

ECE 3510 Lecture 4 notes Inverse Laplace transforms by Partial Fraction Expansion

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1/8/08,
1/15/10
1/8/20

Ex. 1 by Clearing Fractions

Like Example 1-b from page 25, but with more interesting numbers

$$F(s) = \frac{12 \cdot s + 64}{(s+4)^2 \cdot (s+6)} = \frac{A}{s+4} + \frac{B}{(s+4)^2} + \frac{C}{s+6}$$

Multiply both sides by: $(s+4)^2 \cdot (s+6)$

$$12 \cdot s + 64 = A \cdot (s+4) \cdot (s+6) + B \cdot (s+6) + C \cdot (s+4)^2$$

$$\begin{aligned} 12 \cdot s + 64 &= A \cdot s^2 + A \cdot 10 \cdot s + A \cdot 24 + B \cdot s + B \cdot 6 + C \cdot s^2 + C \cdot 8 \cdot s + C \cdot 16 \\ 0 \cdot s^2 &= A \cdot s^2 + 0 \cdot s^2 + C \cdot s^2 \quad A := -C \end{aligned}$$

no s^2 term on the left

$$12 \cdot s = A \cdot 10 \cdot s + B \cdot s + C \cdot 8 \cdot s$$

$$12 = A \cdot 10 + B + -A \cdot 8 \quad B := 12 - 2 \cdot A$$

$$64 = A \cdot 24 + B \cdot 6 + C \cdot 16$$

$$64 = A \cdot 24 + (12 - 2 \cdot A) \cdot 6 + -A \cdot 16$$

$$64 - 72 = -8 = -4 \cdot A \quad A := 2$$

$$C := -2 \quad B := 8$$

$$F(s) = \frac{12 \cdot s + 64}{(s+4)^2 \cdot (s+6)} = \frac{2}{s+4} + \frac{8}{(s+4)^2} + \frac{-2}{s+6}$$

$$f(t) = 2 \cdot e^{-4t} + 8 \cdot t \cdot e^{-4t} + -2 \cdot e^{-6t}$$

Ex. 1 by Residue Method

Like Example 1-a from page 24, but with more interesting numbers

$$12 \cdot s + 64 = A \cdot (s+4) \cdot (s+6) + B \cdot (s+6) + C \cdot (s+4)^2$$

Set $s := -4$

$$12 \cdot (-4) + 64 = 0 + B \cdot (-4 + 6) + 0 \quad B := (s+4)^2 \cdot \frac{F(s)}{(s+4)^2 \cdot (s+6)}$$

$$B := 8$$

Set $s := -6$

$$12 \cdot (-6) + 64 = 0 + 0 + C \cdot (-6 + 4)^2 \quad C := -2$$

See Eq. 2.9, page 22 of Bodson Text

$$A = \frac{d}{ds} \left[(s+4)^2 \cdot \frac{12 \cdot s + 64}{(s+4)^2 \cdot (s+6)} \right] \Bigg|_{s:=-4} = \frac{d}{ds} \frac{12 \cdot s + 64}{(s+6)} \Bigg|_{s:=-4}$$

$$\text{Recall: } \frac{d \cdot \frac{h}{g}}{ds} = \frac{h \cdot \frac{dg}{ds} - g \cdot \frac{dh}{ds}}{g^2}$$

$$\frac{d}{ds} \frac{12 \cdot s + 64}{(s+6)} = \frac{(s+6) \cdot \left[\frac{d}{ds} (12 \cdot s + 64) \right] - (12 \cdot s + 64) \cdot \left[\frac{d}{ds} (s+6) \right]}{(s+6)^2} = \frac{(s+6) \cdot 12 - (12 \cdot s + 64) \cdot 1}{(s+6)^2}$$

$$= \frac{(12 \cdot s + 72) - 12 \cdot s - 64}{(s+6)^2} = \frac{8}{(s+6)^2} \Bigg|_{s:=-4} = \frac{8}{2^2} = \frac{8}{4} = 2 = A$$

Ex. 1 by the Mixed Method

$$F(s) = \frac{12 \cdot s + 64}{(s+4)^2 \cdot (s+6)} = \frac{A}{s+4} + \frac{B}{(s+4)^2} + \frac{C}{s+6}$$

Multiply both sides by: $(s+4)^2 \cdot (s+6)$

$$12 \cdot s + 64 = A \cdot (s+4) \cdot (s+6) + B \cdot (s+6) + C \cdot (s+4)^2$$

Set $s := -4$

$$\frac{12 \cdot (-4) + 64}{16} = 0 + \frac{B \cdot (-4 + 6)}{2} + 0 \quad B := 8$$

Set $s := -6$

$$\frac{12 \cdot (-6) + 64}{-8} = 0 + 0 + \frac{C \cdot (-6 + 4)^2}{(-2)^2} \quad C := -2$$

Back to equation above

$$\begin{aligned} 12 \cdot s + 64 &= A \cdot (s+4) \cdot (s+6) + B \cdot (s+6) + C \cdot (s+4)^2 \\ 12 \cdot s + 64 &= A \cdot s^2 + A \cdot 10 \cdot s + A \cdot 24 + 8 \cdot s + 8 \cdot 6 + C \cdot s^2 + C \cdot 8 \cdot s + C \cdot 16 \\ 0 \cdot s^2 &= A \cdot s^2 + 0 \cdot s^2 + C \cdot s^2 \quad A := -C \end{aligned}$$

no s^2 term on the left

$$A = 2$$

And the rule is: Get as many easy answers as possible before clearing fractions!

$$F(s) = \frac{12 \cdot s + 64}{(s+4)^2 \cdot (s+6)} = \frac{2}{s+4} + \frac{8}{(s+4)^2} + \frac{-2}{s+6}$$

$$f(t) = 2 \cdot e^{-4t} + 8 \cdot t \cdot e^{-4t} + -2 \cdot e^{-6t}$$

$$f(t) = 2 \cdot e^{-4t} + 8 \cdot t \cdot e^{-4t} - 2 \cdot e^{-6t}$$

Same results again

ECE 3510 Partial Fraction Expansion Examples by the Mixed Method

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1/18/20

Ex. 1 Like Example 1 from page 24, but with more interesting numbers

$$F(s) = \frac{2 \cdot s^2 + 9 \cdot s + 15}{(s+2)^2 \cdot (s+3)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s+3}$$

