ECE2280 Fundamentals of Electrical Engineering

Sine wave:
Time domain: \[ v(t) \]
Frequency domain:

amplitude \( \approx 10 \) V
\[ V_{\text{RMS}} = \frac{10 \text{ V}}{\sqrt{2}} \]
\[ f = 200 \text{ Hz} \]
\[ T = 5 \text{ ms} \]
\[ f = \frac{1}{T} \]
\[ \omega = 2\pi f \]

Example
When analyzing a time dependent element (capacitors), translate into frequency domain \( \Rightarrow C = \frac{1}{j\omega C} = \frac{1}{sC} \) and then analyze the circuit using normal circuit analysis techniques. Analyze the circuit to the right to find the transfer function \( \frac{V_o}{V_i} \). Solve the circuit symbolically first (with \( R_1, R_2, R_3, C \)) and then plug in their values.

Get \( V_{th} \) and \( R_{th} \):

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What does this equation mean? By substituting $s = j\omega$ in the above equation. The magnitude of the equation is:

$$\frac{133\pi(\omega)}{266.7\pi(\omega) + 1} = \frac{|133\pi(\omega)|}{|266.7\pi(\omega) + 1|} = \frac{133\pi(\omega)}{(266.7\pi(\omega))^2 + 1^2}$$

This magnitude can now be graphed with the x-axis as $\omega$ and the y-axis as the calculated value. This is one of the graphs used for the Bode plots. To plot this, an understanding of dB is needed.

**Decibels**

Your ears respond to sound logarithmically, both in frequency and in intensity. Musical octaves are in ratios of two. “A” in the middle octave is 94.6 Hz, in the next, 440 Hz, then 880 Hz, etc...

It takes about ten times as much power for you to sense one sound as twice as loud as another.

10x power = 2x loudness

A bel is such a 10x ratio of power. Power ratio expressed in bels = \( \log \frac{P_2}{P_1} \) bels The bel is named for Alexander Graham Bell.

It is a logarithmic expression of a voltage ratio (like gain).

The bel unit is never actually used. Instead, we use the decibel (dB, 1/10 of a bel).

Power ratio expressed in dB = \( 10 \log \left( \frac{P_2}{P_1} \right) \) dB

dB are also used to express voltage and current ratios, which related to power when squared. \( P = \frac{V^2}{R} \)

Voltage ratio expressed in dB = \( 10 \log \left( \frac{V_2}{V_1} \right) \) dB

Current ratio expressed in dB = \( 20 \log \left( \frac{I_2}{I_1} \right) \) dB

These are the most common formulas used for dB

Some common ratios expressed as dB:

$$\begin{align*}
20 \log \left( \frac{1}{\sqrt{2}} \right) &= -3.01 \text{ dB} \\
20 \log \left( \frac{1}{2} \right) &= -6.02 \text{ dB} \\
20 \log \left( \frac{1}{10} \right) &= -20 \text{ dB} \\
20 \log \left( \frac{1}{100} \right) &= -40 \text{ dB}
\end{align*}$$

We will use dB fairly commonly in this class, especially when talking about frequency response curves.

**Example Bode**

The frequency domain expression for the output over the input of a circuit is solved to be:

$$\frac{10^5(s+5)}{(s+1)(s+5000)}$$

Substitute $s = j\omega$ into the above equation and calculate the magnitude (dB) and phase (degrees). Plug in values for $\omega$ equal to $10^{-2}$, $0.1$, $0.9$, $10^{-1}$, $2$, $3$, $4$, $5$, $6$, $7$, $10^1$, $10^2$, $10^3$, $3000$, $4000$, $5000$, $6000$, $7000$, $10^5$, $10^6$ rad/sec and plot these values on a semilog graph for both magnitude and phase. Recall that magnitude, $|a+bi| = \sqrt{a^2 + b^2}$ and the phase, \( \angle(a+bi) = \tan^{-1} \left( \frac{b}{a} \right) \)

Magnitude:

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<table>
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<tr>
<th>ω (rad/sec)</th>
<th>Mag(dB)</th>
<th>Phase(Degrees)</th>
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<td>-4.56598</td>
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<tr>
<td>0.8</td>
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Bode Plots
- 2 plots – both have logarithm of frequency on x-axis
  - y-axis magnitude of transfer function, $H(s)$, in dB
  - y-axis phase angle, in degrees

The plot can be used to interpret how the input affects the output in both magnitude and phase over frequency. To sketch the graphs, the circuit is first analyzed to find output/input (transfer function). This equation is used as the basis for the plots. The equation is analyzed for magnitude and phase as shown in the previous example.

