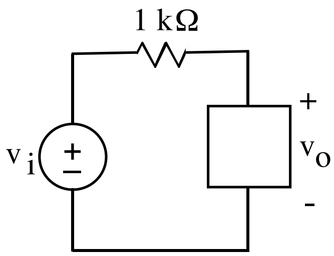
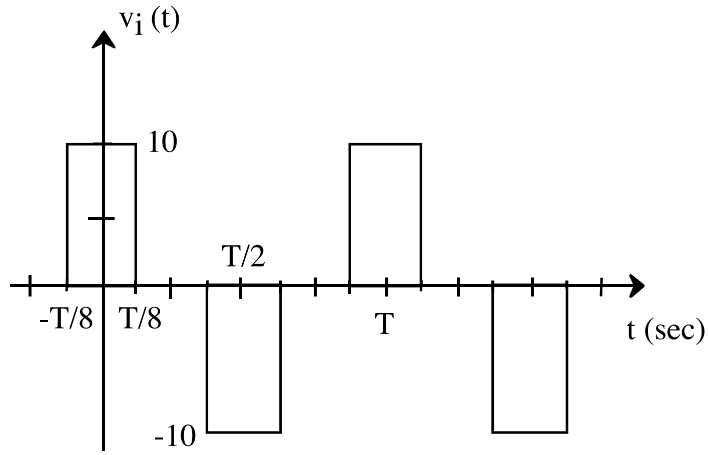
PRACTICE FINAL EXAM SOLUTION: Prob 2

2. (50 points)





$\omega_o = 1\text{ M rad/sec}$

a. Determine the coefficients of the Fourier series, a_v, a_n, and b_n, for the periodic waveform $v_i(t)$. Also, use these Fourier coefficients to find the coefficients of the first five terms of the Fourier series written in terms of inverse phasors:

$$v_1(t) = a_v + \sum_{n=1}^{\infty} A_n \cos\left(n\omega_o t + \theta_n\right)$$

Note any symmetry properties of the waveform that you use to determine coefficients.

b. The circuit on the left is a filter with output $v_o(t)$. Design a circuit to be placed in the box such that the filter rejects the fundamental frequency of $v_i(t)$ and has a bandwidth of 10,000 rad/sec. Specify the component values. Show how the components are connected in the circuit.

ans: a) $a_v = 0$

$$a_n = \begin{cases} \frac{40}{\pi n}\sin\frac{\pi n}{4} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

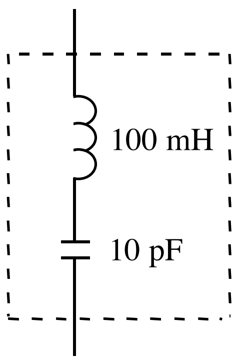
$$b_n = 0 \text{ for all } n$$

$$A_1 = \frac{20\sqrt{2}}{\pi},\ \theta_1 = 0^\circ \quad A_2 = 0,\ \theta_2 = 0^\circ \quad A_3 = \frac{20\sqrt{2}}{3\pi},\ \theta_3 = 0^\circ$$

$$A_4 = 0,\ \ \theta_4 = 0^\circ \qquad A_5 = \frac{-4\sqrt{2}}{\pi},\ \ \theta_5 = 0^\circ$$

Symmetries used: even function, half wave (shift-flip symmetry), and quarter wave symmetry.

b)



sol'n: (a) a_v = ave value of $v_i(t) = 0$ since equal positive and negative areas are under the $v_i(t)$ curve.

$v_i(t)$ is symmetric around vertical axis so $v_i(t)$ is an <u>even</u> function. This means we need only even functions—cosine terms—in our Fourier series.