Multiply both sides by: $(s+2)^2 \cdot (s+3)$

$$2 \cdot s^2 + 9 \cdot s + 15 = A \cdot (s+2) \cdot (s+3) + B \cdot (s+3) + C \cdot (s+2)^2$$

Set $s := -2$

$$\frac{2 \cdot (-2)^2 + 9 \cdot (-2) + 15}{5} = 0 + \frac{B \cdot (-2+3)}{1} + 0 \quad B := 5$$

Set $s := -3$

$$\frac{2 \cdot (-3)^2 + 9 \cdot (-3) + 15}{6} = 0 + 0 + \frac{C \cdot (-3+2)^2}{C \cdot (-1)^2} \quad C := 6$$

Back to equation above

$$2 \cdot s^2 + 9 \cdot s + 15 = A \cdot (s+2) \cdot (s+3) + B \cdot (s+3) + C \cdot (s+2)^2$$

$$2 \cdot s^2 + 9 \cdot s + 15 = A \cdot s^2 + A \cdot 5 \cdot s + A \cdot 6 + 5 \cdot s + 5 \cdot 3 + C \cdot s^2 + C \cdot 4 \cdot s + C \cdot 4$$

$$2 \cdot s^2 = A \cdot s^2 + 0 + C \cdot s^2 \quad A := 2 - C$$

$$A = -4$$

And the rule is: Get as many easy answers as possible before clearing fractions!

$$F(s) = \frac{2 \cdot s^2 + 9 \cdot s + 15}{(s+2)^2 \cdot (s+3)} = \frac{-4}{(s+2)} + \frac{5}{(s+2)^2} + \frac{6}{(s+3)}$$

$$f(t) = -4 \cdot e^{-2 \cdot t} + 5 \cdot t \cdot e^{-2 \cdot t} + 6 \cdot e^{-3 \cdot t}$$

Ex. 2 Like Example 2 from page 25

$$F(s) = \frac{2 \cdot (s+1)}{s \cdot (s^2 + 4 \cdot s + 13)} \quad \text{Try to find factors of } (s^2 + 4 \cdot s + 13): \quad \text{find roots } \rightarrow \frac{-4 + \sqrt{4^2 - 4 \cdot 13}}{2} = -2 + 3j$$

= complex numbers.. STOP!

If there are complex poles, then expect sines and cosines in the time domain. If the poles have real components as well as imaginary components then the sines and cosines are multiplied by exponentials. The entries in the Laplace transform table for these are:

$$e^{at} \cdot \cos(bt) \quad \Leftrightarrow \quad \frac{s-a}{(s-a)^2 + b^2}$$

and $e^{at} \cdot \sin(bt) \quad \Leftrightarrow \quad \frac{b}{(s-a)^2 + b^2}$

DON'T decompose like this:

$$F(s) = \frac{2 \cdot (s+1)}{s \cdot (s^2 + 4 \cdot s + 13)} = \frac{A}{s} + \frac{B'}{(s+2+3j)} + \frac{C'}{(s+2-3j)}$$

Because that will only lead to complex exponentials which then have to be changed to sines and cosine in a separate step.

It's much smarter to decompose to this form:

$$F(s) = \frac{2 \cdot (s+1)}{s \cdot (s^2 + 4 \cdot s + 13)} = \frac{A}{s} + \frac{B \cdot (s-a)}{(s-a)^2 + b^2} + \frac{C \cdot b}{(s-a)^2 + b^2} \quad \text{Because these forms are in the table!}$$

And the rule is: Only decompose to terms that are actually in the table!

Ex. 2, Continued

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Find a and b :

$$s^2 + 4 \cdot s + 13 = (s - a)^2 + b^2 = s^2 - 2 \cdot a \cdot s + a^2 + b^2$$

$$4 \cdot s = -2 \cdot a \cdot s \quad a := -2$$

$$13 = a^2 + b^2 = (-2)^2 + b^2$$

$$b^2 = 9 \quad b := 3$$

Find a and b the easy way: Recall from above, roots of $(s^2 + 4 \cdot s + 13)$:

$$= \frac{-4 + \sqrt{4^2 - 4 \cdot 13}}{2} = -2 + 3j \quad \& \quad -2 - 3j$$

$$a := \text{Re}(-2 + 3j) \quad a = -2 \quad b := \text{Im}(-2 + 3j) \quad b = 3$$

$$F(s) = \frac{2 \cdot (s + 1)}{s \cdot (s^2 + 4 \cdot s + 13)} = \frac{2 \cdot (s + 1)}{s \cdot [(s - a)^2 + b^2]}$$

$$= \frac{2 \cdot (s + 1)}{s \cdot [(s + 2)^2 + 3^2]} = \frac{A}{s} + \frac{B \cdot (s - a)}{(s - a)^2 + b^2} + \frac{C \cdot b}{(s - a)^2 + b^2}$$

Multiply both sides by: $s \cdot [(s + 2)^2 + 3^2]$

$$2 \cdot (s + 1) = A \cdot [(s + 2)^2 + 3^2] + B \cdot (s + 2) \cdot s + C \cdot 3 \cdot s$$

Set $s := 0$

$$2 \cdot (1) = A \cdot [(2)^2 + 3^2] + 0 + 0 \quad A := \frac{2}{13}$$

$$(2)^2 + 3^2 = 13$$

Set $s := -2$

$$2 \cdot (-1) = A \cdot [(0)^2 + 3^2] + 0 + C \cdot 3 \cdot (-2)$$

$$-2 = 9 \cdot A + -6 \cdot C \quad C := \frac{2 + 9 \cdot A}{6} = \frac{22}{39}$$

