$

$$\frac{s + 5}{s \cdot \left(s^2 - 4 \cdot s + 13\right) \cdot (s + 10)}$$

$$\frac{s + 5}{s \cdot \left(s^2 + 4 \cdot s + 13\right)^2 \cdot (s + 10)}$$

$$\frac{s + 5}{s^3 \cdot \left(s^2 + 4 \cdot s + 13\right)^2 \cdot (s + 10)^2}$$

ECE 3510 Lecture 6 & 7 notes Transfer Functions & Systems

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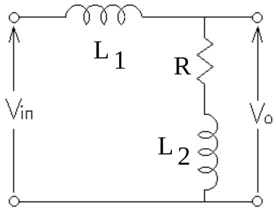
11/22/09,
1/22/15

Now that we've reviewed Laplace transforms of signals, we can move on to systems, the transfer function, and system block diagrams using blocks which contain transfer functions.

Consider a circuit:



$$\mathbf{H}(s) = \frac{\mathbf{V_o}(s)}{\mathbf{V_{in}}(s)} = \frac{R + L_2 \cdot s}{R + L_1 \cdot s + L_2 \cdot s} = \frac{R + L_2 \cdot s}{R + (L_1 + L_2) \cdot s}$$

$$= \frac{L_2 \cdot s + R}{(L_1 + L_2) \cdot s + R}$$

This could be represented in as a block operator:

$$\mathbf{V_{in}}(s) \longrightarrow \boxed{\frac{L_2 \cdot s + R}{(L_1 + L_2) \cdot s + R}} \longrightarrow \mathbf{V_o}(s) = \mathbf{V_{in}}(s) \cdot \mathbf{H}(s)$$

Transfer functions can be written for all kinds of devices and systems, not just electric circuits and the input and output do not have to be similar. For instance, the potentiometers used to measure angular position in the crude servo of lab 1 can be represented like this:

$$\boldsymbol{\theta}_{\mathbf{in}}(s) \longrightarrow \boxed{Kp = 0.7 \cdot \frac{V}{rad} = 0.012 \cdot \frac{V}{deg}} \longrightarrow \mathbf{V_{out}}(s) = K_p \cdot \boldsymbol{\theta}_{\mathbf{in}}(s)$$

In general:

$$\mathbf{H}(s) = \frac{\text{output}}{\text{input}} = \frac{\mathbf{Y}(s)}{\mathbf{X}(s)}$$

$$\mathbf{X}(s) \longrightarrow \boxed{\mathbf{H}(s)} \longrightarrow \mathbf{Y}(s) = \mathbf{X}(s) \cdot \mathbf{H}(s)$$

$$= \frac{\mathbf{N_X}(s)}{\mathbf{D_X}(s)} \cdot \frac{\mathbf{N_H}(s)}{\mathbf{D_H}(s)}$$

X and **Y** could be anything from small electrical signals to powerful mechanical motions or forces.

The output signal has the poles of both the input AND the transfer function.

Serial - path systems

Two blocks with transfer functions $\mathbf{A}(s)$ and $\mathbf{B}(s)$ in a row would look like this:

$$\mathbf{X}(s) \longrightarrow \boxed{\mathbf{A}(s)} \xrightarrow{\mathbf{X}(s) \cdot \mathbf{A}(s)} \boxed{\mathbf{B}(s)} \longrightarrow \mathbf{Y}(s) = \mathbf{X}(s) \cdot \mathbf{A}(s) \cdot \mathbf{B}(s)$$

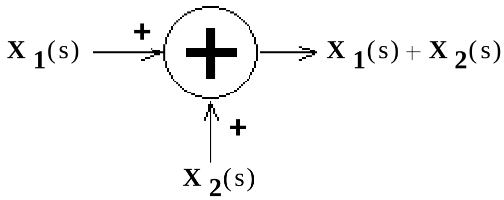
$$= \mathbf{X}(s) \cdot \frac{\mathbf{N_A}(s)}{\mathbf{D_A}(s)} \cdot \frac{\mathbf{N_B}(s)}{\mathbf{D_B}(s)}$$

The two blocks could be replaced by a single equivalent block:

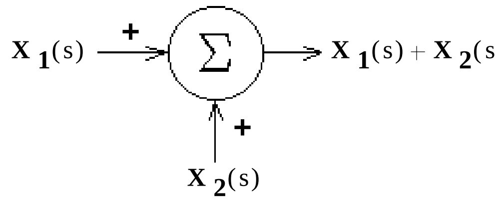
$$\mathbf{X}(s) \longrightarrow \boxed{\mathbf{A}(s) \cdot \mathbf{B}(s)} \longrightarrow \mathbf{Y}(s) = \mathbf{X}(s) \cdot \mathbf{A}(s) \cdot \mathbf{B}(s)$$

The output signal has the poles of the input AND BOTH transfer functions.

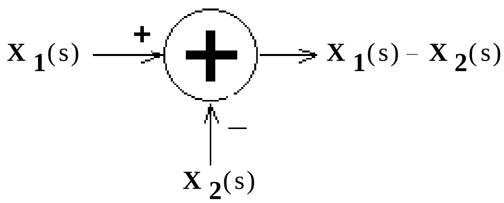
Summer blocks can be used to add signals:



OR



or subtract signals:



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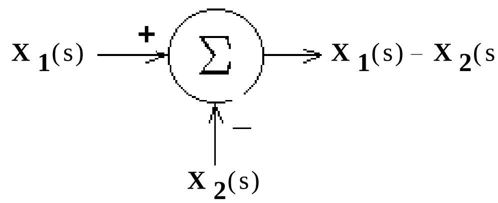
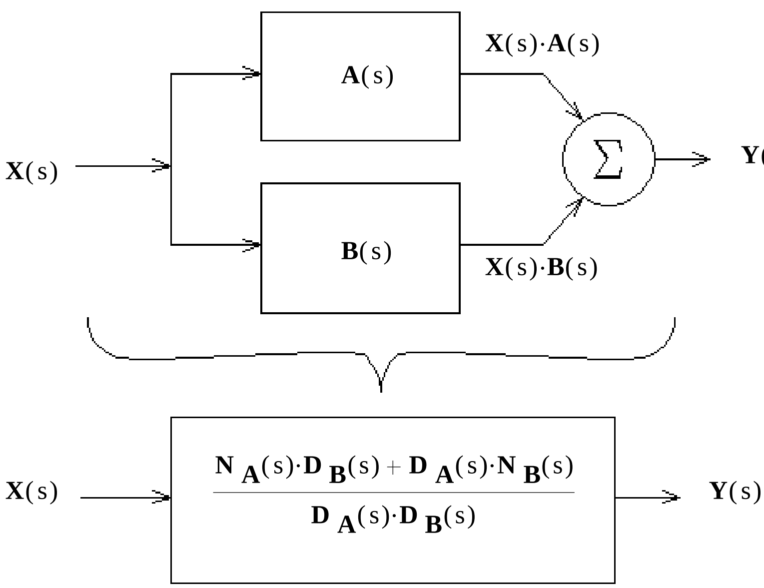


Parallel - path systems



$$\mathbf{Y}(s) = \mathbf{X}(s)\cdot(\mathbf{A}(s)+\mathbf{B}(s))$$

$$= \mathbf{X}(s)\cdot\left(\frac{\mathbf{N}_\mathbf{A}(s)}{\mathbf{D}_\mathbf{A}(s)}+\frac{\mathbf{N}_\mathbf{B}(s)}{\mathbf{D}_\mathbf{B}(s)}\right)$$

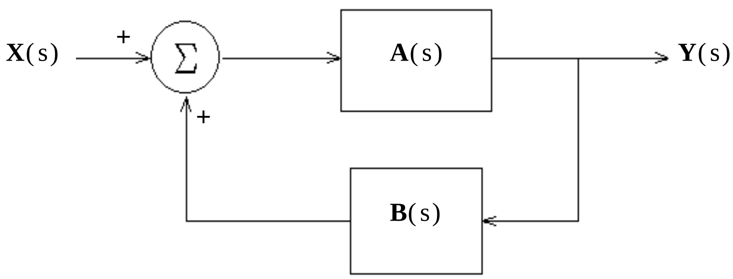
$$= \mathbf{X}(s)\cdot\left(\frac{\mathbf{N}_\mathbf{A}(s)\cdot\mathbf{D}_\mathbf{B}(s)+\mathbf{D}_\mathbf{A}(s)\cdot\mathbf{N}_\mathbf{B}(s)}{\mathbf{D}_\mathbf{A}(s)\cdot\mathbf{D}_\mathbf{B}(s)}\right)$$

$$\frac{\mathbf{N}_\mathbf{A}(s)\cdot\mathbf{D}_\mathbf{B}(s)+\mathbf{D}_\mathbf{A}(s)\cdot\mathbf{N}_\mathbf{B}(s)}{\mathbf{D}_\mathbf{A}(s)\cdot\mathbf{D}_\mathbf{B}(s)}$$

Again, the output signal has the poles of the input AND BOTH transfer functions.