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Frequency response

The “response” of a system or circuit is the output for a given input.

A “transfer function” is a mathematical description of how the output is related to the input.

output = transfer function x input
or...

Transfer function = \frac{output}{input}

No real system or circuit treats all frequencies the same, so all real transfer functions are functions of frequency.

Transfer function = H(e^j\omega) or H(f) or Transfer function = H(s)

The transfer function can be used to describe the “frequency response” of a circuit. That is, how does the circuit respond to inputs of different frequencies.

Summary

1. Be able to analyze a circuit with independent and dependent sources to find all currents and voltages within the circuit.
2. Find the Thevenin Circuit for circuits containing independent and dependent sources.
3. Understand signals: peak value, RMS value, peak to peak value, frequency, AC vs DC, describe waveforms mathematically.
4. Analyze circuits containing capacitors to obtain the transfer function.
5. Convert between dB and V/V.
6. Apply complex math to a transfer function and plot the Bode plots.

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MAGNITUDE PLOT:
1) Determine the Transfer Function of the system:
\[ H(s) = \frac{K(s + z_1)(s + z_2)\cdots}{(s + p_1)(s + p_2)\cdots} \]

2) Rewrite it by factoring both the numerator and denominator into the standard form
\[ H(s) = \frac{Kz_1z_2\cdots(s/z_1 + 1)(s/z_2 + 1)\cdots}{p_1p_2\cdots(s/p_1 + 1)(s/p_2 + 1)\cdots} \]

where the \( z \) s are called zeros and the \( p \) s are called poles.

3) Replace \( s \) with \( j\omega \). Then find the Magnitude of the Transfer Function.
\[ H(j\omega) = \frac{Kz_1z_2\cdots(j\omega/z_1 + 1)(j\omega/z_2 + 1)\cdots}{p_1p_2\cdots(j\omega/p_1 + 1)(j\omega/p_2 + 1)\cdots} \]

If we take the \( \log_{10} \) of this magnitude and multiply it by 20 it takes on the form of
\[ 20 \log_{10}(H(j\omega)) = 20\log_{10}\left(\frac{Kz_1z_2\cdots(j\omega/z_1 + 1)(j\omega/z_2 + 1)\cdots}{p_1p_2\cdots(j\omega/p_1 + 1)(j\omega/p_2 + 1)\cdots}\right) = \]

\[ 20\log_{10}|K| + 20\log_{10}|z_1| + 20\log_{10}|z_2| + \cdots + 20\log_{10}\left|\frac{j\omega}{z_1} + 1\right| + 20\log_{10}\left|\frac{j\omega}{z_2} + 1\right| + \cdots - 20\log_{10}|p_1| - \cdots - 20\log_{10}\left|\frac{j\omega}{p_1} + 1\right| - 20\log_{10}\left|\frac{j\omega}{p_2} + 1\right| - \cdots \]

Recall \( \Rightarrow \log(ab) = \log(a)+\log(b) \) and \( \log(a/b) = \log(a) - \log(b) \)
You can see from this expression that each term contributes a number to the final value at a specific frequency. Therefore, each of these individual terms is very easy to show on a logarithmic plot. The entire Bode log magnitude plot is the result of the superposition of all the straight line terms. This means with a little practice, we can quickly sketch the effect of each term and quickly find the overall effect. To do this we have to understand the effect of the four different types of terms.

These include: 1) Constant terms \( K \)
2) Poles and Zeros at the origin \( |j\omega| \)
3) Poles and Zeros not at the origin \( \left| 1 + \frac{j\omega}{p_1} \right| \) or \( \left| 1 + \frac{j\omega}{z_1} \right| \)
4) Complex Poles and Zeros (not addressed at this time)

Effect of Constant Terms:
Constant terms such as \( K \) contribute a straight horizontal line of magnitude \( 20 \log_{10}(K) \) (not dependent on frequency)

\[ 20 \log_{10}(TF) \]

\[ 20 \log_{10}(K) \]

\[ TF = K \]

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Effect of Individual Zeros and Poles at the origin:
A zero at the origin occurs when there is an \( s \) or \( j\omega \) multiplying the numerator. Each occurrence of this causes a positively sloped line passing through \( \omega = 1 \) with a rise of 20 db over a decade.

