IDENTITY: $$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^n}\right\} = \frac{t^{n-1}}{(n-1)!}e^{-at}$$

PROOF: Use induction.

First, verify the identity for $n = 1$. From straightforward calculations, the following result is known:

$$\mathcal{L}\left\{e^{-at}\right\} = \frac{1}{s+a}$$

For $n = 1$, the identity gives the same result:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^1}\right\} = \frac{t^{1-1}}{(1-1)!}e^{-at} = e^{-at}$$

NOTE: $0! = 1$.

Thus, the identity is valid for $n = 1$.

Now assume the identity is valid for $n > 1$ and show that it holds for $n + 1$.

Thus, we assume the following is true:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^n}\right\} = \frac{t^{n-1}}{(n-1)!}e^{-at}$$

or

$$\mathcal{L}\left\{\frac{t^{n-1}}{(n-1)!}e^{-at}\right\} = \frac{1}{(s+a)^n}$$

Apply the following identity for Laplace transforms:

$$\mathcal{L}\{tf(t)\} = -\frac{dF(s)}{ds} \quad \text{where } F(s) \equiv \mathcal{L}\{f(t)\}$$

This yields the following result:

$$\mathcal{L}\left\{t\frac{t^{n-1}}{(n-1)!}e^{-at}\right\} = -\frac{d}{ds}\frac{1}{(s+a)^n} = -\frac{-n}{(s+a)^{n+1}} = \frac{n}{(s+a)^{n+1}}$$

Since the Laplace transform is linear, we have the following identity:

$$\mathcal{L}\left\{\frac{1}{n}g(t)\right\} = \frac{1}{n}G(s)$$

Applying this to our last result yields an equation that matches the identity we are trying to prove when $n + 1$ is substituted for n:

$$\mathcal{L}\left\{\frac{t}{n}\frac{t^{n-1}}{(n-1)!}e^{-at}\right\} = \frac{1}{(s+a)^{n+1}}$$

or

$$\mathcal{L}\left\{\frac{t^{(n+1)-1}}{((n+1)-1)!}e^{-at}\right\} = \frac{1}{(s+a)^{n+1}}$$

By the axiom of induction, it follows that the identity holds for all $n \geq 1$, and our proof is complete.