Back to equation above

$$2 \cdot (s + 1) = A \cdot (s^2 + 4 \cdot s + 13) + B \cdot (s + 2) \cdot s + C \cdot 3 \cdot s$$

$$2 \cdot s + 2 = A \cdot s^2 + A \cdot 4 \cdot s + A \cdot 13 + B \cdot s^2 + 2 \cdot B \cdot s + 3 \cdot C \cdot s$$

$$0 \cdot s^2 = A \cdot s^2 + B \cdot s^2 \quad B := -A = -\frac{2}{13}$$

no s^2 term on the left

An **Alternate Way** to find B and C With Complex math

Multiply both sides by: $[(s + 2)^2 + 3^2]$

$$\frac{2 \cdot (s + 1)}{s} = \frac{A}{s} \cdot [(s + 2)^2 + 3^2] + B \cdot (s + 2) + C \cdot 3$$

Set $s := -2 + 3j$

$$\frac{2 \cdot (-2 + 3j + 1)}{-2 + 3j} = \frac{A}{s} \cdot (0) + B \cdot (-2 + 3j + 2) + C \cdot 3$$

$$\frac{(-2 - 3j) \cdot 2 \cdot (-1 + 3j)}{(-2 - 3j) \cdot -2 + 3j} = \frac{A}{s} \cdot (0) + B \cdot (3j) + C \cdot 3$$

$$\frac{2 \cdot ((2 + 9) + (3 - 6) \cdot j)}{2^2 + 3^2} = + B \cdot (3j) + C \cdot 3$$

$$\frac{22 - 6 \cdot j}{13 \cdot 3} = C + B \cdot j$$

$$C = \frac{22}{39} \quad B = -\frac{6}{39} = -\frac{2}{13}$$

Ex. 2, Continued

$$\begin{aligned}
 F(s) &= \frac{2 \cdot (s+1)}{s \cdot (s^2 + 4s + 13)} = \frac{A}{s} + \frac{B \cdot (s-a)}{(s-a)^2 + b^2} + \frac{C \cdot b}{(s-a)^2 + b^2} \\
 &= \frac{2}{13} \cdot \frac{1}{s} + \frac{2}{13} \cdot \frac{(s+2)}{(s+2)^2 + 3^2} + \frac{22}{39} \cdot \frac{3}{(s+2)^2 + 3^2} \\
 f(t) &= \left[\frac{2}{13} - \frac{2}{13} \cdot e^{-2t} \cdot \cos(3 \cdot t) + \frac{3}{13} \cdot e^{-2t} \cdot \sin(3 \cdot t) \right] u(t) \\
 &= \frac{1}{39} \cdot (6 - 6 \cdot e^{-2t} \cdot \cos(3 \cdot t) + 22 \cdot e^{-2t} \cdot \sin(3 \cdot t)) \cdot u(t)
 \end{aligned}$$

Ex. 2.2 Another like Example 2, Different numbers and method (Not done in class)

$$F(s) = \frac{130 \cdot (s+1)}{(s+2) \cdot (s^2 + 6 \cdot s + 34)} = \frac{A}{s+2} + \frac{B \cdot (s-a)}{(s^2 + 6 \cdot s + 34)} + \frac{C \cdot b}{(s^2 + 6 \cdot s + 34)}$$

Because these forms are in the table!

$$\begin{aligned}
 -2a=6 & \quad a^2=b^2=34 \\
 a := -3 & \quad b = \sqrt{34 - a^2} = 5
 \end{aligned}$$

Multiply both sides by: $(s+2)$ Set $s := -2$

$$\begin{aligned}
 \frac{130 \cdot (s+1)}{(s^2 + 6 \cdot s + 34)} &= A + \left[\frac{B \cdot (s-a)}{(s^2 + 6 \cdot s + 34)} + \frac{C \cdot b}{(s^2 + 6 \cdot s + 34)} \right] (s+2) \\
 \frac{130 \cdot ((-2)+1)}{[(-2)^2 + 6 \cdot (-2) + 34]} &= A + 0 \\
 \frac{-130}{4 - 12 + 34} = \frac{-130}{26} = -5 &= A
 \end{aligned}$$

Multiply both sides by: $(s^2 + 6 \cdot s + 34)$ Set $s := -3 + 5j$

$$\begin{aligned}
 \frac{130 \cdot (s+1)}{s+2} &= \frac{A}{s+2} \cdot (s^2 + 6 \cdot s + 34) + B \cdot (s+3) + C \cdot 5 \\
 \frac{130 \cdot (-3 + 5j + 1)}{-3 + 5j + 2} &= \frac{A}{s+2} \cdot (0) + B \cdot (-3 + 5j + 3) + C \cdot 5 \\
 \frac{(-1 - 5j) \cdot 130 \cdot (-2 + 5j)}{(-1 - 5j) \cdot (-1 + 5j)} &= 0 + B \cdot (5j) + C \cdot 5 \\
 \frac{130 \cdot ((2 + 25) + (10 - 5) \cdot j)}{1^2 + 5^2} &= + B \cdot (5j) + C \cdot 5 \\
 \frac{130 \cdot (((-1) \cdot (-2) + (-5j) \cdot (5j)) + ((-1) \cdot (5j) + (-5j) \cdot (-2)))}{(1 + 25)} &= B \cdot (5j) + C \cdot 5 \\
 \frac{130 \cdot (27 + 5 \cdot j)}{26 \cdot 5} = 27 + 5j &= B \cdot j + C \\
 &B = 5 \quad C = 27
 \end{aligned}$$

$$\begin{aligned}
 F(s) &= \frac{130 \cdot (s+1)}{(s+2) \cdot (s^2 + 6 \cdot s + 34)} = \frac{5}{s+2} + \frac{-5 \cdot (s+3)}{(s^2 + 6 \cdot s + 34)} + \frac{27 \cdot 5}{(s^2 + 6 \cdot s + 34)} \\
 f(t) &= \left[-5 \cdot e^{-2t} + 5 \cdot e^{-3t} \cdot \cos(5 \cdot t) + 27 \cdot e^{-5t} \cdot \sin(5 \cdot t) \right] u(t)
 \end{aligned}$$

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Ex. 3 $F(s) := \frac{9 \cdot s^2 - 9 \cdot s + 36}{(s^2 + 1) \cdot (s^2 + 4)}$

DON'T decompose like this:

$$F(s) := \frac{9 \cdot s^2 - 9 \cdot s + 36}{(s^2 + 1) \cdot (s^2 + 4)} = \frac{A'}{(s+j)} + \frac{B'}{(s-j)} + \frac{C'}{(s+2j)} + \frac{D'}{(s-2j)}$$

or even this:

$$= \frac{A'' \cdot s + B''}{(s^2 + 1)} + \frac{C'' \cdot s + D''}{(s^2 + 4)}$$

Look at the table first, to see what you should aim for.

