Ex:

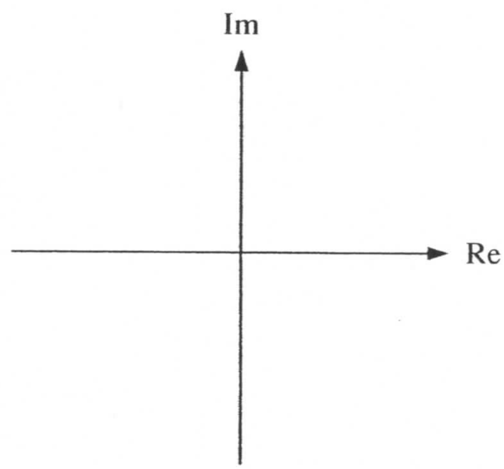
a) Find $\mathcal{L}\left\{\left(\int_0^{t-2} \tau e^{-3\tau}\, d\tau\right) u(t-2)\right\}$.

b) Find $v(t)$ if $V(s) = 2 + \dfrac{s^2 + s + 4}{s(s^2+4)}$.

c) Find $\lim\limits_{t\to 0} v(t)$ if $V(s) = \dfrac{(3s^2+12)(s-4)}{[(s+2)^2+5^2](s^2+5s+6)}$.

d) Plot and label the values of the poles and zeros of $V(s)$ in the s plane.

$$V(s) = \frac{3s^2+12}{[(s+2)^2+5^2](s^2+5s+6)}$$



Sol'n: a) We identify $\int_0^{t-2} \tau e^{-3\tau}\, d\tau$ as $f(t-2)$ and use the delay (or shift) identity:

$$\mathcal{L}\{f(t-2)u(t-2)\} = e^{-2s} F(s)$$

To find $f(t)$, we replace $t-2$ with t in $f(t-2)$:

$$f(t) = \int_0^{t} \tau e^{-3\tau}\, d\tau$$

To find $\mathcal{L}\{\int_0^t \tau e^{-3\tau}\,d\tau\}$, we use the integral identity:

$$\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}F(s)$$

where $F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\tau e^{-3\tau}\}$.

From tables, we have $\mathcal{L}\{\tau e^{-3\tau}\} = \dfrac{1}{(s+3)^2}$.

Back substituting into earlier expressions gives

$$\mathcal{L}\left\{\left(\int_0^{t-2} \tau e^{-3\tau}\,d\tau\right)u(t-2)\right\} = \frac{e^{-2s}}{s(s+3)^2}$$

b) We know $\mathcal{L}^{-1}\{2\} = 2\delta(t)$, so we focus on the inverse transform of $\dfrac{s^2+s+4}{s(s^2+4)}$.

$$\frac{s^2+s+4}{s(s^2+4)} = \frac{A}{s} + \frac{Bs + C(2)}{(s^2+2^2)}$$

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Note: the second term is written as the form for a cosine plus sine (in t-domain):

$$B\frac{s}{s^2+\omega^2} + C\frac{\omega}{s^2+\omega^2} \xrightarrow{\mathcal{L}^{-1}} B\cos\omega t + C\sin\omega t$$

Using the pole cover-up method, we find A:

$$A = \cancel{s}\left(\frac{s^2+s+4}{\cancel{s}(s^2+4)}\right)\Bigg|_{s=0} = \frac{4}{4} = 1$$

Subtracting the $\frac{1}{s}$ term leaves $\dfrac{Bs+C(2)}{s^2+2^2}$:

$$\frac{s^2+s+4}{s(s^2+4)} - \frac{1}{s} = \frac{s^2+s+4}{s(s^2+4)} - \frac{(s^2+4)}{s(s^2+4)} = \frac{\cancel{s}1}{\cancel{s}(s^2+4)}$$

So we have $\dfrac{1}{s^2+4} = \dfrac{\frac{1}{2}(2)}{s^2+4}$, so $B=0$, $C=\frac{1}{2}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\cdot\frac{2}{s^2+4}\right\} = \frac{1}{2}\sin(4t)\,u(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = u(t)$$

Thus,

$$v(t) = 2\delta(t) + \left[1+\frac{1}{2}\sin(4t)\right]u(t)$$

c) We use the Initial Value Theorem:

$$\lim_{t\to 0} v(t) = \lim_{s\to\infty} sV(s)$$

$$= \lim_{s\to\infty} \frac{s(3s^2+12)(s-4)}{\left[(s+2)^2+5^2\right](s^2+5s+6)}$$

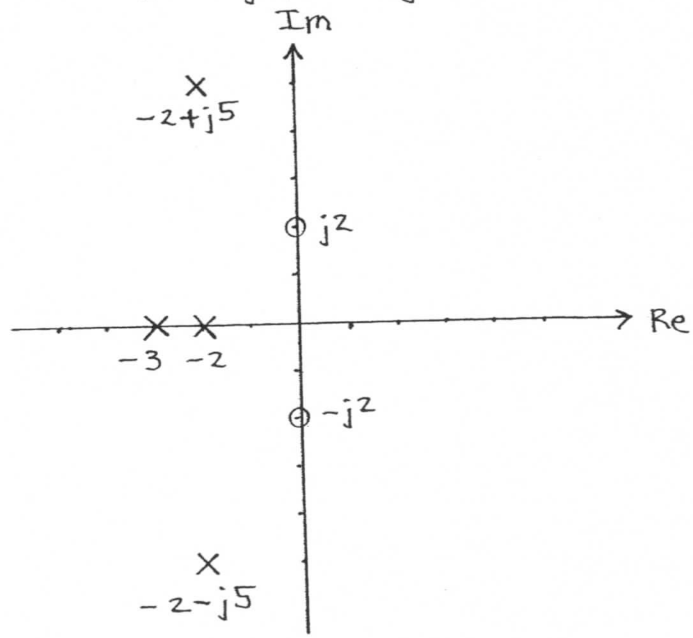
We may ignore additive terms if there is a term with a higher power of s, since $s^2 \gg s$, for example, as $s\to\infty$.

$$\lim_{t\to 0} v(t) = \lim_{s\to\infty} \frac{\cancel{s}(3\cancel{s^2})\cancel{s}}{\cancel{s^2}\cdot\cancel{s^2}} = \frac{3}{1} = 3$$

Thus, $\lim_{t\to 0} v(t) = 3$

d) We factor the numerator and denominator of $V(s)$ to find root terms that correspond to poles (denominator) and zeros (numerator).

$$V(s) = \frac{3(s+j2)(s-j2)}{(s+2+j5)(s+2-j5)(s+2)(s+3)}$$



Note: The scaling factor of 3 in the numerator is missing in the pole-zero diagram. Thus, we are unable to determine $V(s)$ from poles and zeros alone. (The gain term may be added to the diagram as a footnote.)