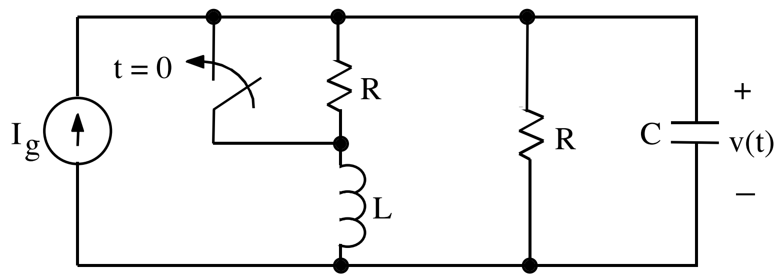
2. (45 points)



$I_g = 1A$

$R = 4\Omega$

$L = 0.1$ mH

$C = 1\mu F$

The current source is a dc current source. After being open for a long time, the switch is closed at t = 0.

a. Write a numerical time-domain expression for v(t).

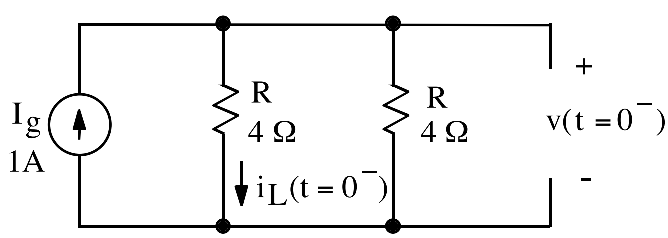
b. From the Laplace transform of v(t), find the numerical values of v(t) for $t = 0^+$ and $t \to \infty$.

ans: a) $v(t > 0) = 2.003\ e^{-0.4kt} - 0.003\ e^{-249.6kt}$ V

b) $v(t = 0^+) = 2V$, $v(t \to \infty) = 0V$

sol'n: (a) First we find initial conditions for L and C. (We need these for s-domain models of L and C.)

For $t = 0^-$, L acts like short, C acts like open circuit.

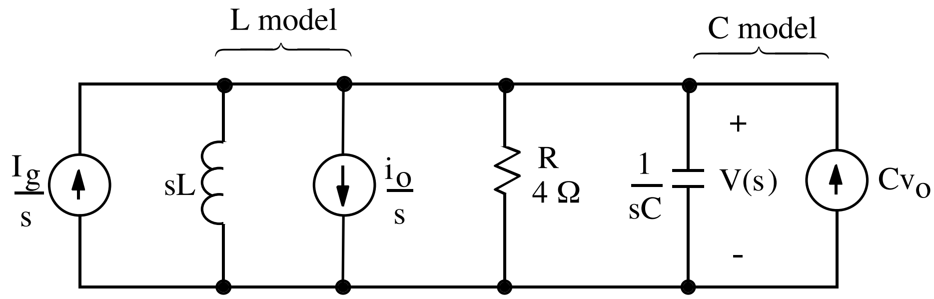


$$v(t = 0^-) = I_g \cdot R \| R = 1\text{A} \cdot 2\Omega = 2\text{V} \equiv v_o$$

$$i_L(t = 0^-) = I_g \cdot \frac{R}{R + R} = \frac{1}{2}\text{A} \equiv i_o$$

When we close the switch, we short out the first R.

s-domain model:



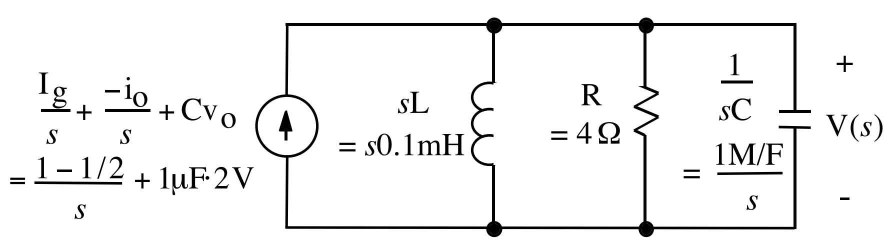
Note: A DC source corresponds to a step function <u>even if there is no switch and the source has the same output for all time.</u> Thus, we have I_g/s as the source in the s-domain. (Conceptually, we only need the current source for $t > 0$ because the initial conditions on L and C account for what the current source did for $t < 0$.)

$$\mathcal{L}\{I_g\} = \mathcal{L}\{I_g u(t)\} = \frac{1}{s}$$

Note: We may choose either a series sL and V-source for L or a parallel sL and I-source for L. Here, the parallel I-source model is more convenient. The same applies to the C.

Normally, we might use superposition at this point, turning on the I-sources one at a time and then summing currents or voltages to get a final answer.