It's much smarter to decompose to this form:

$$F(s) := \frac{9 \cdot s^2 - 9 \cdot s + 36}{(s^2 + 1) \cdot (s^2 + 4)} = \frac{A \cdot s}{(s^2 + a_1^2)} + \frac{B \cdot a_1}{(s^2 + a_1^2)} + \frac{C \cdot s}{(s^2 + a_2^2)} + \frac{D \cdot a_2}{(s^2 + a_2^2)}$$

$$a_1 := \sqrt{1} \quad a_1 = 1 \quad a_2 := \sqrt{4} \quad a_2 = 2$$

$$= \frac{A \cdot s}{(s^2 + 1^2)} + \frac{B \cdot 1}{(s^2 + 1^2)} + \frac{C \cdot s}{(s^2 + 2^2)} + \frac{D \cdot 2}{(s^2 + 2^2)}$$

or, if you prefer:

$$= \frac{A \cdot s + B \cdot a_1}{(s^2 + 1)} + \frac{C \cdot s + D \cdot a_2}{(s^2 + 4)}$$

which is the same thing

Multiply both sides by: $(s^2 + 1) \cdot (s^2 + 4)$

$$9 \cdot s^2 - 9 \cdot s + 36 = A \cdot s \cdot (s^2 + 2^2) + B \cdot 1 \cdot (s^2 + 2^2) + C \cdot s \cdot (s^2 + 1^2) + D \cdot 2 \cdot (s^2 + 1^2)$$

Set $s := \sqrt{-1}$

$$9 \cdot s^2 - 9 \cdot s + 36 = A \cdot j \cdot \frac{(-1 + 2^2)}{3} + B \cdot 1 \cdot \frac{(-1 + 2^2)}{3} + 0 + 0$$

$$9 \cdot j^2 - 9 \cdot j + 36 = A \cdot j \cdot 3 + B \cdot 3$$

$$-9 \cdot j + 27 \quad A := -3 \quad B := 9$$

A and B must be real because of the way that we have decomposed the transform. The time functions cannot have unreal coefficients

Set $s := \sqrt{-4}$

$$9 \cdot (2 \cdot j)^2 - 9 \cdot (2 \cdot j) + 36 = 0 + 0 + C \cdot 2 \cdot j \cdot \frac{(-4 + 1^2)}{-3} + D \cdot 2 \cdot \frac{(-4 + 1^2)}{-3}$$

$$-36 - 18 \cdot j + 36 \quad -6 \cdot j \cdot C \quad -6 \cdot D$$

$$C := 3 \quad D := 0$$

$$F(s) := \frac{9 \cdot s^2 - 9 \cdot s + 36}{(s^2 + 1) \cdot (s^2 + 4)} = \frac{-3 \cdot s}{(s^2 + 1^2)} + \frac{9 \cdot 1}{(s^2 + 1^2)} + \frac{3 \cdot s}{(s^2 + 2^2)} + \frac{0 \cdot 2}{(s^2 + 2^2)}$$

$$f(t) = ((-3) \cdot \cos(1 \cdot t) + 9 \cdot \sin(1 \cdot t) + 3 \cdot \cos(2 \cdot t) + 0 \cdot \sin(2 \cdot t)) \cdot u(t)$$

$$= ((-3) \cdot \cos(1 \cdot t) + 9 \cdot \sin(1 \cdot t) + 3 \cdot \cos(2 \cdot t)) \cdot u(t)$$

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Ex. 4 Like Example 3 from page 27 (Not done in class)

$$F(s) = \frac{1}{(s^2 + 2 \cdot s + 2)^2} \quad \text{Try to find factors of } (s^2 + 2 \cdot s + 2) : \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = -1 + j \quad \& \quad -1 - j$$

= complex numbers.. STOP!

Decompose to: $e^{a \cdot t} \cdot \cos(b \cdot t) \iff \frac{s - a}{(s - a)^2 + b^2}$ $a := \text{Re}(-1 + j) \quad a = -1$

$e^{a \cdot t} \cdot \sin(b \cdot t) \iff \frac{b}{(s - a)^2 + b^2}$ $b := \text{Im}(-1 + j) \quad b = 1$

and $t \cdot e^{a \cdot t} \cdot \cos(b \cdot t) \iff \frac{(s - a)^2 - b^2}{[(s - a)^2 + b^2]^2}$

$t \cdot e^{a \cdot t} \cdot \sin(b \cdot t) \iff \frac{2 \cdot b \cdot (s - a)}{[(s - a)^2 + b^2]^2}$

Decompose to this form:

$$F(s) = \frac{1}{[(s - a)^2 + b^2]^2} = \frac{A \cdot (s - a)}{(s - a)^2 + b^2} + \frac{B \cdot b}{(s - a)^2 + b^2} + \frac{C \cdot [(s - a)^2 - b^2]}{[(s - a)^2 + b^2]^2} + \frac{D \cdot (2 \cdot b \cdot (s - a))}{[(s - a)^2 + b^2]^2}$$

Multiply both sides by: $[(s + 1)^2 + 1^2]^2$

$$1 = A \cdot (s + 1) \cdot [(s + 1)^2 + 1^2] + B \cdot 1 \cdot [(s + 1)^2 + 1^2] + C \cdot [(s + 1)^2 - 1^2] + D \cdot (2 \cdot 1 \cdot (s + 1))$$

