1. (25 points)

a. Find \( f(t) \) if
\[
F(s) = \frac{s + 2}{(s + 1)^2 (s + 4)}
\]

b. Plot the poles and zeros of \( G(s) \) in the s plane
\[
G(s) = \frac{12 + 4s}{(s + 2)(s^2 + 25)(s^2 + 6s + 25)}
\]

c. Find \( \lim_{t \to 0^+} f(t) \) if
\[
F(s) = \frac{3(s^3 + 7s^2 + 14s + 8)}{s^4 + 14s^3 + 98s^2 + 350s + 625}
\]

d. Find \( \lim_{t \to \infty} f(t) \) if
\[
F(s) = \frac{2s^4 + 6s^3 + 30s^2 + 25s + 120}{s^6 + 14s^5 + 112s^4 + 448s^3 + 975s^2 + 625s}
\]
(All poles of \( F(s) \) are in the left-half plane.)

e. Write an expression for \( H(s) \).
ans: a) \[ f(t) = -\frac{2}{9} e^{-4t} + \frac{1}{3} t e^{-t} + \frac{2}{9} e^{-t} \]

b)

\[
\begin{array}{c}
\text{Im} \\
\text{0} \\
\text{X} \\
\text{j5}
\end{array}
\]

\[
\begin{array}{c}
\text{Re} \\
\text{-3} \\
\text{-2} \\
\text{X} \\
\text{-3 - j4} \\
\text{X - j5}
\end{array}
\]

c) 3

d) \[ \frac{120}{625} = \frac{24}{125} = 0.192 \]
e) \[ H(s) = \frac{3e^{-2s} + 2e^{-5s} - 7e^{-7s} + 2e^{-9s}}{s} \]

sol'n: (a) Use partial fractions.

\[
F(s) = \frac{k_1}{s + 4} + \frac{k_2}{(s + 1)^2} + \frac{k_3}{s + 1}
\]

\[
k_1 = F(s)(s + 4)\bigg|_{s=-4} = \frac{s + 2}{(s + 1)^2} \frac{s + 4}{s + 4} = \frac{-4 + 2}{(-4 + 1)^2} = -\frac{2}{9}
\]

\[
k_2 = F(s)(s + 1)^2\bigg|_{s=-1} = \frac{s + 2}{(s + 1)^2} \frac{(s + 1)^2}{s + 4} = \frac{-1 + 2}{-1 + 4} = \frac{1}{3}
\]

\[
k_3 = \frac{1}{1!} \frac{d}{ds} \left[ F(s)(s + 1)^2 \right]_{s=-1} = \frac{d}{ds} \left[ \frac{s + 2}{s + 4} \right]_{s=-1}
\]

or

\[
k_3 = 1 \cdot (s + 4)^{-1} + (s + 2)(-1)(s + 4)^{-2} \bigg|_{s=-1}
\]

or
\[
k_3 = \frac{1}{-1 + 4} + \frac{(-1 + 2)(-1)}{(-1 + 4)^2} = \frac{1}{3} + \frac{-1}{9} = \frac{2}{9}
\]

Plugging in \(k_1, k_2, \) and \(k_3\) gives our expression for \(F(s)\):

\[
F(s) = \frac{-2/9}{s + 4} + \frac{1/3}{(s + 1)^2} + \frac{2/9}{s + 1}
\]

Use inverse Laplace transform for each term to get final answer:

\[
\mathcal{L}^{-1}\left\{ \frac{k}{s + a} \right\} = k e^{-at} \quad \text{and} \quad \mathcal{L}^{-1}\left\{ \frac{k}{(s + a)^2} \right\} = k t e^{-at}
\]

Thus, our answer is

\[
f(t) = -\frac{2}{9} e^{-4t} + \frac{1}{3} t e^{-t} + \frac{2}{9} e^{-t}.
\]

**sol'n:** (b)

\[
G(s) = \frac{4(3 + s)}{(s + 2)(s + j5)(s - j5)(s + 3 + j4)(s + 3 - j4)}
\]

The zeros are the roots of the numerator, (i.e., the values of \(s\) where \(G(s)\) goes to zero:

\[
4(3 + s) = 0 \quad \Rightarrow \quad 3 + s = 0 \quad \Rightarrow \quad s = -3
\]

We plot zeros as 0's in s-plane. (See answer plot.)

The poles are the roots of the denominator, (i.e., the values of \(s\) where \(G(s)\) goes to infinity).

The root for a factor of form \(s + a\) is \(s = -a\).

Therefore, poles are at \(s = -2, -j5, j5, -3 - j4, \) and \(-3 + j4\).

