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HW 5 prob 2 solution

IDENTITY: $\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$ for $a > 0$

PROOF: By definition, we have the following equation:

$$\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(at)e^{-st}dt$$

We change variables to $\tau = at$.

At $t = 0^-$, $\tau = a0^- = 0^-$.

For t -> ∞, $\tau = a\infty = \infty$.

Inside the integral, $st = s\tau/a$.

For dt we have $d\tau/dt = a$, so $dt = \tau/a$.

Making these substitutions, our identity is verified:

$$\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(\tau)e^{-\frac{s}{a}\tau}\frac{d\tau}{a} = \frac{1}{a}\int_{0^-}^{\infty} f(\tau)e^{-\frac{s}{a}\tau}d\tau = \frac{1}{a}F\left(\frac{s}{a}\right).$$