Here, however, we have parallel I-sources that sum:



Combining the parallel impedances and using **V** = **I**z, we have

$$V(s) = \left(\frac{1}{2s} + 2\mu\text{FV}\right) \cdot sL \parallel R \parallel \frac{1}{sC}$$

To compute the parallel z value, we factor out $1/s$C to remove fractions:

$$sL \parallel R \parallel \frac{1}{sC} = sL \parallel \frac{1}{sC} \parallel R = \frac{1}{sC} \cdot s^2 LC \parallel 1 \parallel sRC$$

Now multiply through by sC:

$$sL \,||\, R \,||\, \frac{1}{sC} = \frac{1}{sC} \cdot \frac{s^2 LC}{s^2 LC + 1} \,||\, sRC = \frac{sL}{s^2 LC + 1} \,||\, R$$

Factor out the denominator again:

$$sL \,||\, R \,||\, \frac{1}{sC} = \frac{1}{s^2 LC + 1} \cdot sL \,||\, R\left(s^2 LC + 1\right)$$

$$sL \,||\, R \,||\, \frac{1}{sC} = \frac{1}{s^2 LC + 1} \cdot \frac{sLR\left(s^2 LC + 1\right)}{sL + R\left(s^2 LC + 1\right)}$$

Multiply through by s^2LC + 1:

$$sL \,||\, R \,||\, \frac{1}{sC} = \frac{sLR}{sL + R\left(s^2 LC + 1\right)}$$

Now divide top and bottom by RLC to make the coefficient of the highest power of *s* in the denominator equal to unity.

$$sL \,||\, R \,||\, \frac{1}{sC} = \frac{s/C}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

Check: Using the numerator and the first term in the denominator, we have the following units analysis:

$$\frac{s/C}{s^2} = \frac{1}{sC}$$

Thus, we have an impedance as we should have. The other terms in the denominator have the same units as s^2 since the units of *s* are, ironically, 1/sec or 1/s.

Now we plug in numbers to compute V(*s*):

$$V(s) = \underbrace{\left(\frac{1}{2s} + 2\mu\right)}_{= \frac{1 + s4\mu}{2\cancel{s}}} \left(\frac{\overset{1/C}{1\text{M}\cancel{s}}}{\underbrace{s^2 + \underset{1/RC}{\frac{1\text{M}}{4}} s + \underset{1/LC}{10\text{G}}}_{\text{quadratic poles term}}}\right)$$

Find poles for quadratic term in preparation for partial fractions:

$$s_{1,2} = -\frac{1M}{8} \pm \sqrt{\left(\frac{1M}{8}\right)^2 - 10G}$$ <u>not</u> complex poles

$$\parallel$$

$$\sqrt{(125k)^2 - (100k)^2}$$

$s_{1,2}$ = -125k ± 75k rad/s (based on $5^2 - 4^2 = 3^2$ pythagorean triple)

s_1 = –50k, s_2 = –200 rad/s

Now use partial fractions:

$$V(s) = \frac{k_1}{s + 50k} + \frac{k_2}{s + 200k}$$

$$k_1 = V(s)(s + 50k)\Big|_{s=-50k} = \frac{1 - 50k \cdot 4\mu}{2} \cdot \frac{1M}{-50k + 200k}$$

$$= \frac{1 - 200m}{2} \frac{1M}{150k}$$

$$= \frac{800\cancel{m}}{2} \frac{1\cancel{M}}{150\cancel{k}} = \frac{8}{3}$$

$$k_2 = V(s)(s + 50k)\Big|_{s=-200k} = \frac{1 - 200k4\mu}{2} \frac{1M}{-200k + 50k}$$

$$= \frac{\cancel{m} \cdot 1\cancel{M}}{-2(150\cancel{k})} = -\frac{2}{3}$$

$$V(s) = \frac{8/3}{s + 50k} - \frac{2/3}{s + 200k}$$

Use the standard inverse Laplace transform term:

$$\mathcal{L}^{-1}\left\{\frac{k}{s + a}\right\} = ke^{-at}$$

This gives the final answer:

v(t > 0) = 2.003 $e^{-0.4kt}$ - 0.003 $e^{-249.6kt}$ V

sol'n: (b) Use the initial value theorem to find v(t=0$^+$):

$$v(t = 0^+) = \lim_{s\to\infty} sV(s) = \lim_{s\to\infty} s \frac{1 + s4\mu}{2} \frac{1M}{s^2 + \frac{1M}{4}s + 10G}$$

The largest power of s dominates in the numerator and in the denominator.

$$v(t = 0^+) = \lim_{s\to\infty} sV(s) = \lim_{s\to\infty} \frac{s^2 4\mu 1M}{2s^2} = \frac{4}{2} = 2\text{V} \ \surd$$

Note: We expect $v(0^+) = 2V$ since this is the initial capacitor voltage.

Use the final value theorem to find $v(t \rightarrow \infty)$:

$$v(t \rightarrow \infty) = \lim_{s \rightarrow 0} sV(s) = \lim_{s \rightarrow 0} s \frac{1 + s4\mu}{2} \frac{1M}{s^2 + \frac{1M}{4}s + 10G}$$

$$= 0 \cdot \frac{1}{2} \cdot \frac{1M}{10G} = 0V \quad \checkmark$$

Note: We expect v(t) to decay, since L becomes a short.