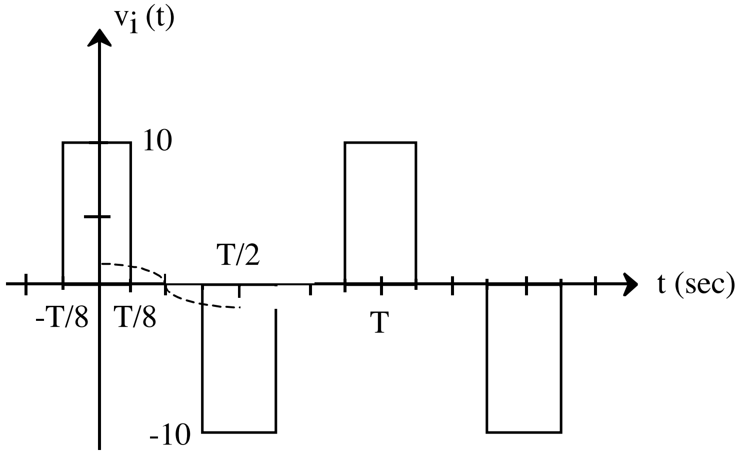
$\therefore$ $b_n = 0$ for all n (no $\sin(n\omega_o t)$ terms in Fourier series)

If we shift $v_i(t)$ one-half period and flip it upside down, we have $v_i(t)$ again. Thus, we have half-wave symmetry or, as refer to it, shift-flip symmetry.

$\therefore$ $a_n = 0$ for n even ($b_n = 0$ for n even, too, but we already know $b_n = 0$ all n)

For the question of quarter wave symmetry, we look for symmetry around T/4 and 3T/4. What we find is that $v_i(t)$ is odd around T/4 and 3T/4. In other words, if the vertical axis for T = 0 were shifted to T/4 or 3T/4, $v_i(t)$ would be an odd function. If we superimpose the $\cos(n\omega_o t)$ term for $n = 1$ on $v_i(t)$ and consider the signs of the product $v_i(t)\cos(n\omega_o t)$, as shown below, we discover that we can calculate a_1 by quadrupling the integral from 0 to T/4 in the formula for a_1:

$$a_1 = 4 \cdot \frac{2}{T} \int_0^{T/4} v_i(t) \cos(1 \cdot \omega_o t)\ dt$$



$\omega_o = 1\text{ M rad/sec}$

The same will hold true for every odd numbered *n*.

Now we define $v_i(t)$ from 0 to T/4:

$$v_i(t) = \begin{cases} 10 & 0 \le t \le T/8 \\ 0 & T/8 < t \le T/4 \end{cases}$$

Thus,

$$a_n = \frac{8}{T} \left[\int_0^{T/8} 10 \cos\left(n\omega_o t\right) dt + \underbrace{\int_{T/8}^{T/4} 0 \cdot \cos\left(n\omega_o t\right) dt}_{\int 0\, dt = 0} \right]$$

or

$$a_n = \frac{8}{T} \int_0^{T/8} 10 \cos\left(n\omega_o t\right) dt$$

$$= \frac{8}{T} \left. \frac{10 \sin\left(n\omega_o t\right)}{n\omega_o} \right|_0^{T/8}$$

Now substitute:

$$\omega_o \equiv \frac{2\pi}{T}$$

$$a_n = \frac{\cancel{8}}{\cancel{T}} \left. \frac{10 \sin \; n\frac{2\pi}{T}}{n\frac{\cancel{2}\pi}{\cancel{T}}} \right|_0^{T/8}$$

$$= \frac{40}{\pi n} \left[\sin \; \frac{2\pi n}{\cancel{T}} \frac{\cancel{T}}{8} - \cancelto{0}{\sin 0} \right]$$

$$a_n = \frac{40}{\pi n} \sin\left(\frac{\pi n}{4}\right) \quad \text{for } n \text{ odd}$$

If we compute the values of $\sin(\pi n/4)$ for $n = 0, 1, \ldots$ we get $0, 1/\sqrt{2}, 1, 1/\sqrt{2}, 0, -1/\sqrt{2}, -1/\sqrt{2}, 0$, in a repeating pattern.