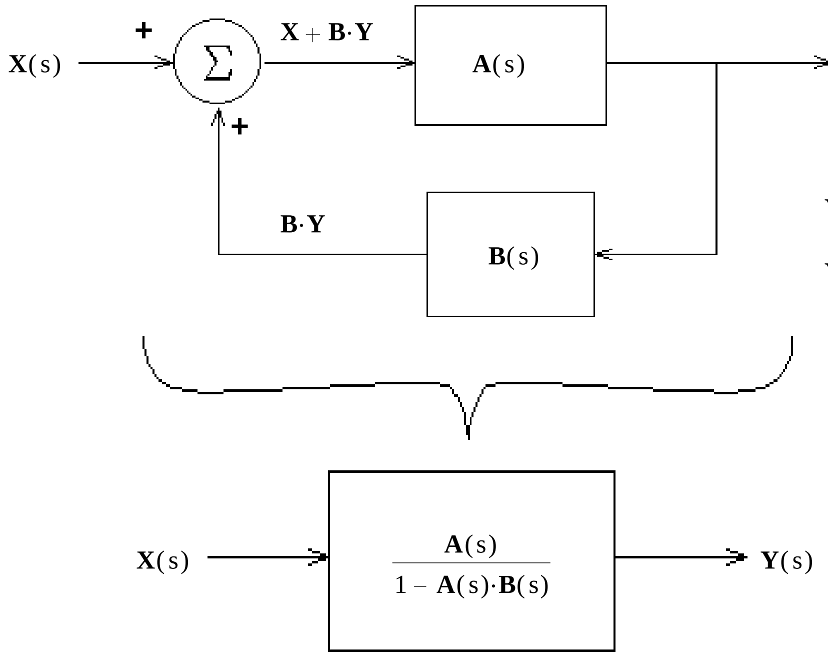
The two blocks could be replaced by a single equivalent block:

A feedback loop system is particularly interesting and useful:



The entire loop can be replaced by a single equivalent block:

Note that I've begun to drop the (s)



$$\mathbf{Y}(s) = \mathbf{A}\cdot(\mathbf{X}+\mathbf{B}\cdot\mathbf{Y})$$

$$= \mathbf{A}\cdot\mathbf{X}+\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{Y}$$

$$\mathbf{Y}-\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{Y} = \mathbf{A}\cdot\mathbf{X}$$

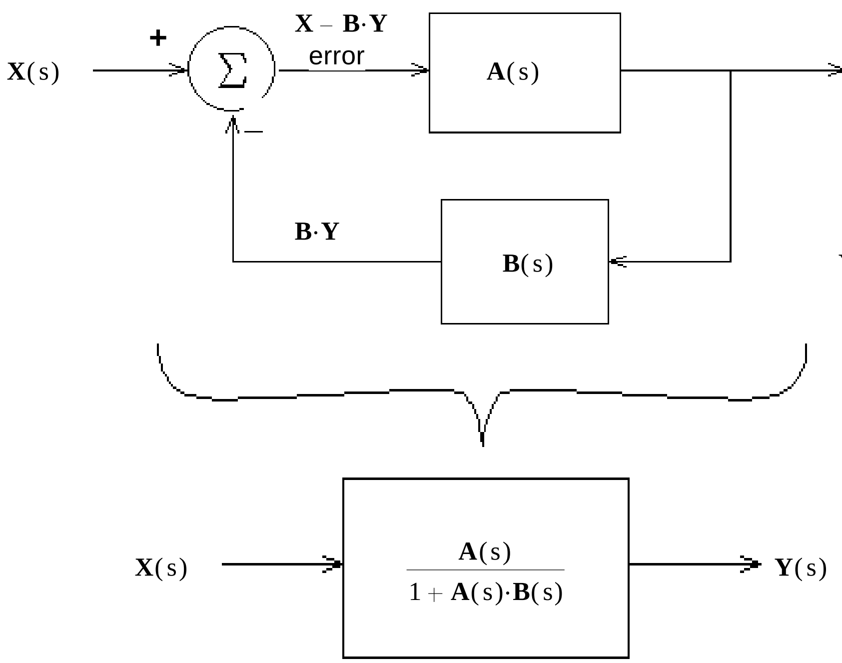
$$\mathbf{Y}\cdot(1-\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{X}$$

$$\frac{\mathbf{Y}}{\mathbf{X}} = \frac{\mathbf{A}}{1-\mathbf{A}\cdot\mathbf{B}} = \mathbf{H}(s)$$

The equivalent transfer function

$\mathbf{A}(s)\cdot\mathbf{B}(s)$ is called the "loop gain" or "open loop gain"

Negative feedback is more common and is used as a control system:



$$\mathbf{Y}(s) = \mathbf{A}\cdot(\mathbf{X}-\mathbf{B}\cdot\mathbf{Y})$$

$$= \mathbf{A}\cdot\mathbf{X}-\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{Y}$$

$$\mathbf{Y}+\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{Y} = \mathbf{A}\cdot\mathbf{X}$$

$$\mathbf{Y}\cdot(1+\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{X}$$

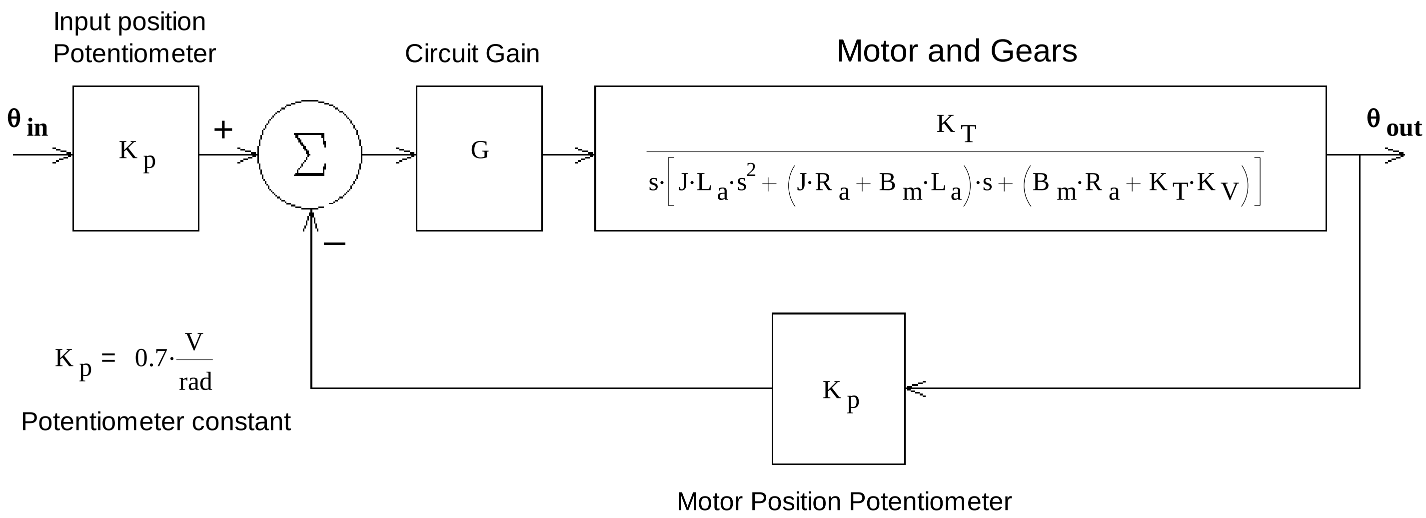
$$\frac{\mathbf{Y}}{\mathbf{X}} = \frac{\mathbf{A}}{1+\mathbf{A}\cdot\mathbf{B}} = \mathbf{H}(s)$$

The equivalent transfer function

This is called a "closed loop" system, whereas a a system without feedback is called "open loop".
The term "open loop" is often used to describe a system that is out of control.

The output signal poles are <u>different</u> than either the poles of the input or the transfer functions.
Different poles means different characteristics! This implies that you might start with a stable system and make an unstable system or (more productively) start with an unstable system and make a stable system.

The servo used in our lab can be represented by:



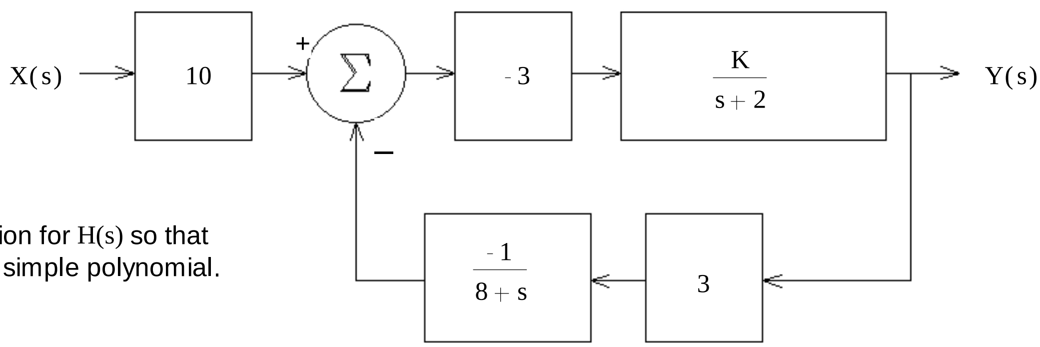
$K_p = 0.7\cdot\dfrac{V}{rad}$

Potentiometer constant

$$\mathbf{H}(s) = \frac{\theta_{out}(s)}{\theta_{in}(s)} = \frac{G\cdot K_T\cdot K_p}{s\cdot\left[J\cdot L_a\cdot s^2+\left(J\cdot R_a+B_m\cdot L_a\right)\cdot s+\left(B_m\cdot R_a+K_T\cdot K_V\right)\right]+K_p\cdot G\cdot K_T}$$

See the appendix to lab 1 for the complete analysis

Ex. 1 a) A feedback system is shown in the figure. What is the transfer function of the whole system, with feedback.



$H(s) = \dfrac{Y(s)}{X(s)}$ = ?

Simplify your expression for $H(s)$ so that the denominator is a simple polynomial.

