HW 5 prob 3d solution

EX: Find the inverse Laplace transform for the following expression:

$$F(s) = \frac{9s^2 + 35s + 49}{(s+2)(s^2 + 2s + 5)}$$

SOL'N: We first find the root terms of the quadratic term in the denominator. Since the square of one-half the middle term equals 1^2, which is less than the constant term, 5, the roots are complex.

We find the roots, $s_{1,2} = -a \pm j\omega$, as follows:

$$s^2 + 2s + 5 = (s+a)^2 + \omega^2 = s^2 + 2as + a^2 + \omega^2$$

The value of a is half the middle coefficient:

$$a = \frac{2}{2} = 1$$

Using this value of a and the constant term of the quadratic, we find the value of ω:

$$\omega = \sqrt{5 - a^2} = \sqrt{5-1} = 2$$

Our partial fraction expansion is as follows:

$$F(s) = \frac{A_1}{(s+2)} + \frac{A_2}{s+1-j2} + \frac{A_2^*}{s+1+j2}$$

We multiply by root terms and evaluate at roots to find the partial fraction coefficients:

$$A_1 = (s+2)F(s)\Big|_{s=-2} = \left.\frac{9s^2 + 35s + 49}{s^2 + 2s + 5}\right|_{s=-2} = \frac{9 \cdot 4 - 35 \cdot 2 + 49}{4 - 2 \cdot 2 + 5}$$

or

$$A_1 = \frac{15}{5} = 3$$

The A_2 coefficient is complex:

$$A_2 = (s+1-j2)F(s)\Big|_{s=-(1-j2)} = \left.\frac{9s^2 + 35s + 49}{(s+2)(s+1+j2)}\right|_{s=-(1-j2)}$$

or

$$A_2 = \frac{9(1-j2)^2 - 35(1-j2) + 49}{[-(1-j2)+2][-(1-j2)+1+j2]} = \frac{9(-3-j4) - 35(1-j2) + 49}{(1+j2)j4}$$

or

$$A_2 = \frac{-13+j34}{-8+j4} = \frac{-13+j34}{-8+j4} \cdot \frac{-8-j4}{-8-j4} = \frac{104+136+j(52-272)}{80}$$

or

$$A_2 = \frac{240-j220}{80} = 3 - j\frac{11}{4}$$

We now have the following expansion:

$$F(s) = \frac{3}{(s+2)} + \frac{3-j\dfrac{11}{2}}{s+1-j2} + \frac{3+j\dfrac{11}{2}}{s+1+j2}$$

For the complex poles we use the following identity:

$$\mathcal{L}^{-1}\left\{\frac{c+jd}{s+a-j\omega} + \frac{c-jd}{s+a+j\omega}\right\} = 2ce^{-at}\cos(\omega t) - 2de^{-at}\sin(\omega t)$$

Our final result is a decay plus a decaying cosine and decaying sine:

$$f(t) = 3e^{-2t} + 6e^{-t}\cos(2t) + \frac{11}{2}e^{-2t}\sin(2t)$$

An alternative approach is to find A_1 and then write the original expansion in terms of the decaying cosine and decaying sine transforms:

$$F(s) = \frac{A_1}{(s+2)} + \frac{K_2(s+a)}{s^2+2s+5} + \frac{K_3 \cdot \omega}{s^2+2s+5}$$

or

$$F(s) = \frac{3}{(s+2)} + \frac{K_2(s+1)}{s^2+2s+5} + \frac{K_3 \cdot 2}{s^2+2s+5}$$

We then put everything over a common denominator and match the numerator to the known numerator of $F(s)$:

$$F(s) = \frac{3(s^2 + 2s + 5) + K_2(s+1)(s+2) + K_3 \cdot 2(s+2)}{(s+2)(s^2 + 2s + 5)} = \frac{9s^2 + 35s + 49}{(s+2)(s^2 + 2s + 5)}$$

By matching the coefficients of the s^2 terms, we have that $K_2 = 6$. Using this value and matching the coefficients of the s terms, we have the following:

$$3(2s) + 6(3s) + K_3(2s) = 35s$$

or

$$K_2 = \frac{11}{2}$$

As a check, we verify that the constant terms match:

$$3(5) + 6(2) + \frac{11}{2}(4) = 15 + 12 + 22 = 49 \quad \surd$$

This approach is clearly more efficient.