Therefore, a_n coefficients for n odd up to the fifth harmonic are:

$$a_1 = \frac{\sqrt{2}}{2} \cdot \frac{40}{\pi}, \quad a_3 = \frac{\sqrt{2}}{2} \cdot \frac{40}{3\pi}, \quad a_5 = \frac{-\sqrt{2}}{2} \cdot \frac{40}{5\pi}$$

Now we convert to phasor form, $a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t)$. The time-domain rectangular representation of the nth term of the Fourier series is

$a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t)$

Recalling that the phasor for pure cos() is 1 and for pure sin() is –j, the phasor for the nth term of the Fourier series is

a_n (or $a_n \angle 0°$) + $-jb_n$ (or $b_n \angle -90°$)

Thus, our phasor is $a_n - jb_n$. Incidentally, if we convert to polar form, $A_n \angle \theta_n$, we have:

$$A_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

Here, however, all $b_n = 0$. So we have $A_n = a_n$, $\theta_n = 0°$. In other words, we have only cos( ) terms, and the phase angle for cos( ) terms is zero since they are real.

$$A_1 = a_1 = \frac{20\sqrt{2}}{\pi}, \quad \theta_1 = 0^\circ$$

$$A_3 = a_3 = \frac{20\sqrt{2}}{3\pi}, \quad \theta_3 = 0^\circ$$

$$A_5 = a_5 = \frac{-20\sqrt{2}}{5\pi}, \quad \theta_5 = 0^\circ$$

Note: You may find it easier to derive symmetry results by drawing $v_i(t)$ and the cos( ) or sin( ) waveforms on a plot and multiplying them point by point (a rough sketch will do). The area under the curve corresponds to

$$\int_0^T v_i(t) \cos(\ ) \text{ or } \int_0^T v_i(t) \sin(\ )$$

If the positive and negative areas under the product curves cancel, a_n (or b_n) = 0.

sol'n: (b) We want a band reject filter with center frequency = ω_o = 1M rad/s, (see diagram in problem statement), and bandwidth β = 10k rad/s (see problem statement).

Note: By coincidence, in this problem ω_o for the Fourier series (which is determined by the value of the period, T), happens to be the same as the center frequency, ω_o, of the filter (which is determined the values of R, L, and C). This need not always be the case.

Our transfer function is $H(s) \equiv V_o(s)/V_i(s)$.

We use V-divider formula for $V_o(s)$ in terms of $V_i(s)$, letting z_L denote the impedance in the box.

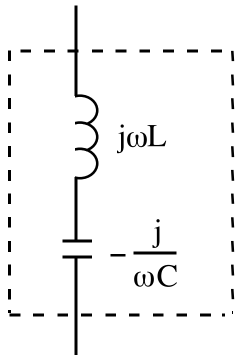
$$V_o(s) = V_i(s) \cdot \frac{z_L}{1\,k\Omega + z_L}$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{z_L}{1\,k\Omega + z_L}$$

We need $z_L = 0$ at ω = 1M to get

$$\frac{V_o(s = j\omega = j1\text{Mr/s})}{V_i(s = j\omega = j1\text{Mr/s})} = 0$$

We use an L in series with a C to get z cancellation:



$$z_L = j\omega L - \frac{j}{\omega C}$$

To get cancellation, ωL = 1/ωC at ω = 1M or

$$LC = \frac{1}{\omega^2} = \frac{1}{(1\text{M})^2} = 1 \text{ ps}$$

We have RLC in series, and for a series RLC band-reject filter, we have β = R/L. For β = 10k rad/s and R = 1 kΩ, we get

L = R/β = 0.1 H.

Knowing L, we can now solve for C:

$$C = \frac{1}{L\omega^2} = \frac{1}{0.1\text{H}(1\text{M/s})^2}$$

$$\therefore \quad C = 10 \text{ pF}$$