Feedback loop:

Loop gain: $L = \left(\dfrac{-3\cdot K}{s+2}\right)\cdot\left(\dfrac{-1\cdot 3}{8+s}\right)$

Simplification:

$$A_f = \frac{\left(\dfrac{-3\cdot K}{s+2}\right)}{1+\left(\dfrac{-3\cdot K}{s+2}\right)\cdot\left(\dfrac{-3}{8+s}\right)}$$

$$A_f = \frac{\left(\dfrac{-3\cdot K}{s+2}\right)}{1+\left(\dfrac{-3\cdot K}{s+2}\right)\cdot\left(\dfrac{-3}{8+s}\right)}\cdot\left[\frac{(s+2)\cdot(8+s)}{(s+2)\cdot(8+s)}\right] = \frac{(-3\cdot K)\cdot(s+8)}{(s+2)\cdot(8+s)+(3\cdot K)\cdot 3}$$

$$= \frac{(-3\cdot K)\cdot s-K\cdot 24}{s^2+10\cdot s+16+9\cdot K}$$

Whole system:

$$H(s) = 10\cdot\frac{-3\cdot K\cdot s-24\cdot K}{s^2+10\cdot s+16+9\cdot K} = \frac{-30\cdot K\cdot(s+8)}{s^2+10\cdot s+16+9\cdot K}$$

b) Find the value of K to make the transfer function critically damped. Answer may be left as a fraction.

characteristic eq.: $0 = s^2 + 10\cdot s + 16 + 9\cdot K$

to solve for the poles: $s = \dfrac{-10 + \sqrt{10^2 - 4\cdot(16 + 9\cdot K)}}{2}$ at critical damping, the part under the radical is zero.

thus: $10^2 = 4\cdot(16 + 9\cdot K)$

$100 = 64 + 36\cdot K$

$K = \dfrac{100 - 64}{36} = \dfrac{36}{36} = 1$ solve for K

c) If K is less than the value found in part b), will the system be under-, critical-, or overdamped?

$10^2 - 4\cdot(16 + 9\cdot K) > 0$ so it will be overdamped

d) If $K := 5$, find the pole(s) of the transfer function:

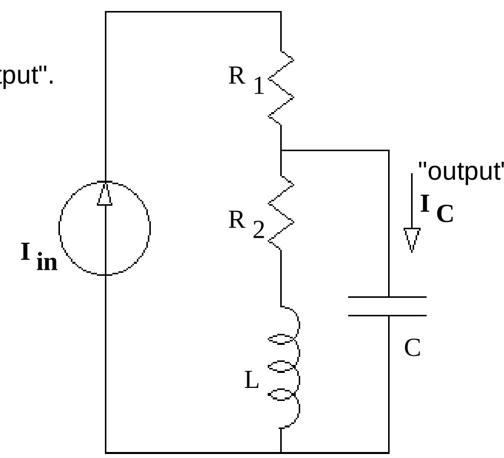
characteristic eq.: $0 = s^2 + 10\cdot s + 16 + 9\cdot K = s^2 + 10\cdot s + 61$

$\dfrac{-10 + \sqrt{10^2 - 4\cdot 61}}{2} = -5 + 6j$ $\dfrac{-10 - \sqrt{10^2 - 4\cdot 61}}{2} = -5 - 6j$

e) If $K := 5$, find the zero(s) of the transfer function: $s + 8 = 0$ $s = -8$

Ex.2 a) Find the transfer function of the circuit shown. Consider I_C as the "output".

Properly simplify all your expressions for $\mathbf{H}(s)$. By this I mean that he numerator and denominator should both be simple polynomials or factored polynomials. There should be no $1/s^n$ terms in either the numerator or denominator. Also, there should be no coefficient on the highest-order term in the denominator



$$\mathbf{H}(s) = \frac{\mathbf{I_C}(s)}{\mathbf{I_{in}}(s)} = ?$$

Current divider:

$$\mathbf{I_C}(s) = \mathbf{I_{in}}(s)\cdot\frac{\dfrac{1}{\left(\dfrac{1}{C\cdot s}\right)}}{\dfrac{1}{\left(\dfrac{1}{C\cdot s}\right)} + \dfrac{1}{R_2 + L\cdot s}} = \mathbf{I_{in}}(s)\cdot\frac{C\cdot s}{C\cdot s + \dfrac{1}{R_2 + L\cdot s}}$$

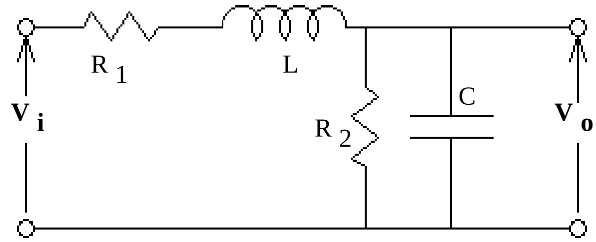
$$\mathbf{H}(s) = \frac{\mathbf{I_C}(s)}{\mathbf{I_{in}}(s)} = \frac{C\cdot s}{C\cdot s + \dfrac{1}{R_2 + L\cdot s}}\cdot\frac{(R_2 + L\cdot s)}{(R_2 + L\cdot s)} = \frac{C\cdot R_2 + L\cdot C\cdot s^2}{C\cdot R_2\cdot s + L\cdot C\cdot s^2 + 1}\cdot\frac{\left(\dfrac{1}{L\cdot C}\right)}{\left(\dfrac{1}{L\cdot C}\right)} = \frac{\dfrac{R_2}{L}\cdot s + s^2}{\dfrac{R_2}{L}\cdot s + s^2 + \dfrac{1}{L\cdot C}}$$

$$= \frac{s\cdot\left(s + \dfrac{R_2}{L}\right)}{s^2 + \dfrac{R_2}{L}\cdot s + \dfrac{1}{L\cdot C}}$$

b) How many zeroes does this transfer function have? 2 , 0 and $-R_2/L$

c) How many poles does this transfer function have? 2 at: $-\dfrac{R_2}{2\cdot L} \pm \dfrac{1}{2}\cdot\sqrt{\left(\dfrac{R_2}{L}\right)^2 - \dfrac{4}{L\cdot C}}$

Ex.3 a) Find the transfer function of the circuit shown.
$\mathbf{V_i}$ is the input and $\mathbf{V_o}$ is the output.



$$\mathbf{H}(s) = \frac{\mathbf{V_o}(s)}{\mathbf{V_i}(s)} = \frac{\dfrac{1}{\dfrac{1}{R_2}+C\cdot s}}{R_1+L\cdot s+\dfrac{1}{\dfrac{1}{R_2}+C\cdot s}} \cdot \frac{\dfrac{1}{R_2}+C\cdot s}{\dfrac{1}{R_2}+C\cdot s} = \frac{1}{R_1\cdot\left(\dfrac{1}{R_2}+C\cdot s\right)+L\cdot s\cdot\left(\dfrac{1}{R_2}+C\cdot s\right)+1}$$

$$= \frac{1}{\dfrac{R_1}{R_2}+R_1\cdot C\cdot s+\dfrac{L\cdot s}{R_2}+L\cdot s\cdot C\cdot s+1} \cdot \frac{\dfrac{1}{L\cdot C}}{\dfrac{1}{L\cdot C}} = \frac{\dfrac{1}{L\cdot C}}{\dfrac{R_1}{R_2}\cdot\dfrac{1}{L\cdot C}+\dfrac{R_1\cdot C}{L\cdot C}\cdot s+\dfrac{L\cdot s}{R_2}\cdot\dfrac{1}{L\cdot C}+s^2+\dfrac{1}{L\cdot C}}$$

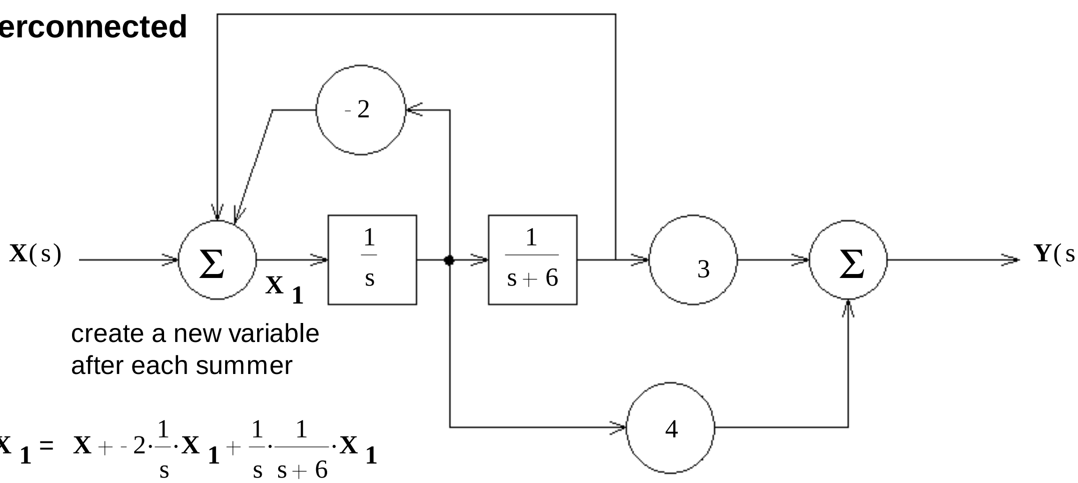
$$= \frac{\dfrac{1}{L\cdot C}}{s^2+\left(\dfrac{R_1}{L}+\dfrac{1}{R_2\cdot C}\right)\cdot s+\left(1+\dfrac{R_1}{R_2}\right)\cdot\dfrac{1}{L\cdot C}}$$

b) Find the characteristic equation of the circuit shown. $\quad 0 = s^2+\left(\dfrac{R_1}{L}+\dfrac{1}{R_2\cdot C}\right)\cdot s+\left(1+\dfrac{R_1}{R_2}\right)\cdot\dfrac{1}{L\cdot C}$

c) The solutions to the characteristic equation are called the _______ of the transfer function. Poles

d) Does the transfer function have one or more zeros? If yes, express it (them) in terms of R_1, R_2, C, & L. NO

General Interconnected System



$$\mathbf{X_1} = \mathbf{X} + -2\cdot\frac{1}{s}\cdot\mathbf{X_1}+\frac{1}{s}\cdot\frac{1}{s+6}\cdot\mathbf{X_1}$$

$$\mathbf{X_1}\cdot\left[1+\frac{2}{s}-\frac{1}{s\cdot(s+6)}\right] = \mathbf{X} = \mathbf{X_1}\cdot\left[\frac{s\cdot(s+6)}{s\cdot(s+6)}+\frac{2\cdot(s+6)}{s\cdot(s+6)}-\frac{1}{s\cdot(s+6)}\right] = \mathbf{X_1}\cdot\left[\frac{s^2+8\cdot s+11}{s\cdot(s+6)}\right]$$