\[
TF = |J\omega| \quad \frac{20 \log(TF)}{db} \\
\text{If } \omega = 0.1 \Rightarrow |J(0.1)| = 0.1V/V = 20\log(0.1) = -20 \db \\
\text{If } \omega = 1 \Rightarrow |J(1)| = 1V/V = 20\log(1) = 0 \db \\
\text{If } \omega = 10 \Rightarrow |J(10)| = 10V/V = 20\log(10) = 20 \db
\]

A pole at the origin occurs when there are \( s \) or \( j\omega \) multiplying the denominator. Each occurrence of this causes a negatively sloped line passing through \( \omega = 1 \) with a drop of 20 db over a decade.

\[
TF = \frac{1}{|J\omega|} \\
\text{If } \omega = 0.1 \Rightarrow \frac{1}{|J(0.1)|} = 10V/V = 20\log(10) = 20 \db \\
\text{If } \omega = 1 \Rightarrow \frac{1}{|J(1)|} = 1V/V = 20\log(1) = 0 \db \\
\text{If } \omega = 10 \Rightarrow \frac{1}{|J(10)|} = 0.1V/V = 20\log(0.1) = -20 \db
\]

Effect of Individual Zeros and Poles Not at the Origin:
Zeros and Poles not at the origin are indicated by the \( 1+J\omega/z \) and \( 1+J\omega/p \). The values \( z_i \) and \( p_i \) in each of these expressions is called a break frequency. Below their break frequency these terms do not contribute to the log magnitude of the overall plot. Above the break frequency, they represent a ramp function of 20 db per decade. Zeros give a positive slope. Poles produce a negative slope.

\[
\begin{align*}
TF &= \left| \frac{1 + J\omega/z_i}{1 + J\omega/p_i} \right| \\
\text{If } \omega = 0.1 \Rightarrow \left| \frac{1 + J(0.1)/z_i}{1 + J(0.1)/p_i} \right| = 1/V/V = 20\log(1) = 0 \db \\
\text{If } \omega = 1 \Rightarrow \left| \frac{1 + J(1)/z_i}{1 + J(1)/p_i} \right| = 10V/V = 20\log(10) = 20 \db
\end{align*}
\]

Before looking at the effect of the \( 2^{nd} \) order terms, let's learn how to plot with the three terms already described. We will work several examples where we show how the Bode log magnitude plot is sketched. To complete the log magnitude vs. frequency plot of a Bode diagram, superposition all the lines of the different terms on the same plot.

**Phase Plot:**
For our original transfer function,

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\[
H(jw) = \frac{K z_1 z_2 \cdots (\frac{jw}{z_1} + 1)(\frac{jw}{z_2} + 1) \cdots}{p_1 p_2 \cdots (\frac{jw}{p_1} + 1)(\frac{jw}{p_2} + 1) \cdots}
\]

the cumulative phase angle associated with this function are given by

\[
\angle H(jw) = \angle \frac{K z_1 z_2 \cdots (\frac{jw}{z_1} + 1)(\frac{jw}{z_2} + 1) \cdots}{p_1 p_2 \cdots (\frac{jw}{p_1} + 1)(\frac{jw}{p_2} + 1) \cdots}
\]

Then the cumulative phase angle as a function of the input frequency may be written as

\[
\angle H(jw) = \angle \left[ K + z_1 + z_2 + \cdots - p_1 - p_2 - \cdots \right] + \tan^{-1}\left(\frac{w}{z_1}\right) + \tan^{-1}\left(\frac{w}{z_2}\right) + \cdots - \tan^{-1}\left(\frac{w}{p_1}\right) - \tan^{-1}\left(\frac{w}{p_2}\right) - \cdots
\]

Once again, to show the phase plot of the Bode diagram, lines can be drawn for each of the different terms. Then the total effect may be found by superposition.

**Effect of Constants on Phase:**
A positive constant, \(K>0\), has no effect on phase. A negative constant, \(K<0\), will set up a phase shift of \(\pm 180^\circ\).

**Effect of Zeros at the origin on Phase Angle:**
Zeros at the origin, \(s\), cause a constant +90 degree shift for each zero.

\[
\angle TF \quad \text{+90 deg} \quad \angle jw = +90
\]

\[
\omega (\log)
\]

**Effect of Poles at the origin on Phase Angle:**
Poles at the origin, \(s^{-1}\), cause a constant -90 degree shift for each pole.

\[
\angle TF \quad \text{-90 deg}
\]

\[
\frac{1}{jw} = -90
\]

**Effect of Zeros not at the origin on Phase Angle:**
Zeros not at the origin, like \(1 + \frac{jw}{z_1}\), have no phase shift for frequencies much lower than \(z_1\), have a +45 deg shift at \(z_1\), and have a +90 deg shift for frequencies much higher than \(z_1\).

