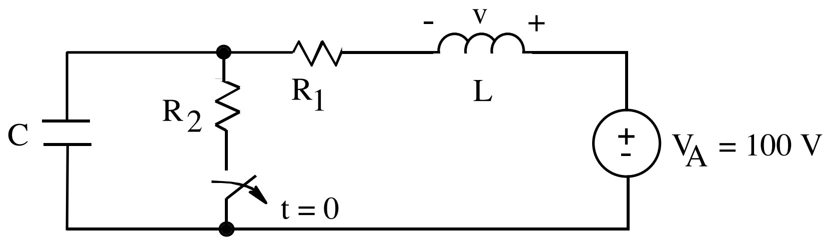
PRACTICE FINAL EXAM SOLUTION Prob 1

1. (50 points)



After having been open for a long time, the switch is closed at t = 0.

$R_1 = 12.5\Omega$ $R_2 = 12.5\Omega$ $L = 6.25\ \mu H$

a. Two capacitances are available: 250 nF and 2 nF. Specify the value of C that will make v(t) overdamped.

b. Using the value of C found in (a), write a time-domain expression for v(t).

ans: a) C = 250 nF

b) $v(t) = 13.3\,(e^{-0.4Mt} - e^{-1.6Mt})$ V

sol'n: (a) To make the response overdamped, we must have two real characteristic roots. We use the circuit for t > 0, consisting of C, R_1, L, and v_A in series. We may find the characteristic equation by looking it up in a textbook or by setting the impedance of R_1, C, and L in series to zero.

$$z = R_1 + \frac{1}{sC} + sL = s^2 + \frac{R_1}{L}s + \frac{1}{LC} = 0$$

The characteristic roots for the quadratic equation are

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

or

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2} \qquad \alpha \equiv \frac{R}{2L} \qquad \omega_o \equiv \frac{1}{\sqrt{LC}}$$

We want an overdamped response, (real roots $\alpha^2 > \omega_o^2$).

$$\alpha = \frac{R}{2L} = \frac{12.5\Omega}{(2)\,6.25\ \mu H} = \frac{12.5}{12.5}\text{M rad/s} = 1\text{ M rad/s}$$

Try each C value in turn.

C = 2 nF:

$$\omega_o = \frac{1}{\sqrt{6.25\ \mu H \cdot 2\ nF}} = \frac{1}{\sqrt{12.5\ m \cdot 1\mu}} = \frac{1M}{1.1118}\ \text{rad/s}$$

$\omega_o = 8.9\text{M rad/s} > \alpha^2$ (underdamped)

C = 250 nF:

$$\omega_o = \frac{1}{\sqrt{6.25\ \mu H \cdot 250\ nF}} = \frac{1}{\sqrt{1562.5\ m \cdot 1\mu}} = \frac{1M}{1.25}\ \text{rad/s}$$

$\omega_o = 0.8\text{M rad/s} < \alpha^2$ (overdamped)

We need C = 250 nF for an overdamped solution.

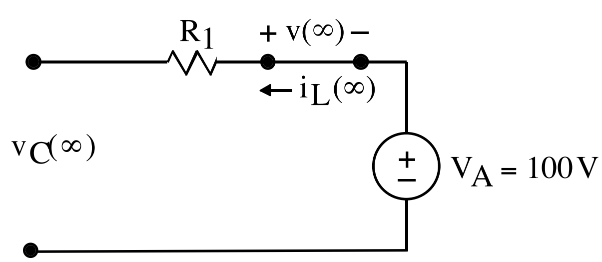
sol'n: (b) We use the exponential solution for the overdamped case:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3$$

Because the value of A_3 is all that is left of v(t) as $t \to \infty$, we first find the constant term, A_3. (The other terms decay because the characteristic roots always have negative real parts in a passive RLC circuit. When the switch opens, the energy sloshing back and forth in the L and C will decay owing to power dissipated by the series resistor R_1.)

As $t \to \infty$, the circuit reaches equilibrium. C acts like an open circuit, L acts like a short circuit or wire.

Model:



$i_L(\infty) = 0\,A$

$v_C(\infty) = 100V$

Since L acts like a wire, there is no voltage drop across it.

Thus, $A_3 = v(t\to\infty) = 0$.

We find coefficients A_1 and A_2 by matching initial conditions in the circuit. We find initial conditions by examining the circuit at $t = 0^-$, when the circuit has reached equilibrium. We find the values of i_L and v_C, the energy variables, at $t = 0^-$ and use the same values at $t = 0^+$ (since the energy in the circuit cannot change instantly).

Mathematically, our general form of solution for the overdamped case gives the following values for $v(0^+)$ and $dv(t)/dt|_{t=0+}$:

$$v(0^+) = A_1 + A_2 + A_3 = A_1 + A_2$$

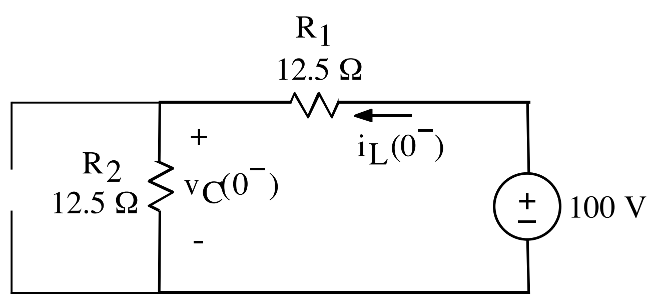
$$\left.\frac{dv(t)}{dt}\right|_{t=0^+} = A_1 s_1 + A_2 s_2.$$

Note: We must always differentiate first and then plug in $t = 0^+$. Otherwise, we always get zero.