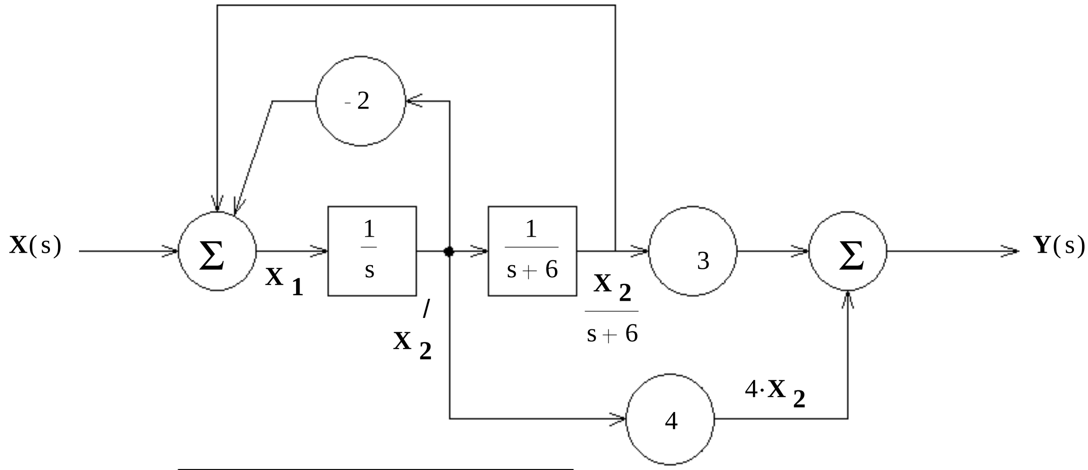
$$\mathbf{X_1} = \frac{s\cdot(s+6)}{s^2+8\cdot s+11}\cdot\mathbf{X}$$

$$\mathbf{Y} = 3\cdot\left(\frac{1}{s}\cdot\frac{1}{s+6}\right)\cdot\mathbf{X_1}+4\cdot\frac{1}{s}\cdot\mathbf{X_1} = 3\cdot\left[\frac{1}{s\cdot(s+6)}\right]\cdot\left[\frac{s\cdot(s+6)}{s^2+8\cdot s+11}\cdot\mathbf{X}\right]+4\cdot\frac{1}{s}\cdot\left[\frac{s\cdot(s+6)}{s^2+8\cdot s+11}\cdot\mathbf{X}\right]$$

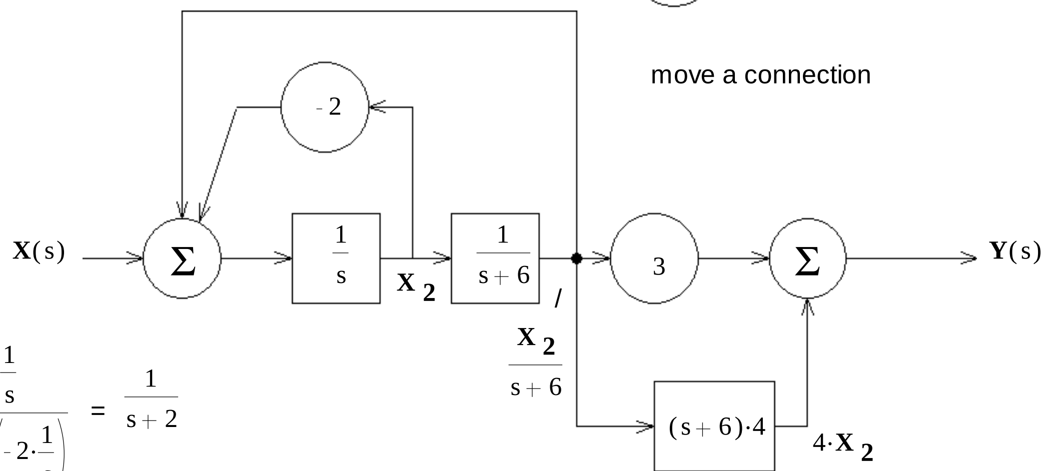
$$= \frac{3+(4\cdot s+24)}{s^2+8\cdot s+11}\cdot\mathbf{X} = \frac{4\cdot s+27}{s^2+8\cdot s+11}\cdot\mathbf{X}$$

$$\mathbf{H}(s) = \frac{\mathbf{Y}}{\mathbf{X}} = \frac{4\cdot s+27}{s^2+8\cdot s+11}$$

General Interconnected System, Another approach



move a connection



parallel paths $3+4\cdot(s+6)$

inner loop $\dfrac{\frac{1}{s}}{1-\left(-2\cdot\frac{1}{s}\right)} = \dfrac{1}{s+2}$

outer loop $\dfrac{\frac{1}{(s+2)}\cdot\frac{1}{(s+6)}}{1-1\cdot\frac{1}{(s+2)}\cdot\frac{1}{(s+6)}} = \dfrac{1}{(s+2)\cdot(s+6)-1} = \dfrac{1}{s^2+8\cdot s+12-1}$

$= \dfrac{1}{s^2+8\cdot s+11}$

whole transfer function $\mathbf{H}(s) = \dfrac{1}{s^2+8\cdot s+11}\cdot(3+4\cdot(s+6)) = \dfrac{4\cdot s+27}{s^2+8\cdot s+11}$

Bounded-Input Bounded-Output (BIBO) Stable

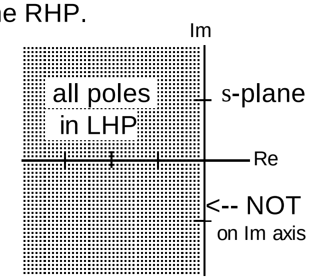
A system is considered BIBO stable if the output in bounded for any bounded input.

A bounded input could have single poles on the imaginary axis at any location.
A bounded output may not have double poles on the imaginary axis or any poles in the RHP (Right-half-plane).
The output will have all the poles of the input plus all the poles of the system. (except in rare pole-zero cancellations.)

Therefore: A BIBO system may not have any poles on the imaginary axis or any poles in the RHP.

Examples of systems with poles on the imaginary axis: If the output of a DC motor is angular position of the shaft then it has a pole at the origin. The response to a DC input is a shaft that keeps turning and the position grows without bounds. This system is not BIBO stable. (If the output is shaft speed, then it would be BIBO stable.)

If a system has a pair of imaginary poles at $\pm j\omega$, then it has a resonant frequency of ω.
If the input also had a pair of imaginary poles at $\pm j\omega$ then it would excite that resonance and the output would grow without bounds.



system transfer function poles

ECE 3510 BIBO System Responses to an Impulse or a Step input

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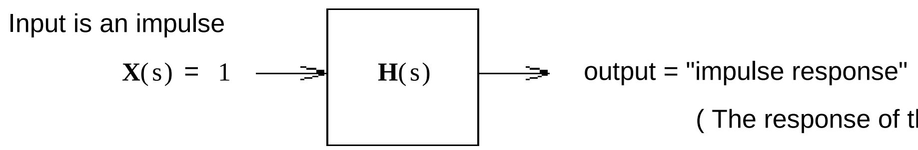
1/22/15

Impulse Response

The Impulse response of a system is the output when the input is an impulse (delta function).

The simplest possible input: $\mathbf{X}(s) = 1$



(The response of the system to an impulse)

$$\mathbf{Y}(s) = \mathbf{X}(s)\cdot\mathbf{H}(s) = 1\cdot\mathbf{H}(s) = \mathbf{H}(s)$$

A signal who's transform is
the system's transfer function

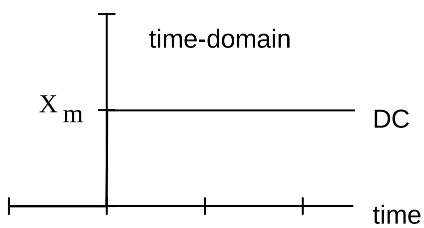
Of course, an impulse is a little impractical in real life.
But, if you can approximate one, than you may be able to use it to characterize an unknown system.

Sometimes the term "impulse response" is used in place of the term "transfer function"

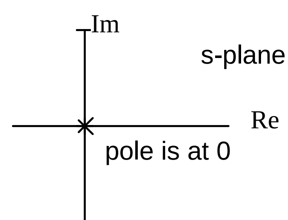
Step Responses

The step response of a system is the output when the input is a step (DC which starts at time-zero).