\[
\angle TF \quad \text{+45 deg} \quad \text{+90 deg} \quad \text{If } z_1 = 1 \Rightarrow \angle (1 + jw) = \tan^{-1}\left(\frac{w}{z_1}\right)
\]

\[
\text{If } \omega = 0.1 \Rightarrow \tan^{-1}(0.1) = \text{If } \omega = 1 \Rightarrow \tan^{-1}(1) = \text{If } \omega = 10 \Rightarrow \tan^{-1}(10) = \text{If } \omega = 100 \Rightarrow \tan^{-1}(100) =
\]

To draw the lines for this type of term, the transition from 0° to +90° is drawn over 2 decades, starting at 0.1\(z_1\) and ending at 10\(z_1\).

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Effect of Poles not at the origin on Phase Angle:

Poles not at the origin, like \( \frac{1}{1 + \frac{j\omega}{p_1}} \), have no phase shift for frequencies much lower than \( p_1 \), have a -45 deg shift at \( p_1 \), and have a -90 deg shift for frequencies much higher than \( p_1 \).

To draw the lines for this type of term, the transition from 0\(^\circ\) to -90\(^\circ\) is drawn over 2 decades, starting at 0.1\( p_1 \) and ending at 10\( p_1 \).

When drawing the phase angle shift for not-at-the-origin zeros and poles, first locate the break frequency of the zero or pole. Then start the transition 1 decade before, following a slope of ±45\(^\circ\)/decade. Continue the transition until reaching the frequency one decade past the break frequency.

**SUMMARY OF STRAIGHT-LINE APPROXIMATION PROCEDURE STEPS (NO COMPLEX):**

(Note that a decade is a multiple of 10 - 1, 10, 100, 1000, etc)

1. Rearrange the equation into standard form:

\[
H(s) = \frac{K_z z_2 \cdots (s/z_1 + 1)(s/z_2 + 1) \cdots}{p_1 p_2 \cdots (s/p_1 + 1)(s/p_2 + 1) \cdots}
\]

where \( K, z_1, z_2, \) etc are all constant values.

2. Determine the poles and zeros.

   *Note: If there are more than one poles/zeros at the same break frequency (say there are \( r \)), just multiply the slope/phase changes by \( r \). (ex. \( 1+s/10 \) \(^2\) => it is a negative zero (numerator) and so it will change the slope by 2*20dB/dec and have a 2*45\(^\circ\) slope/dec.)*

3. Draw the magnitude plot:
   a. Determine starting value:

   **Case 1:** No pole or zero at the origin:

   \[
   \text{starting value} = 20 \log_{10} \left( \frac{K_z z_2 \cdots}{p_1 p_2 \cdots} \right)
   \]

   **Case 2:** A pole or zero at the origin:

   - Pick a frequency value less than the lowest pole or zero value.

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* Plug in the frequency in the standard form equation above and take the magnitude. This value is for that frequency only. There is a constant slope going through this point.
  +20dB/dec slope if the location is a zero. -20dB/dec slope if the location is a pole.

b. Begin at the starting point. Start with the slope (0 slope if a constant, +20dB/dec slope if zero at origin, -20dB/dec slope if pole at origin). From left to right, at each zero add +20dB/dec to the current slope and at each pole -20dB/dec. Continue through each frequency.

4. Draw the phase plot:
   a. Determine the starting value:

   **Case 1: No pole or zero at the origin:**
   
   If constant > 0 then starting value = 0°
   If constant < 0 then starting value = ±180°

   **Case 2: A pole or zero at the origin:**

   starting value = +90° if zero at origin
   starting value = -90° if pole at origin

   b. Label each range of frequency according to the following (suggest putting on graph):

   zero => from 1 decade before frequency to 1 decade after frequency: 45°slope/dec
   pole => from 1 decade before frequency to 1 decade after frequency: -45°slope/dec

   (eg if ω=10 and is a pole then range is 1<ω<100 with a slope of -45°slope/dec)

c. Look at each frequency range that has a slope. Add all slopes within that region. From left to right: start with starting value and slope of 0, continue until first region of change. Add all slopes within that region. Continue until the end is met. If no slope during a region the slope is constant (0).

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Example:

$$H(s) = \frac{(s+100)(s+1K)}{(s+10)(s+10K)}$$
Example

\[ H(s) = \frac{100(s+100)(s+10)}{s^2(s+10k)} \]