Set $s := -1$

$$1 = 0 + B \cdot 1 + C \cdot (0 - 1) + 0$$

$1 = B - C$
 $-C = 1 - B$

Back to equation above

$$1 = A \cdot s^3 + A \cdot 2 \cdot s^2 + A \cdot 2 \cdot s + A \cdot s^2 + A \cdot 2 \cdot s + A \cdot 2 + B \cdot s^2 + B \cdot 2 \cdot s + B \cdot 2 + C \cdot s^2 + C \cdot 2 \cdot s + D \cdot 2 \cdot s + D \cdot 2$$

$A := 0$

$$0 \cdot s^3 = A \cdot s^3$$

$$1 = 0 + B \cdot s^2 + B \cdot 2 \cdot s + B \cdot 2 + C \cdot s^2 + C \cdot 2 \cdot s + D \cdot 2 \cdot s + D \cdot 2$$

$B := -C = 1 - B$

$$0 \cdot s^2 = B \cdot s^2 + C \cdot s^2$$

$B := \frac{1}{2}$

$$1 = B \cdot 2 + D \cdot 2$$

$C := -B = -\frac{1}{2}$

$$1 = \frac{B \cdot 2}{2} + D \cdot 2$$

$\frac{1}{2} \cdot 2 = 1$
 $D := 0$

$$F(s) = \frac{1}{(s^2 + 2 \cdot s + 2)^2} = 0 + \frac{\frac{1}{2} \cdot 1}{(s - 1)^2 + 1^2} + \frac{-\frac{1}{2} \cdot [(s + 1)^2 - 1^2]}{[(s + 1)^2 + 1^2]^2} + 0$$

$$f(t) = \left[\frac{1}{2} \cdot e^{-t} \cdot \sin(t) - \frac{1}{2} \cdot t \cdot e^{-t} \cdot \cos(t) \right] u(t)$$

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Ex. 4.2 Another like Example 3, but with more interesting numbers

$$F(s) = \frac{256 \cdot (s+5)}{(s^2 + 4 \cdot s + 68)^2}$$

Try to find factors of $(s^2 + 4 \cdot s + 68)$: $\frac{-4 + \sqrt{4^2 - 4 \cdot 68}}{2} = -2 + 8j$ & $-2 - 8j$
 = complex numbers.. STOP!

From Laplace table: $e^{a \cdot t} \cdot \sin(b \cdot t) \iff \frac{b}{(s-a)^2 + b^2}$ $a := \text{Re}(-2 + 8j) \quad a = -2$
 $b := \text{Im}(-2 + 8j) \quad b = 8$

$e^{a \cdot t} \cdot \cos(b \cdot t) \iff \frac{s-a}{(s-a)^2 + b^2}$ $s^2 + 4 \cdot s + 68 =$

$t \cdot e^{a \cdot t} \cdot \cos(b \cdot t) \iff \frac{(s-a)^2 - b^2}{[(s-a)^2 + b^2]^2}$ $(s-a)^2 + b^2 =$
 $(s+2)^2 + 8^2 =$
 $s^2 + 4 \cdot s + 4 + 64$

$t \cdot e^{a \cdot t} \cdot \sin(b \cdot t) \iff \frac{2 \cdot b \cdot (s-a)}{[(s-a)^2 + b^2]^2}$

Decompose to:

$$F(s) = \frac{256 \cdot (s+5)}{(s^2 + 4 \cdot s + 68)^2} = \frac{A \cdot (s-a)}{(s-a)^2 + b^2} + \frac{B \cdot b}{(s-a)^2 + b^2} + \frac{C \cdot [(s-a)^2 - b^2]}{[(s-a)^2 + b^2]^2} + \frac{D \cdot (2 \cdot b \cdot (s-a))}{[(s-a)^2 + b^2]^2}$$

Multiply both sides by: $[(s+2)^2 + 8^2]^2$

$$256 \cdot (s+5) = A \cdot (s+2) \cdot [(s+2)^2 + 8^2] + B \cdot 8 \cdot [(s+2)^2 + 8^2] + C \cdot [(s+2)^2 - 8^2] + D \cdot (2 \cdot 8 \cdot (s+2))$$

Set $s := -2$

$$256 \cdot (-2 + 5) = 768 = 0 + B \cdot 8 \cdot (0 + 8^2) + C \cdot (0 - 8^2) + 0$$

$$8^3 = 512 \quad -64 \quad 768 = 512 \cdot B - 64 \cdot C$$

$$C = -12 + 8 \cdot B$$

Back to equation above

$$256 \cdot (s+5) = A \cdot [s \cdot (s^2 + 4 \cdot s + 68) + 2 \cdot (s^2 + 4 \cdot s + 68)] + B \cdot 8 \cdot (s^2 + 4 \cdot s + 68) + C \cdot [(s+2)^2 - 8^2] + D \cdot (2 \cdot 8 \cdot (s+2))$$

$$= A \cdot s^3 + A \cdot 4 \cdot s^2 + A \cdot 68 \cdot s + A \cdot 2 \cdot s^2 + A \cdot 8 \cdot s + A \cdot 136 + B \cdot 8 \cdot s^2 + B \cdot 32 \cdot s + B \cdot 544 + C \cdot s^2 + C \cdot 4 \cdot s - C \cdot 60 + D \cdot 16 \cdot s + D \cdot 32$$

$$0 \cdot s^3 = A \cdot s^3 \quad A := 0$$

$$0 \cdot s^2 = 0 \cdot 2 \cdot s^2 + 0 \cdot 2 \cdot s^2 + B \cdot 8 \cdot s^2 + C \cdot s^2$$