We plot poles as x's in s-plane. (See answer plot.)

**sol'n:** (c) Initial value theorem:

\[
\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)
\]

\[
\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} \left[ sF(s) = \frac{s \cdot 3(s^3 + 7s^2 + 14s + 8)}{s^4 + 14s^3 + 98s^2 + 350s + 635} \right]
\]
The highest power of $s$ in numerator and denominator dominates as $s$ becomes large. In other words, $s^2$ becomes much larger than $s$ or a constant term as $s$ approaches infinity. Thus, for terms that are summed, we need only consider the term with the highest power of $s$.

$$\lim_{s \to \infty} \frac{3 \cdot s^2}{s^4} = 3$$

\[\therefore f(t = 0^+) = 3 \]

**sol'n: (d)** Final value theorem:

$$\lim_{s \to \infty} f(t) = \lim_{s \to \infty} sF(s)$$

$$= \lim_{s \to 0} \frac{s \cdot (2s^4 + 6s^3 + 30s^2 + 25s + 120)}{s^6 + 14s^5 + 112s^4 + 448s^3 + 975s^2 + 625s}$$

We factor out the highest power of $s$ that is common to every term in the numerator and refer to this as $s^n$. Since we multiply $F(s)$ by $s$, we always can factor out $s^1$ from the numerator of $sF(s)$. Here, $s^1$ is the highest power of $s$ we can factor out from the numerator of $sF(s)$.

**Note:** If the numerator of $F(s)$ ends with a term such as $3s$, (for example), then we can factor out $s^2$ from the numerator of $sF(s)$.

We also factor out the highest power of $s$ that is common to every term in the denominator of $sF(s)$ and refer to this as $s^m$. Here, $s^1$ is the highest power of $s$ we can factor out from the denominator of $sF(s)$.

We now write

$$sF(s) = \frac{s^n}{s^m} \frac{p(s)}{q(s)}$$

where $p(s)$ and $q(s)$ are polynomials with nonzero constant terms.

In the limit as $s \to 0$, we have $p(s) = p(0)$ and $q(s) = q(0)$. We also have $s^n/s^m = 1/s^{n-m}$, and we can easily determine the behavior of this term as $s \to 0$. The following equation encapsulates these results:
\[
\lim_{s \to 0} sF(s) = \begin{cases} 
0 & n > m \\
\infty & n < m \\
p(0) = \text{constant term of } p(s) & n = m
\end{cases}
\]

Here,
\[
\lim_{s \to \infty} f(t) = \lim_{s \to 0} s \frac{s^1}{s^5 + 14s^4 + 112s^3 + 448s^2 + 975s + 625} = \frac{p(s)}{q(s)}
\]

This reduces to
\[
\lim_{s \to \infty} f(t) = \frac{p(0)}{q(0)} = \frac{120}{625} = \frac{24}{125} = 0.192.
\]

**Note:** \( s^n \) \( s^m \) term gives \( m - n = \# \) poles at origin (net).
\[
\lim_{t \to \infty} f(t) = \infty
\]

if \( F(s) \) has two more poles than zeros at origin.

**Note:** We must have all poles in the left-half plane. Otherwise, our time-domain solution will contain a term of form \( e^{at} \), \( a > 0 \), in \( f(t) \). This is a growing exponential that causes
\[
\lim_{t \to \infty} f(t) = \infty
\]

Thus, the first step in applying the final value theorem is to verify that poles are in the left-half plane, (i.e. stable system).

**sol'n:** (e) Use delayed step-functions to create windows for piecewise definition of \( h(t) \).
\[
h(t) = 3[u(t-2) - u(t-5)] + 5[u(t-5) - u(t-7)] - 2[u(t-7) - u(t-9)]
\]

Gather coefficients for each step-function:
\[
h(t) = 3u(t-2) + (5-3)u(t-5) - (2+5)u(t-7) + 2u(t-9)
\]
or

\[ h(t) = 3u(t - 2) + 2u(t - 5) - 7u(t - 7) + 2u(t - 9) \]

Now use the identity for delayed functions:

\[ \mathcal{L}\{f(t - a)u(t - a), \ a > 0\} = e^{-as} \ F(s) \]

where

\[ f(t) = u(t), \quad \mathcal{L}\{f(t) = u(t)\} = \frac{1}{s}. \]

Plugging the various values of delay for \( a \), we get our final answer:

\[ H(s) = \frac{3e^{-2s} + 2e^{-5s} - 7e^{-7s} + 2e^{-9s}}{s} \]