Step input

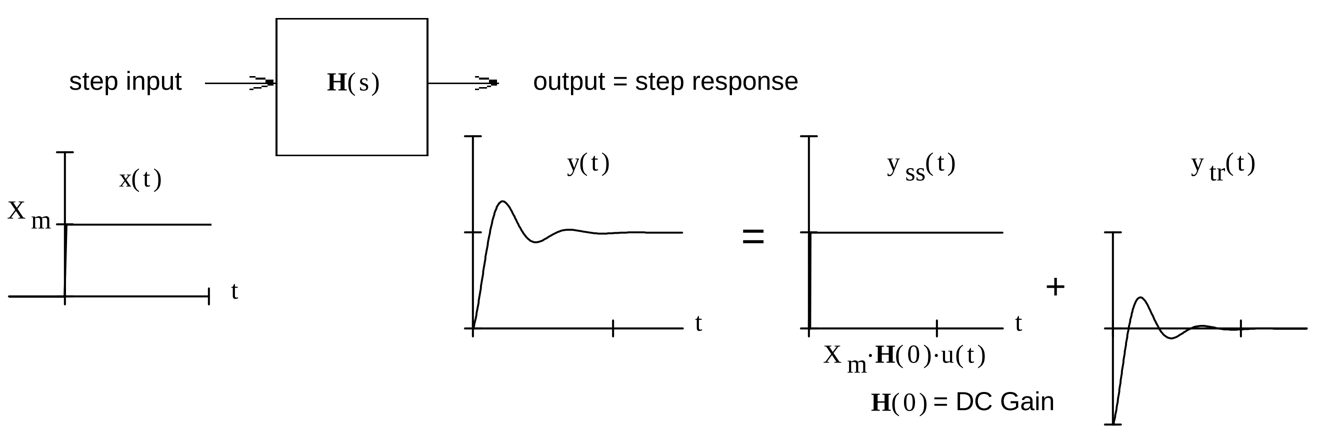


$x(t) = X_m \cdot u(t)$



$X(s) = X_m \cdot \frac{1}{s}$

System Step Response



Complete step response = steady-state response + transient response

Steady-State Response & DC Gain

$$\mathbf{Y}(s) = \mathbf{X}(s)\cdot\mathbf{H}(s) = \frac{X_m}{s}\cdot\mathbf{H}(s)$$

Complete step response

partial fraction expansion: $\mathbf{Y}(s) = \frac{X_m}{s}\cdot\mathbf{H}(s) = \frac{A}{s} + \frac{B}{(\blacksquare)} + \frac{C}{(\blacksquare)} + \frac{D}{(\blacksquare)} + \ldots$

steady-state response + transient response

multiply both sides by s $\quad X_m\cdot\mathbf{H}(s) = A + \left[\frac{B}{(\blacksquare)} + \frac{C}{(\blacksquare)} + \frac{D}{(\blacksquare)}\right]\cdot s$

set $s := 0$ $\quad X_m\cdot\mathbf{H}(0) = A + \left[\frac{B}{(\blacksquare)} + \frac{C}{(\blacksquare)} + \frac{D}{(\blacksquare)}\right]\cdot 0$

$$\mathbf{Y}_{ss}(s) = \frac{A}{s} = \frac{X_m\cdot\mathbf{H}(0)}{s} \qquad y_{ss}(t) = X_m\cdot\mathbf{H}(0)\cdot u(t)$$

$\mathbf{H}(0)$ = DC Gain

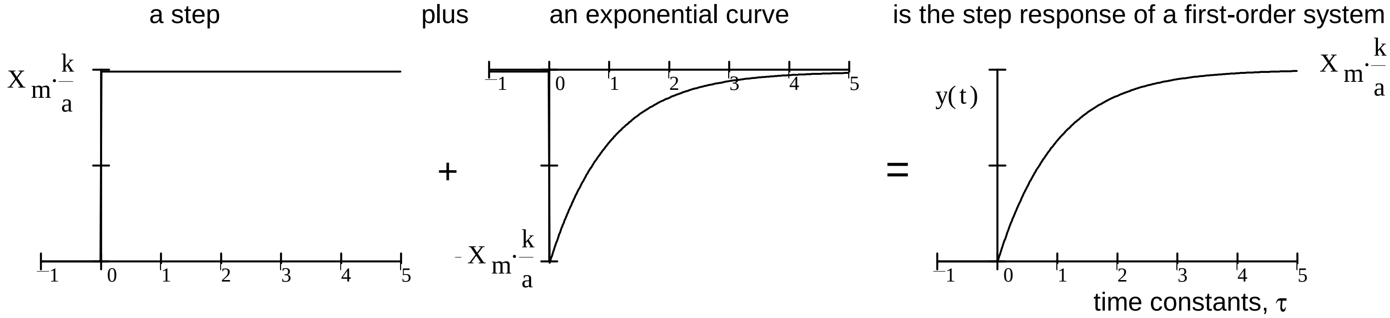
The **transient part** would be found by finishing the partial-fraction expansion.

Step Response of First-Order Systems

$$\mathbf{H}(s) = \frac{k}{s+a} = \frac{k}{s+\frac{1}{\tau}} \qquad \mathbf{Y}(s) = \frac{X_m}{s}\cdot\frac{k}{s+a}$$

$$y(t) = X_m\cdot\left(\frac{k}{a} - \frac{k}{a}\cdot e^{-a\cdot t}\right)\cdot u(t)$$ (ignoring initial conditions)

a step plus an exponential curve is the step response of a first-order system



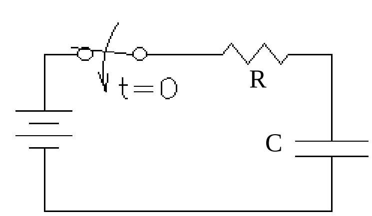
All first-order systems have the same time-domain response:

$$y(t) = y(\infty) + (y(0) - y(\infty))\cdot e^{-\frac{t}{\tau}}$$

$y(0)$ = the initial condition

$y(\infty)$ = the final condition

A simple example of a first-order system

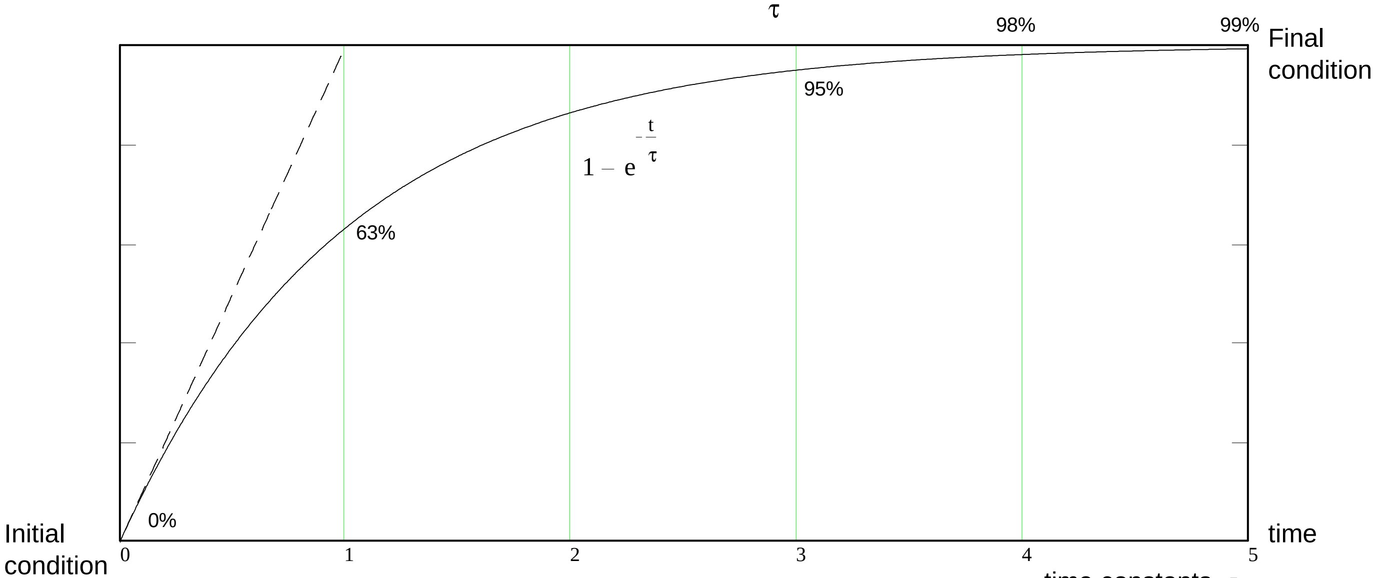


$$v_C(t) = v_C(\infty) + \left(v_C(0) - v_C(\infty)\right)\cdot e^{-\frac{t}{\tau}} \qquad \tau = R\cdot C$$

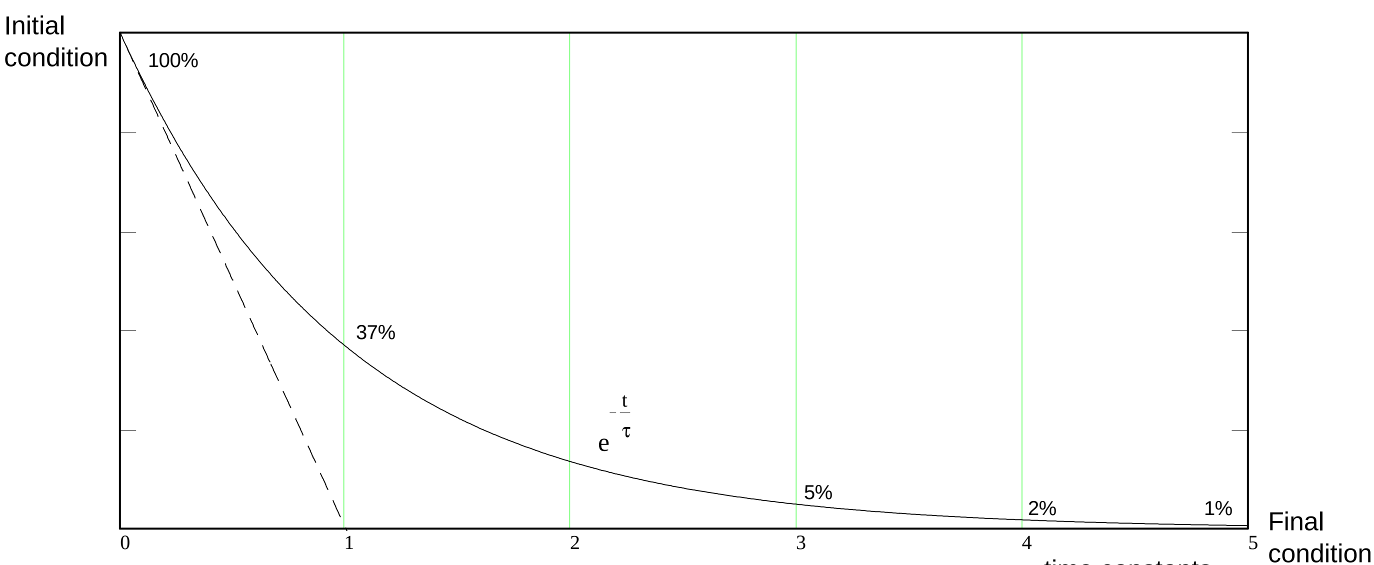
Exponential Curves

Let's take a close look at some of the characteristics of exponential curves, the output of stable first order system. The transient effects always die out after some time, so the exponents are always negative.