$$0 = B \cdot 8 + C = B \cdot 8 + (-12 + 8 \cdot B)$$

$$B = \frac{12}{16} = \frac{3}{4}$$

$$C = -12 + 8 \cdot B = -12 + 8 \cdot \frac{3}{4} = -6$$

$$256 \cdot s = 0 \cdot 68 \cdot s + 0 \cdot 8 \cdot s + B \cdot 32 \cdot s + C \cdot 4 \cdot s + D \cdot 16 \cdot s$$

$$256 = 0 + 0.75 \cdot 32 - 6 \cdot 4 + D \cdot 16$$

$$D = \frac{256 - 24 + 24}{16} = 16$$

$$F(s) = \frac{256 \cdot (s+5)}{(s^2 + 4 \cdot s + 68)^2} = 0 + \frac{\frac{3}{4} \cdot b}{(s+2)^2 + 8^2} + \frac{C \cdot [(s+2)^2 - 8^2]}{[(s+2)^2 + 8^2]^2} + \frac{D \cdot (2 \cdot 8 \cdot (s+2))}{[(s+2)^2 + 8^2]^2}$$

$$f(t) = \left[\frac{3}{4} \cdot e^{-t} \cdot \sin(t) - 6 \cdot t \cdot e^{-t} \cdot \cos(t) + 16 \cdot t \cdot e^{-t} \cdot \sin(t) \right] u(t)$$

Non-strictly-proper transforms section 2.2.5, p.16 in Bodson text

What if the order of the numerator is equal to or even greater than the order of the denominator? $m \geq n$?

Example: $F(s) = \frac{2 \cdot s^2 + 100}{s^2 + 8 \cdot s + 41}$ $m := 2$
 $n := 2$

First divide, before partial fraction expansion $s^2 + 8 \cdot s + 41 \left\{ \begin{array}{l} \overline{2 \cdot s^2 + 0 \cdot s + 100} \\ \hline \end{array} \right.$

"remainder"

$$F(s) = \frac{2 \cdot s^2 + 100}{s^2 + 8 \cdot s + 41} =$$

$$f(t) =$$

Delta functions are not very common in real life.

Non-strictly-proper transforms are just as common.

Properties of Signals

Can you tell what $f(t)$ must be just by looking at $F(s)$? YES, somewhat...

$$\frac{s+5}{s \cdot (s^2 + 4 \cdot s + 13) \cdot (s-10)}$$

$$\frac{s+5}{s \cdot (s^2 + 64) \cdot (s+10)}$$

$$\frac{s+5}{s \cdot (s^2 - 4 \cdot s + 13) \cdot (s+10)}$$

$$\frac{s+5}{s \cdot (s^2 + 4 \cdot s + 13)^2 \cdot (s+10)}$$

$$\frac{s+5}{s^3 \cdot (s^2 + 4 \cdot s + 13)^2 \cdot (s+10)^2}$$

ECE 3510 homework # 3a

Due: Sat, 1/30/21

c

1. Find the inverse Laplace transform of each of the following functions:

Use partial fraction expansion and the tables.

a) $F(s) = \frac{1}{s^2 + 5s + 6}$

b) $F(s) = \frac{s-1}{s \cdot (s+2)}$

ECE 3510 homework # 3b

Due: Tue, 2/2/21

2. Find the inverse Laplace transform of each of the following functions:

Use the mixed method and the tables.

a) $F(s) = \frac{3s+6}{(s^2+1) \cdot (s^2+4)}$

b) $F(s) = \frac{1}{(s+2) \cdot (s+1)^2}$

c) $F(s) = \frac{2s}{s^2 + 2s + \frac{5}{4}}$

d) $F(s) = \frac{8s+4}{s^2 \cdot (s+1)^2}$

e) $F(s) = \frac{\frac{1}{2} \cdot s^3 + s^2 + s + \frac{5}{2}}{s^2 \cdot (s^2 + 2s + 5)}$

3. $F(s) = \frac{s-1}{s^3 \cdot (s^2 + 2s + 5)^2}$

Show the form of $f(t)$ without actually finding it.
Indicate which of the coefficients may not be 0

4. Problem 2.3a - f in textbook (p.33)

As part of your work to reach a solution, draw the pole diagram for each.

Answers (time functions below valid for $t \geq 0$ only)

1. a) $(e^{-2t} - e^{-3t}) \cdot u(t)$

b) $\left(\frac{3}{2} \cdot e^{-2t} - \frac{1}{2}\right) \cdot u(t)$

2. a) $(\cos(t) + 2 \cdot \sin(t) - \cos(2t) - \sin(2t)) \cdot u(t)$

b) $(e^{-2t} + t \cdot e^{-t} - e^{-t}) \cdot u(t)$

c) $\left(2 \cdot e^{-t} \cdot \cos\left(\frac{1}{2} \cdot t\right) - 4 \cdot e^{-t} \cdot \sin\left(\frac{1}{2} \cdot t\right)\right) \cdot u(t)$

d) $(4 \cdot t - 4 \cdot t \cdot e^{-t}) \cdot u(t)$

e) $\left(\frac{1}{2} \cdot t + \frac{1}{2} \cdot e^{-t} \cdot \cos(2t)\right) \cdot u(t)$

3. $(A + B \cdot t + C \cdot t^2 + D \cdot e^{at} \cdot \cos(bt) + E \cdot e^{at} \cdot \sin(bt) + F \cdot t \cdot e^{at} \cdot \cos(bt) + G \cdot t \cdot e^{at} \cdot \sin(bt)) \cdot u(t)$

C may not be 0 & Either F or G may be 0, but **NOT BOTH**

Alternate solution:

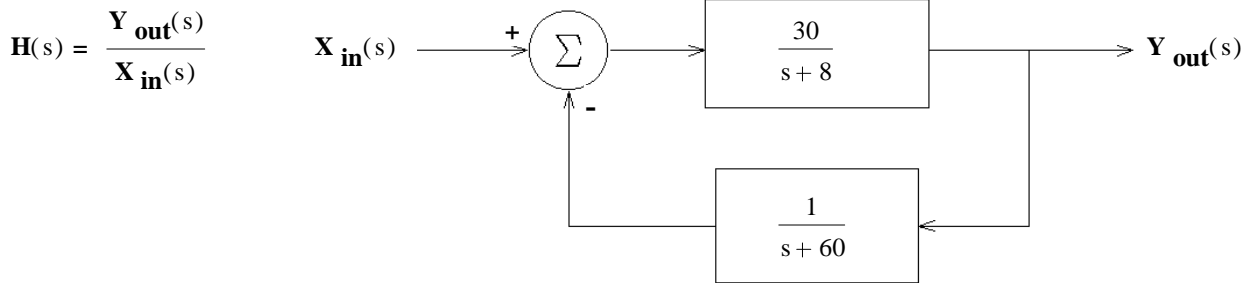
$$\left(A + B \cdot t + C \cdot t^2 + \sqrt{D^2 + E^2} \cdot e^{at} \cdot \cos(bt + \theta) + \sqrt{F^2 + G^2} \cdot t \cdot e^{at} \cdot \cos(bt + \phi)\right) \cdot u(t)$$

Can't be 0: C & $\sqrt{F^2 + G^2}$

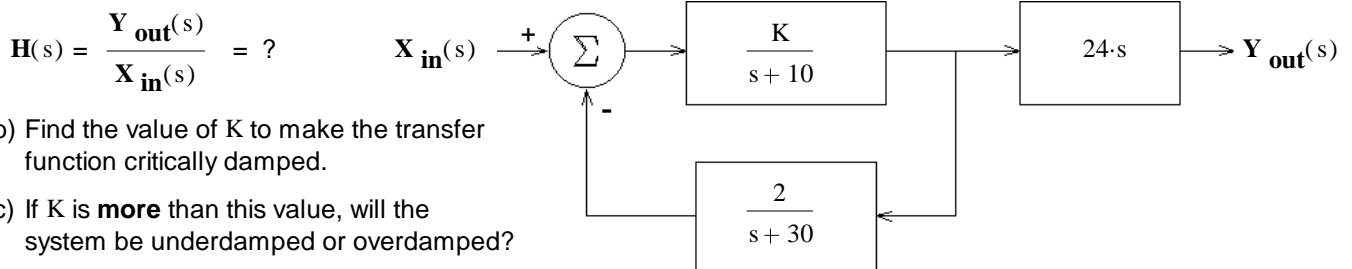
| 4. | <u>Bounded</u> | <u>Converges</u> | <u>f(∞)</u> |
|----|----------------|------------------|----------------|
| a) | yes | yes | 0 |
| b) | yes | yes | $-\frac{1}{2}$ |
| c) | no | | |
| d) | yes | yes | 5 |
| e) | yes | no | |
| f) | no | | |

Properly simplify all your expressions for $H(s)$. By this I mean that the numerator and denominator should both be simple polynomials or factored polynomials. There should be no $1/s^n$ terms in either the numerator or denominator. Also, there should be no coefficient on the highest-order term in the denominator

1. For the feedback system shown below, find the transfer function of the whole system, with feedback.

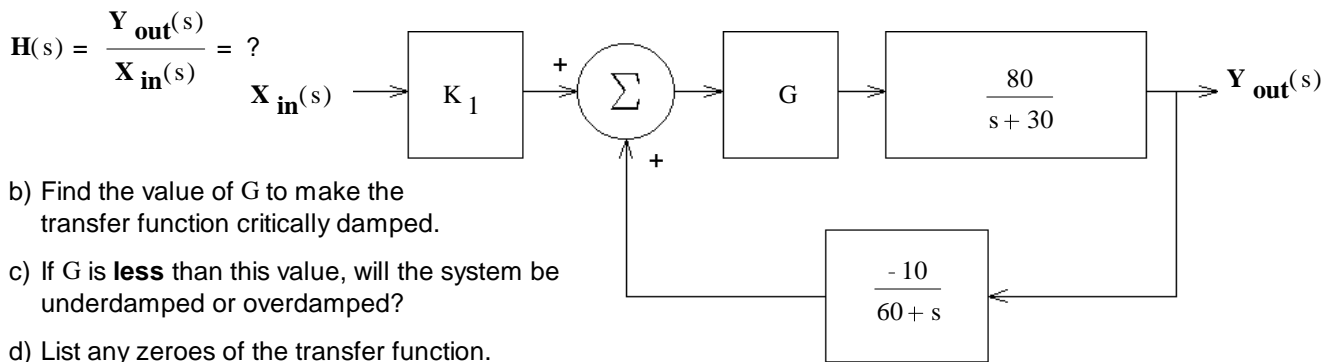


2. a) For the feedback system shown below, find the transfer function of the whole system, with feedback.



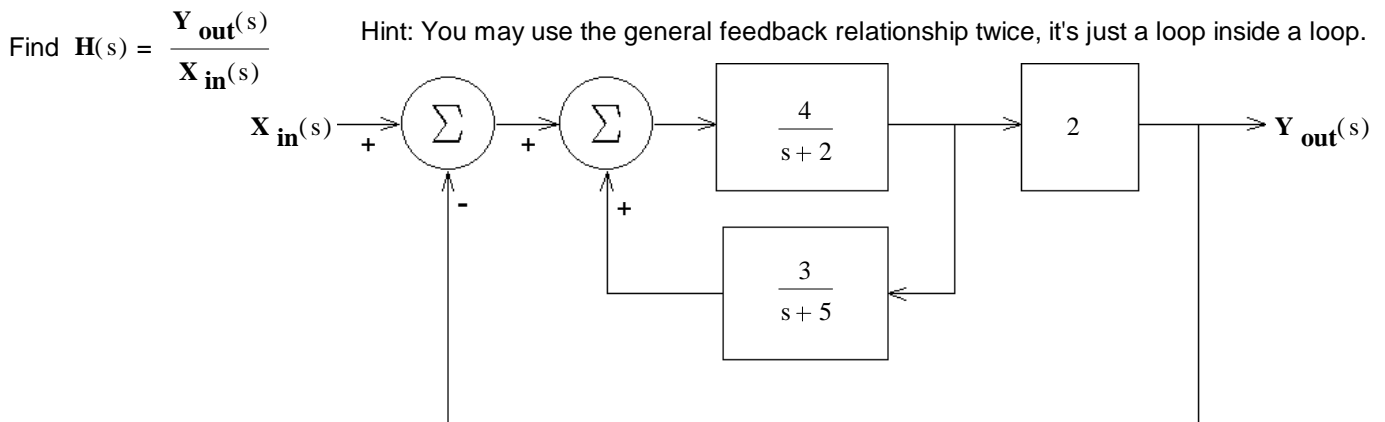
- b) Find the value of K to make the transfer function critically damped.
- c) If K is **more** than this value, will the system be underdamped or overdamped?
- d) List any zeroes of the transfer function.