Now we find the numerical values of $v(0^+)$ and $dv(t)/dt|_{t=0+}$.

At $t = 0^-$, C acts like an open circuit and L acts like a short circuit.

Model:



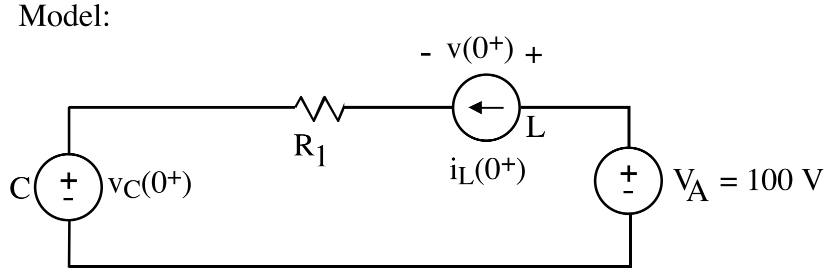
$$i_L(0^-) = \frac{100\text{V}}{25\Omega} = 4\,\text{A}$$

$$v_C(0^-) = 100\text{V} \cdot \frac{12.5\Omega}{25\Omega} = 50\text{V}$$

At time $t = 0^+$, we have $i_L(0^+) = i_L(0^-) = 4$ A and $v_C(0^+) = v_C(0^-) = 50$ V. We solve the circuit at $t = 0^+$, treating $i_L(0^+)$ as a current source and $v_C(0^+)$ as a voltage source.

We now solve for $v(0^+)$ and $dv(t)/dt|_{t=0+}$. From these we find A_1 and A_2.

Model:



We may apply any standard method to solve the circuit, but we can solve the above circuit using a voltage loop.

$$v(0^+) = v_A - i_L(0^+)R_1 - v_C(0^+) = 100\text{V} - 4\,\text{A}\cdot 12.5\Omega - 50\text{V} = 0\ \text{V}$$

The same equation applies for t > 0, and we may differentiate to find dv(t)/dt in terms of energy (or state) variables i_L and v_C.

$$v(t) = v_A - i_L(t)R_1 - v_C(t)$$

$$\frac{dv(t)}{dt} = -\frac{di_L(t)}{dt}R_1 - \frac{dv_C(t)}{dt}$$

The basic equations for L and C, rearranged, allow us to translate the derivatives on the right side of this equation into non-derivatives we can calculate numerically.

$$\frac{di_L(t)}{dt} = \frac{1}{L}v_L(t)$$

$$\frac{dv_C(t)}{dt} = \frac{1}{C}i_C(t)$$

Applying these identities, we have

$$\frac{dv(t)}{dt} = -\frac{1}{L}v_L(t)R_1 - \frac{1}{C}i_C(t).$$

Only now that we have differentiated do we finally evaluate the derivative we seek at t = 0:

$$\left.\frac{dv(t)}{dt}\right|_{t=0^+} = -\frac{1}{L}v_L(0^+)R_1 - \frac{1}{C}i_C(0^+).$$

$$\left.\frac{dv(t)}{dt}\right|_{t=0^+} = -\frac{1}{6.25\mu\text{H}}\cdot 0\text{V}\cdot 12.5\Omega - \frac{1}{250\text{nF}}i_C(0^+).$$

Since i_C is in series with i_L, we have $i_C(0^+) = i_L(0^+) = 4\text{A}$.

$$\left.\frac{dv(t)}{dt}\right|_{t=0^+} = -\frac{4A}{250\text{nF}} = -16\ \text{MV/s}$$

Now we find A_1 and A_2.

$$v(0^+) = 0 = A_1 + A_2 \Rightarrow A_2 = -A_1$$

$$\left.\frac{dv(t)}{dt}\right|_{t=0^+} = -16\text{MV/s} = A_1 s_1 + A_2 s_2 = A_1(s_1 - s_2).$$

$$s_1 - s_2 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} - \left(-\alpha - \sqrt{\alpha^2 - \omega_o^2}\right)$$

$$= 2\sqrt{\alpha^2 - \omega_o^2}$$

$$= 2\sqrt{(1\ M)^2 - (0.8\ M)^2}$$

$$= (2)\, 0.6\ M = 1.2\ M$$

Concluding the algebra, we find the numerical values of the coefficients A_1 and A_2.

$$A_1 = \frac{16\text{ M v/s}}{1.2\text{ M}} = 13.3\text{ v/s}$$

$$A_2 = -13.3\text{ v/s}$$

Using the values of α and ω_o from above, we find the values of s_1 and s_2.

$s_1 = -1\text{ M} + 0.6\text{ M} = -0.4\text{ M}$

$s_2 = -1\text{ M} - 0.6\text{ M} = -1.6\text{ M}$

Plugging into the general form of underdamped solution completes our answer:

$v(t) = 13.3\ (e^{-0.4Mt} - e^{-1.6Mt})\text{ V}$