Step response of: $\mathbf{H}(s) = \dfrac{k}{s+\dfrac{1}{\tau}}$



Rising Exponential Curve



Decaying Exponential Curve

Some Important Features:

1) These curves proceed from an initial condition to a final condition. If the final condition is greater than the initial, then the curve is said to be a "rising" exponential. If the final condition is less than the initial, then the curve is called a "decaying" exponential.

2) The curves' initial slope is $\pm$ $1/\tau$. If they continued at this initial slope they'd reach the final condition in one time constant.

3) In the first time constant the curve goes 63% from initial to the final condition.

4) By four time constants the curve is within 2% of the final condition and is usually considered finished. Mathematically, the curve approaches the final condition asymptotically and never reaches it. In reality, of course, this is nonsense. Whatever difference there may be between the mathematical solution and the final condition will soon be overshadowed by random fluctuations (called noise) in the real system.

$e^{-4} = 0.018 \quad < 2\%$

Step Response of Second-Order Systems

Real poles (over and critically damped)

A first-order system for reference

$$\mathbf{H_1}(s) = \frac{k}{s+a} \qquad a := 1 \qquad k := a \qquad y_1(t) := \left(\frac{k}{a} - \frac{k}{a}\cdot e^{-a\cdot t}\right)$$

\
normalization   to make curves below easier to compare
/

Second-order system, critically damped

$$\mathbf{H_2}(s) = \frac{k}{(s+a)^2} \qquad a := 1 \qquad k := a^2 \qquad y_2(t) := \left(\frac{k}{a^2} - \frac{k}{a^2}\cdot e^{-a\cdot t} - \frac{k}{a}\cdot t\cdot e^{-a\cdot t}\right)$$

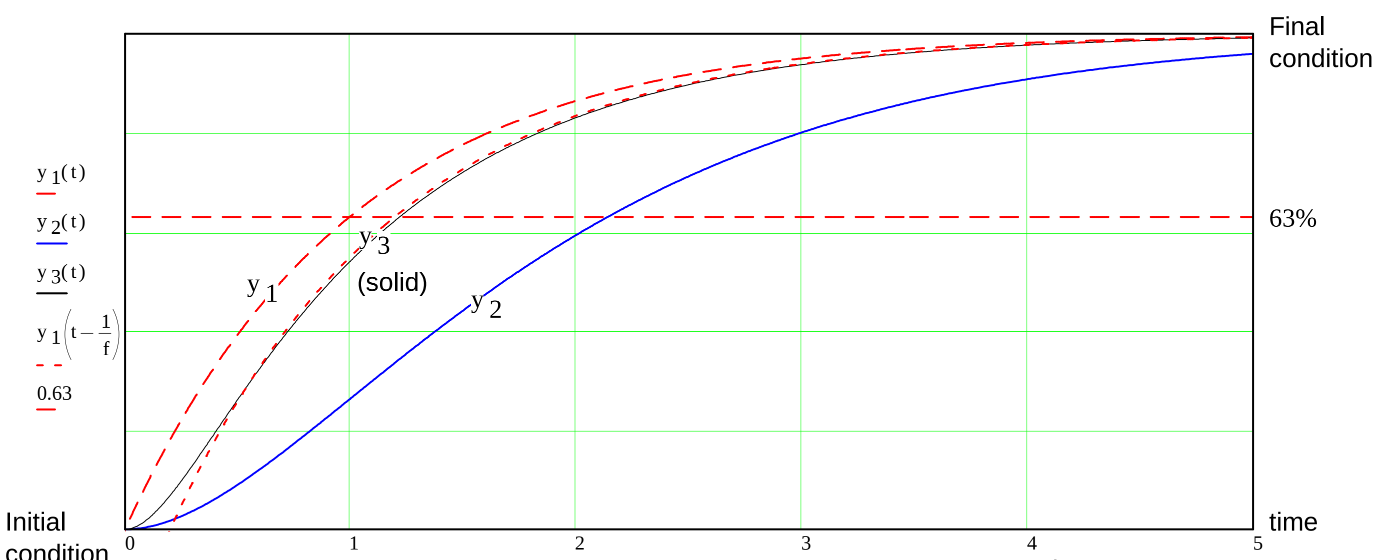
double pole on real axis

Second-order system, over damped

$$\mathbf{H_3}(s) = \frac{k}{(s+a_1)\cdot(s+a_2)}$$

$a_1 := a \qquad f := 5 \qquad a_2 := f\cdot a_1 \qquad k := a_1\cdot a_2$ normalization

$$y_3(t) := \left[\frac{k}{a_1\cdot a_2} + \frac{k}{a_1\cdot(a_1 - a_2)}\cdot e^{-a_1\cdot t} + \frac{k}{a_2\cdot(a_2 - a_1)}\cdot e^{-a_2\cdot t}\right]$$



Rising Curves

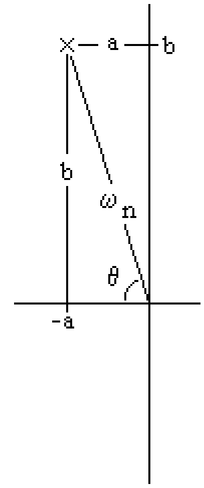
Some Important Features:

1) The poles closest to the jω axis are the **dominant** poles.

2) Poles to the left of the dominant poles may introduce an effect that looks like **time delay**.

3) Conversely, the effects of a time delay (non-linear) can sometimes be modeled by an extra pole (linear) to the left of the dominant poles.

Step Responses of Under-Damped 2nd order Systems (Complex poles)



$$\mathbf{H}(s) = \frac{k}{s^2 + 2\cdot a\cdot s + a^2 + b^2} = \frac{k}{\omega_n^{\,2}}\cdot\frac{\omega_n^{\,2}}{s^2 + 2\cdot\zeta\cdot\omega_n\cdot s + \omega_n^{\,2}}$$

$\omega_n^{\,2} = a^2 + b^2$ $\qquad \omega_n$ = natural frequency

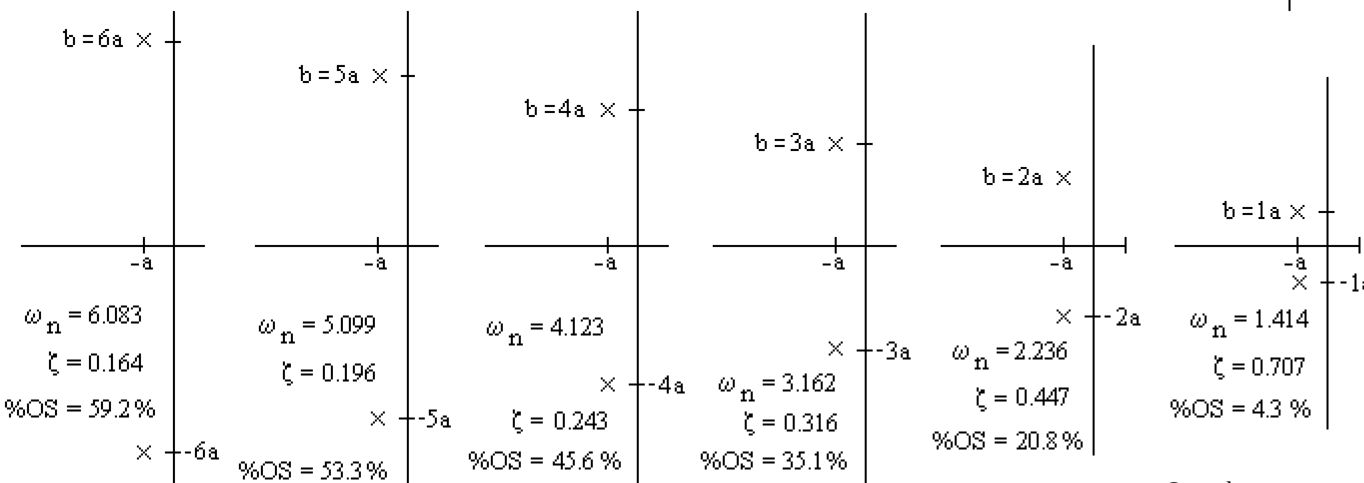
DC gain $\qquad \zeta\cdot\omega_n = a$

$$\mathbf{H}(0) = \frac{k}{a^2 + b^2} = \frac{k}{\omega_n^{\,2}} \qquad \zeta = \frac{a}{\omega_n} = \frac{a}{\sqrt{a^2 + b^2}}$$ = damping factor