3. a) For the feedback system shown below, find the transfer function of the whole system, with feedback.

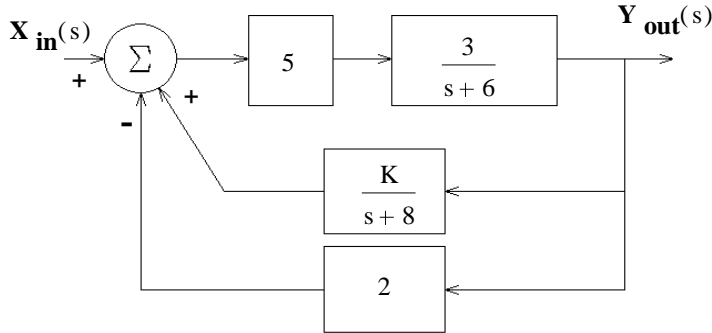


- b) Find the value of G to make the transfer function critically damped.
- c) If G is **less** than this value, will the system be underdamped or overdamped?
- d) List any zeroes of the transfer function.

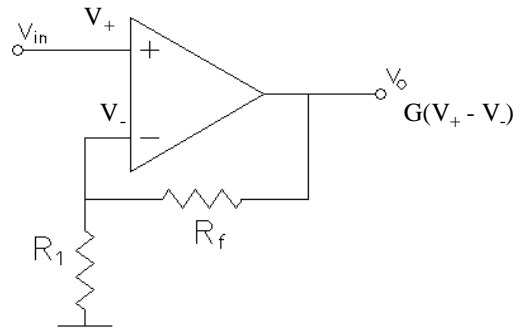
4. For the feedback system shown below, find the transfer function of the whole system, with feedback.



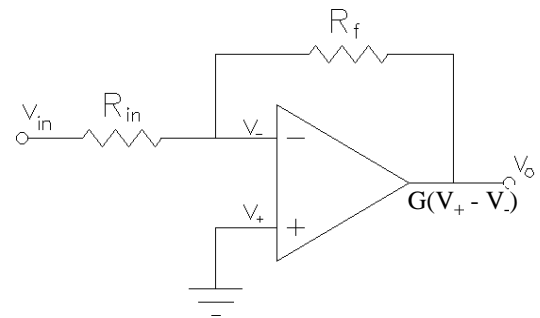
5. Redraw the feedback system below so that it is just one simple loop.



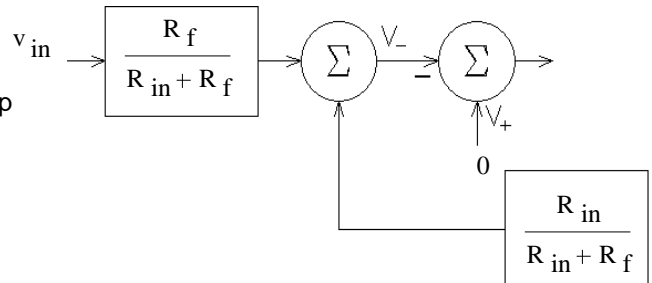
6. a) Draw a standard feedback loop for the noninverting op amp amplifier. Assume no current flows into the op-amp inputs.
 b) Use the standard feedback loop expression to find the transfer function for this amplifier.
 c) Show that this expression simplifies to the standard gain expression for this amplifier if G is very large.



7. a) Draw a standard feedback loop for the inverting op amp amplifier. There also will be an extra block before the loop. This amplifier is trickier than the noninverting amp, so I've done part of the loop for you. The first block determines v_in's contribution to V- (by superposition). The bottom block determines v_o's contribution to V- (by superposition). You will have to combine the summation circles together into one and complete the loop. Assume no current flows into the op-amp inputs.



- b) Combine the leading block with the standard feedback loop expression to find the transfer function for this amplifier.



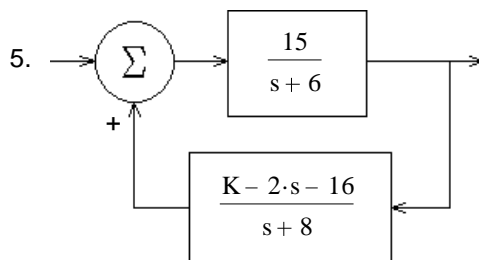
- c) Show that this expression simplifies to the standard gain expression for this amplifier if G is very large.

Answers

1. a) $\frac{30 \cdot s + 1800}{s^2 + 68 \cdot s + 510}$ 2. a) $\frac{K \cdot 24 \cdot s \cdot (s + 30)}{s^2 + 40 \cdot s + 300 + 2 \cdot K}$ b) 50 c) underdamped d) 0, -30

3. a) $K_1 \cdot \frac{G \cdot 80 \cdot s + G \cdot 4800}{s^2 + 90 \cdot s + 800 \cdot G + 1800}$ b) 0.28125 c) overdamped d) -60

4. $\frac{8 \cdot s + 40}{s^2 + 15 \cdot s + 38}$



6. b) $\frac{G \cdot (R_1 + R_f)}{R_1 + R_f + R_1 \cdot G}$ c) $1 + \frac{R_f}{R_1}$

7. b) $-\frac{R_f \cdot G}{R_1 + R_f + R_1 \cdot G}$ c) $-\frac{R_f}{R_1}$