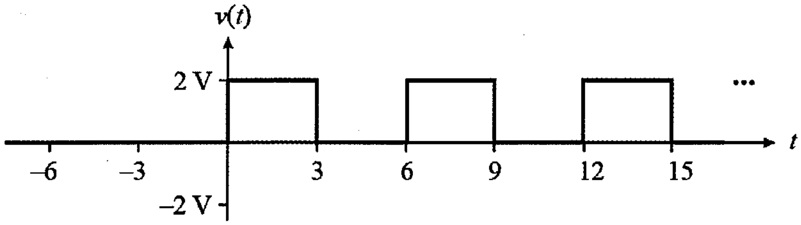
Ex: Find the Laplace transform, if possible, of the following square wave:



SOL'N: One approach is to use step functions to write $v(t)$:

$$v(t) = [u(t) - u(t-3) + u(t-6) - u(t-9) + \ldots] \cdot 2$$

or

$$v(t) = \sum_{n=0}^{\infty} 2u(t-3n)(-1)^n$$

$$\mathcal{L}\{v(t)\} = 2\sum_{n=0}^{\infty} \mathcal{L}\{u(t-3n)(-1)^n\}$$

Now $\mathcal{L}\{u(t-a)\} = 2e^{-as}\mathcal{L}\{u(t)\} = \frac{e^{-as}}{s}$

$$\mathcal{L}\{v(t)\} = 2\sum_{n=0}^{\infty} \frac{e^{-3ns}(-1)^n}{s}$$

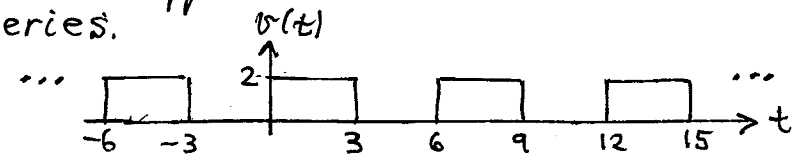
This is a geometric sum: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$

$$\mathcal{L}\{v(t)\} = 2\sum_{n=0}^{\infty} \frac{(-e^{-3s})^n}{s} = \frac{2}{s}\sum_{n=0}^{\infty}(-e^{-3s})^n$$

$$\mathcal{L}\{v(t)\} = \frac{2}{s}\frac{1}{1--e^{-3s}} = \frac{2}{s(1+e^{-3s})}$$

$$\mathcal{L}\{v(t)\} = \frac{2}{s(1+e^{-3s})}$$

Another approach is to use a Fourier series.



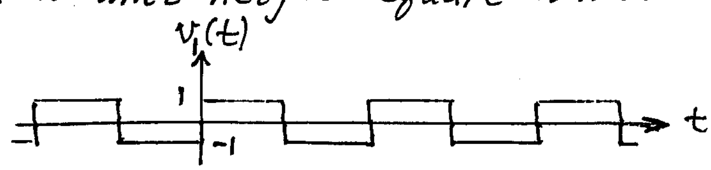
Note: we may assume $v(t)$ is a square wave for $t<0$ since the Laplace transform only requires values of $v(t)$ for $t>0$.

By inspection, the DC or average value of $v(t)$ equals unity.

$$v(t) = a_\nu + \sum_{k=1}^{\infty} a_k \cos(k\omega_o t) + b_k \sin(k\omega_o t)$$

$$a_\nu = 1$$

If we subtract out a_ν, we are left with a unit-height square wave:



From a table of common Fourier series, [1], we have

$$v_1(t) = \sum_{k \text{ odd}} \frac{4}{\pi k} \sin(k\omega_o t)$$

where $\omega_o = \frac{2\pi}{T}$ and $T = 6$ (period of $v_1(t)$)

$$v_1(t) = \sum_{k \text{ odd}} \frac{4}{\pi k} \sin\left[\frac{k(2\pi)t}{6}\right]$$

We use $\mathcal{L}\{\sin(\omega t)\} = \dfrac{\omega}{s^2+\omega^2}$

$$\mathcal{L}\{v(t)\} = \mathcal{L}\left\{1 + \sum_{k\,\text{odd}}^{\infty} \frac{4}{\pi k}\sin\left(\frac{1}{3}\pi k t\right)\right\}$$

$$= \mathcal{L}\{1\} + \sum_{k\,\text{odd}}^{\infty} \frac{4}{\pi k}\,\mathcal{L}\left\{\sin\left(\frac{1}{3}\pi k t\right)\right\}$$

$$= \frac{1}{s} + \sum_{k\,\text{odd}}^{\infty} \frac{4}{\pi k}\cdot\frac{\frac{\pi}{3}k}{s^2+\left(\frac{\pi k}{3}\right)^2}$$

$$\mathcal{L}\{v(t)\} = \frac{1}{s} + \frac{4}{3}\sum_{k\,\text{odd}}^{\infty} \frac{1}{s^2+\left(\frac{\pi k}{3}\right)^2}$$

This looks very different from the previous answer but should be equivalent.