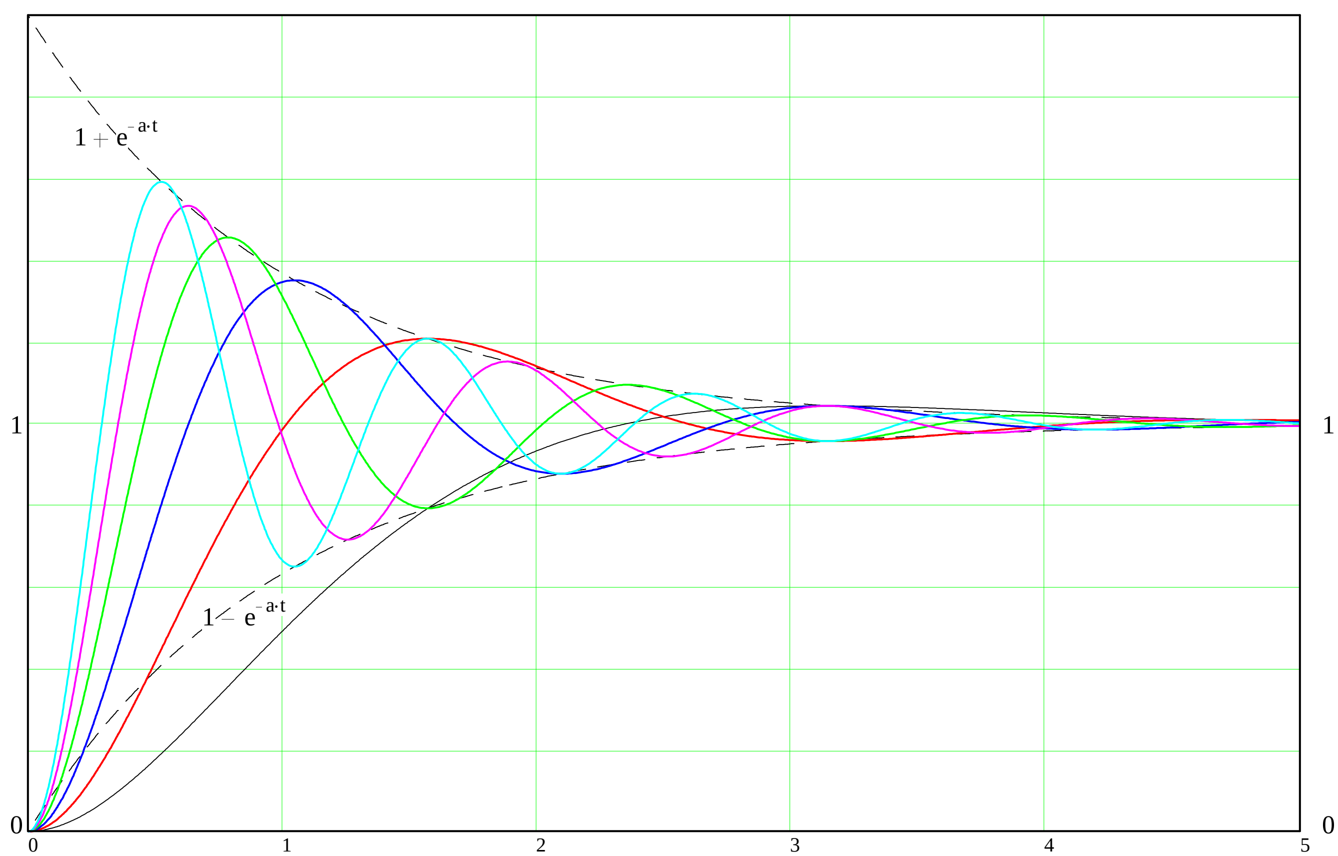
$= \cos(\theta)$

$$y(t) = x_m\cdot\mathbf{H}(0)\cdot\left(1 - e^{-a\cdot t}\cdot\cos(b\cdot t) - \frac{a}{b}\cdot e^{-a\cdot t}\cdot\sin(b\cdot t)\right)$$

(curves below are normalized so $\mathbf{H}(0) = 1$)



Overshoot
$$\%OS = 100\cdot\%\cdot e^{-\frac{a}{b}\cdot\pi}$$



Underdamped Curves

A first-order system for reference

$$\mathbf{H}_1(s) = \frac{k}{s+a} \qquad a := 1 \qquad k := a \qquad y_1(t) := \left(\frac{k}{a} - \frac{k}{a}\cdot e^{-a\cdot t}\right)$$

An overdamped system with a single zero

$$\mathbf{H}(s) = \frac{k\cdot(s+z)}{(s+a_1)\cdot(s+a_2)} \qquad \mathbf{Y}(s) = \frac{X_m}{s}\cdot\frac{k\cdot(s+z)}{(s+a_1)\cdot(s+a_2)}$$

k is normalized so the curves below will not reach the same final condition.

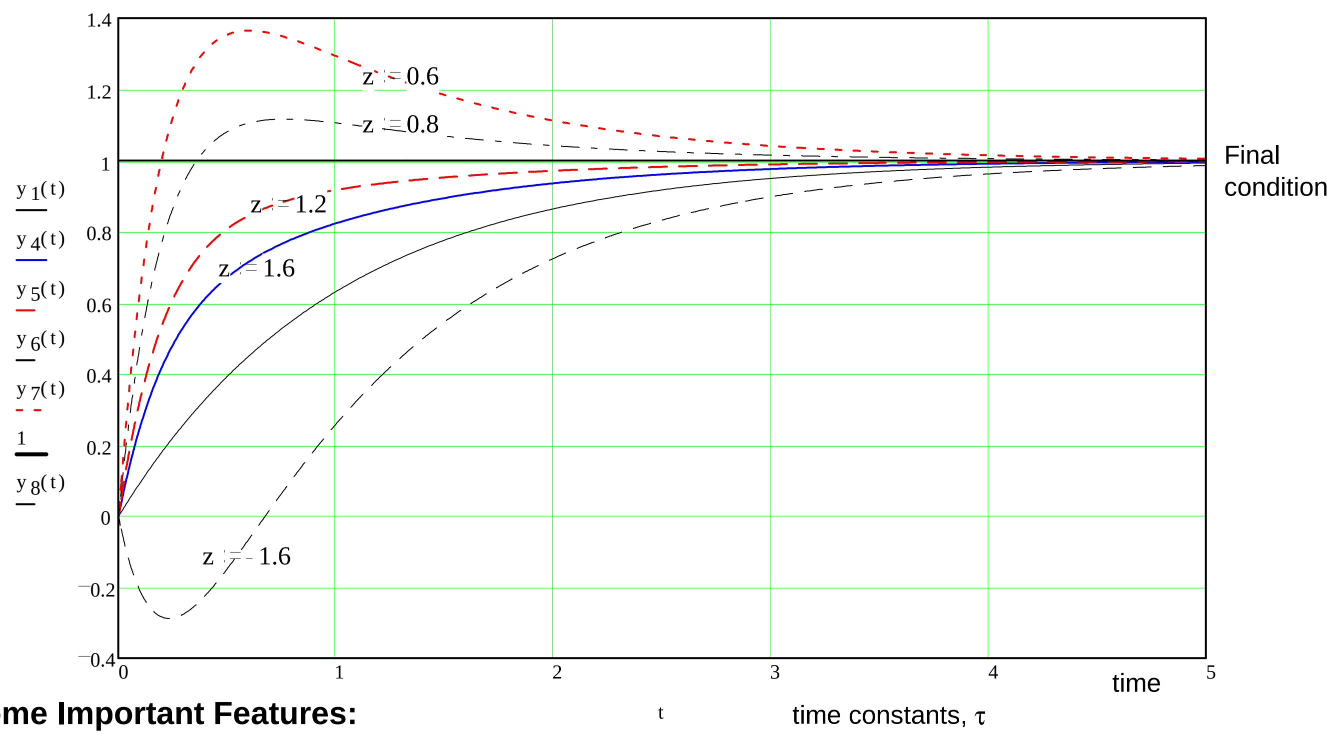
$$z := 1.6 \qquad k := \frac{a_1\cdot a_2}{z} \qquad y_4(t) := \left[\frac{k\cdot z}{a_1\cdot a_2} + \frac{k\cdot(z-a_1)}{a_1\cdot(a_1-a_2)}\cdot e^{-a_1\cdot t} + \frac{k\cdot(z-a_2)}{a_2\cdot(a_2-a_1)}\cdot e^{-a_2\cdot t}\right]$$

$$z := 1.2 \qquad k := \frac{a_1\cdot a_2}{z} \qquad y_5(t) := \left[\frac{k\cdot z}{a_1\cdot a_2} + \frac{k\cdot(z-a_1)}{a_1\cdot(a_1-a_2)}\cdot e^{-a_1\cdot t} + \frac{k\cdot(z-a_2)}{a_2\cdot(a_2-a_1)}\cdot e^{-a_2\cdot t}\right]$$

$$z := 0.8 \qquad k := \frac{a_1\cdot a_2}{z} \qquad y_6(t) := \left[\frac{k\cdot z}{a_1\cdot a_2} + \frac{k\cdot(z-a_1)}{a_1\cdot(a_1-a_2)}\cdot e^{-a_1\cdot t} + \frac{k\cdot(z-a_2)}{a_2\cdot(a_2-a_1)}\cdot e^{-a_2\cdot t}\right]$$

$$z := 0.6 \qquad k := \frac{a_1\cdot a_2}{z} \qquad y_7(t) := \left[\frac{k\cdot z}{a_1\cdot a_2} + \frac{k\cdot(z-a_1)}{a_1\cdot(a_1-a_2)}\cdot e^{-a_1\cdot t} + \frac{k\cdot(z-a_2)}{a_2\cdot(a_2-a_1)}\cdot e^{-a_2\cdot t}\right]$$

$$z := -1.6 \qquad k := \frac{a_1\cdot a_2}{z} \qquad y_8(t) := \left[\frac{k\cdot z}{a_1\cdot a_2} + \frac{k\cdot(z-a_1)}{a_1\cdot(a_1-a_2)}\cdot e^{-a_1\cdot t} + \frac{k\cdot(z-a_2)}{a_2\cdot(a_2-a_1)}\cdot e^{-a_2\cdot t}\right]$$



Some Important Features:

1) The zero (z) is in the LHP if z is positive.

2) If the zero is closer to the origin than the poles, than it can cause overshoot and/or significant steady-state error.

Remember this one

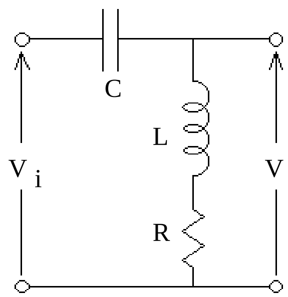
3) The steady-state error will be 100% (no DC gain) if the zero is at the origin. The zero is at the origin cancels the pole of the DC (step) input. (The system has a differentiator.)

4) A zero in the RHP (non-minimum phase zero) can cause undershoot or a negative DC gain.

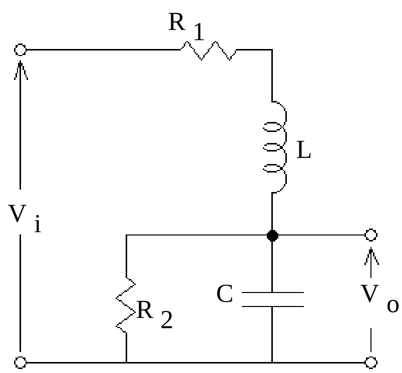
ECE 3510 homework # 5 Due: Tue, 2/9/21

1. Problem 3.2b, p.68 in Bodson text.
2. Problem 3.3 in Bodson text. As part of your work to reach a solution, draw the pole diagram for each.
3. Find the transfer function $\mathbf{H}(s) = \dfrac{\mathbf{V_o}(s)}{\mathbf{V_i}(s)}$ for these circuits. Properly simplify all your expressions for $\mathbf{H}(s)$ like you did in HW 4.

a)



b)



4. Find the step response of: $\mathbf{H}(s) = \dfrac{k}{(s+a_1)\cdot(s+a_2)}$ Step input: $x(t) = x_m \cdot u(t)$ $\mathbf{X}(s) = \dfrac{x_m}{s}$

   Show the steps necessary to arrive at the steady-state and transient responses shown as equation(s) 3.37 on p.49 of the text.

5. Find the step response of: $\mathbf{H}(s) = \dfrac{k\cdot s}{(s+a)^2 + b^2} = \dfrac{k\cdot s}{s^2 + 2\cdot s\cdot a + (a^2+b^2)}$ where b is real

   Show the steps necessary to arrive at the steady-state and transient responses.

6. For the transfer functions below, find the DC gain and the full step responses. You may use the results found in section 3.3.2 of the text as well as problem 3, above.

   a) $\mathbf{H}(s) = \dfrac{2}{s^2 + 2\cdot s + 1}$ b) $\mathbf{H}(s) = \dfrac{-s-2}{s^2+2\cdot s+2}$ Hint: Notice how easily this will split into two parts that you already have answers for.

Answers

1. $\dfrac{H_1\cdot H_4 + H_2\cdot H_4 - H_1\cdot H_2\cdot H_3 + H_1\cdot H_3}{1+H_1}$

2.

	Stable	Example of a Problem input
a)	yes	
b)	no	$\cos(2\cdot t)$
c)	yes	
d)	no	any input, even noise
e)	no	$\dfrac{1}{1}$ DC
f)	no	
		u(t) is assumed

3. a) $\dfrac{s^2 + \dfrac{R}{L}\cdot s}{s^2 + \dfrac{R}{L}\cdot s + \dfrac{1}{L\cdot C}}$ b) $\dfrac{\dfrac{1}{L\cdot C}}{s^2 + \left(\dfrac{1}{C\cdot R_2} + \dfrac{R_1}{L}\right)\cdot s + \left(1 + \dfrac{R_1}{R_2}\right)\cdot\dfrac{1}{L\cdot C}}$

4. $y(\infty) = \dfrac{x_m\cdot k}{a_1\cdot a_2}$ $y_{tr}(t) = x_m\cdot\left[\dfrac{k}{a_1\cdot(a_1-a_2)}\cdot e^{-a_1\cdot t} + \dfrac{k}{a_2\cdot(a_2-a_1)}\cdot e^{-a_2\cdot t}\right]$

   OR: $y(t) = x_m\cdot\left[\dfrac{k}{a_1\cdot a_2} + \dfrac{k}{a_1\cdot(a_1-a_2)}\cdot e^{-a_1\cdot t} + \dfrac{k}{a_2\cdot(a_2-a_1)}\cdot e^{-a_2\cdot t}\right]$

5. $y(\infty) = 0$ $y_{tr}(t) = x_m\cdot\dfrac{k}{b}\cdot e^{-a\cdot t}\cdot\sin(b\cdot t)$ OR: $y(t) = 0 + x_m\cdot\dfrac{k}{b}\cdot e^{-a\cdot t}\cdot\sin(b\cdot t)$

6. a) $x_m\cdot\left(2 - 2\cdot e^{-t} - 2\cdot t\cdot e^{-t}\right)$ b) $x_m\cdot\left(-1 + e^{-t}\cdot\cos(t)\right)$

ECE 3510 homework # 6 Review of Steady-State AC Due: Fri, 2/12/21 b

1. Convert the following complex numbers to polar form ($m\underline{/\theta}$ or $me^{j\theta}$). a) $2.6 + 8.7j$ b) $3 + 4j$ c) $-3 - 4j$

2. Convert the following complex numbers to rectangular form (a + bj). a) $10\cdot e^{j\cdot 60\cdot deg}$ b) $10\cdot e^{-j\cdot 45\cdot deg}$ c) $20\cdot e^{j\cdot 120\cdot deg}$

3. Add or subtract the complex numbers. a) $(3 + 2j) + (6 + 9j)$ b) $(9 - 10j) - (9 + 10j)$

4. Multiply the complex numbers. a) $\left(20\cdot e^{j\cdot 40\cdot deg}\right)\cdot\left(10\cdot e^{j\cdot 60\cdot deg}\right)$ b) $(-2 - j)\cdot(-6 - 9j)$

5. Divide the complex numbers. a) $\dfrac{20\cdot e^{j\cdot 40\cdot deg}}{10\cdot e^{j\cdot 60\cdot deg}}$ b) $\dfrac{12 + 10j}{6 + 9j}$

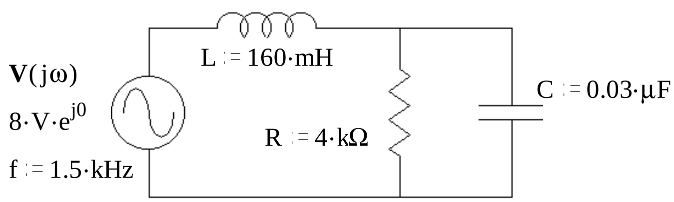
6. Add and subtract the sinusoidal voltages using phasors. Draw a phasor diagram which shows all 4 phasors, and give your final answer in time domain form.

   $v_1(t) = 1.5\cdot V\cdot\cos(\omega\cdot t + 10\cdot deg)$ $v_2(t) = 3.2\cdot V\cdot\cos(\omega\cdot t + 25\cdot deg)$

   a) Find $v_3(t) = v_1(t) + v_2(t)$ b) Find $v_4(t) = v_1(t) - v_2(t)$

7. a) Find $\mathbf{Z_{eq}}$.

   b) Find the current $\mathbf{I_L}(j\omega)$.



8. Find the steady-state magnitude and phase of each of the following transfer functions. $|H(j\cdot\omega)|$ = ? $\underline{/H(j\omega)}$ = ?

   a) $\omega := 10\cdot\dfrac{rad}{sec}$ $s = j\cdot\omega$ $H(s) = \dfrac{\dfrac{40}{sec}\cdot s}{s^2 + \dfrac{10}{sec}\cdot s + \dfrac{200}{sec^2}}$

   b) $f := 50\cdot Hz$ $H(s) = \dfrac{s^2 + \dfrac{1000}{sec}\cdot s}{s^2 + \dfrac{300}{sec}\cdot s + \dfrac{10000}{sec^2}}$

9. Find the following outputs. Express them in the time domain, first as a cosine with a phase angle and then as a sum of cosine and sine with no phase angles:

   a) The input $x(t) = 3\cdot\cos(10\cdot t)$ is the input for the transfer function of 8a), above.

   b) The input $x(t) = 5\cdot\sin(2\cdot\pi\cdot 50\cdot t)$ is the input for the transfer function of 8b), above.

   remember, sine is -j

Answers

1. a) $9.08\cdot e^{j\cdot 73.4\cdot deg}$ b) $5\cdot e^{j\cdot 53.1\cdot deg}$ c) $5\cdot e^{-j\cdot 126.9\cdot deg}$

2. a) $5 + 8.66\cdot j$ b) $7.071 - 7.071\cdot j$ c) $-10 + 17.321\cdot j$ 3. a) $9 + 11\cdot j$ b) $-20\cdot j$

4. a) $200\cdot e^{j\cdot 100\cdot deg}$ b) $24.2\cdot e^{j\cdot 82.9\cdot deg}$ 5. a) $2\cdot e^{-j\cdot 20\cdot deg}$ b) $1.385 - 0.41\cdot j$

6. a) $v_1(t) + v_2(t) = 4.67\cdot\cos(\omega\cdot t + 20.2\cdot deg)\cdot V$ b) $v_1(t) - v_2(t) = 1.794\cdot\cos(\omega\cdot t - 142.5\cdot deg)\cdot V$

7. a) $1.82\cdot k\Omega$ $-15.2\cdot deg$ b) $4.4\cdot mA$ $15.2\cdot deg$

8. a) M = 2.828 $45\cdot deg$ b) M = 2.544 $-25.8\cdot deg$

9. a) $y(t) = 8.484\cdot\cos\left(10\cdot\dfrac{rad}{sec}\cdot t + 45\cdot deg\right) = 6\cdot\cos\left(10\cdot\dfrac{rad}{sec}\cdot t\right) - 6\cdot\sin\left(10\cdot\dfrac{rad}{sec}\cdot t\right)$

   b) $y(t) = 12.72\cdot\cos(2\cdot\pi\cdot 50\cdot t - 115.82\cdot deg) = -5.54\cdot\cos(2\cdot\pi\cdot 50\cdot t) + 11.45\cdot\sin(2\cdot\pi\cdot 50\cdot t)$