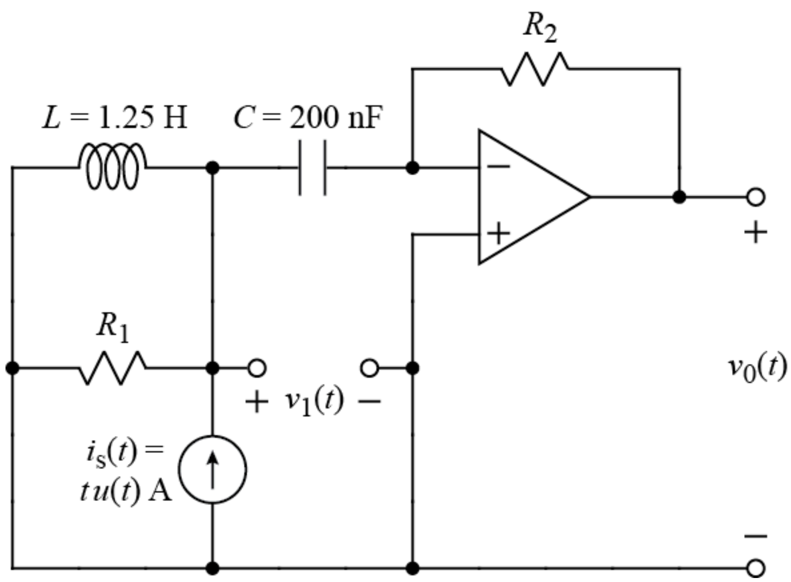
HOMEWORK #10 probs 1 and 2 solution

Ex:



The current source in the above circuit is off for $t < 0$.

a) Find a symbolic expression for the Laplace-transformed output, $\mathbf{V}_0(s)$, in terms of not more than R_1, R_2, L, C, and values of sources or constants.

b) Choose a numerical value for R_1 for the circuit in problem 1 to make

$$v_1(t) = v_m - v_m e^{-\alpha t}\left[\cos(\beta t) + \frac{1}{2}\sin(\beta t)\right]$$

where v_m, α, and β are real-valued constants.

Hint: C behaves as though it is in parallel with L and R_1.

sol'n: a) We define $\mathbf{I}(s)$ to the current flowing thru C toward the minus input of the op-amp. Since $\mathbf{I}(s)$ cannot flow into the op-amp, it also flows thru R_2.

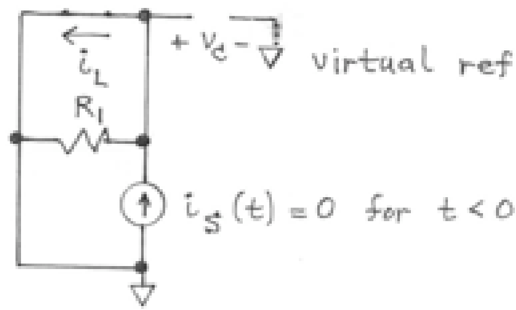
We can add a refernce on the bottom rail, making V_+ at the positive input of the op-amp equal to 0V. Because of R_2 and the negative feedback it provides, we have

$$V_- = V_+ = 0\text{V}$$

Given the direction of $\mathbb{I}(s)$ and $V_- = 0V$, Ohm's law gives the value of $V_o(s)$:

$$V_o(s) = -\mathbb{I}(s)\,R_2$$

To find $\mathbb{I}(s)$, we need initial conditions for L and C. We consider time $t = 0^-$:



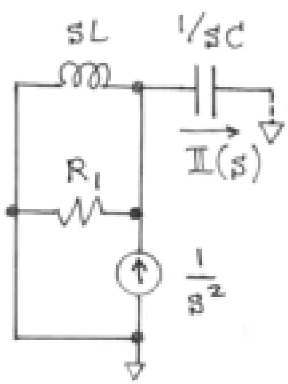
Since there is no current, we have

$$i_L(0^-) = 0, \qquad v_C(0^-) = 0.$$

Thus, our initial conditions are zero.

The Laplace transform of $i_S(t)$ for $t > 0$ is

$$\mathcal{L}\{t\,u(t)\,A\} = \frac{1}{s^2}$$



R_1, L, and C are effectively in parallel.

Thus, we have an I-divider.

$$I(s) = \frac{1}{s^2} \frac{R_1 \| sL}{R_1 \| sL + \frac{1}{sC}}$$

$$= \frac{1}{s^2} \frac{1}{1 + \frac{1}{sC}\left(\frac{1}{R_1} + \frac{1}{sL}\right)}$$

$$I(s) = \frac{1}{s^2 + \frac{1}{R_1 C} s + \frac{1}{LC}}$$

Now we can write an expression for $V_o(s)$:

$$V_o(s) = \frac{-R_2}{s^2 + \frac{1}{R_1 C} s + \frac{1}{LC}}$$

b) The Laplace transform of $v_1(t)$, as specified in the problem, is as follows:

$$\mathcal{L}\{v_1(t) = v_m - v_m e^{-\alpha t}\left[\cos\beta t - \frac{1}{2}\sin\beta t\right]\}$$

$$= \frac{v_m}{s} - v_m \frac{(s+\alpha) + \frac{1}{2}\beta}{(s+\alpha)^2 + \beta^2} = V_1(s)$$

For the circuit, using $I(s)$ from part (a), we have the following for $V_1(s)$:

$$V_1(s) = I(s)\frac{1}{sC} = \frac{1/C}{s\left(s^2 + \frac{1}{R_1 C} s + \frac{1}{LC}\right)}$$

To equate the two expressions for $V_1(s)$, we use a common denominator.

$$s^2 + 2\alpha s + \alpha^2$$

$$\parallel$$

$$V_1(s) = V_m \left[\frac{(s+\alpha)^2 + \beta^2 - s(s+\alpha) - s\beta/2}{s\,(s^2 + 2\alpha s + \alpha^2 + \beta^2)} \right]$$

We now match the coefficients of each power of s for both the numerator and denominator.

coeff		expressions	
s^2	numerator	$0 = 0$	(1)
s	numerator	$V_m(2\alpha - \alpha - \beta/2) = 0$	(2)
1	numerator	$V_m(\alpha^2 + \beta^2) = 1/C$	(3)
$s \cdot s^2$	denominator	$1 = 1$	(4)
$s \cdot s$	denominator	$2\alpha = \dfrac{1}{R_1 C}$	(5)
$s \cdot 1$	denominator	$\alpha^2 + \beta^2 = \dfrac{1}{LC}$	(6)

From the last eq'n and the 3rd eq'n, we have

$$V_m(\alpha^2 + \beta^2) = V_m \frac{1}{LC} = 1/C$$

From the 5th eq'n we have an expression for α:

$$\alpha = \frac{1}{2R_1C}$$

From the 2nd eq'n we have an expression for β in terms of α:

$$\alpha - \beta/2 = 0 \quad \Rightarrow \quad \beta = 2\alpha$$

Using this expression for β, we substitute into the 6th eq'n to find R_1:

$$\alpha^2 + (2\alpha)^2 = \frac{1}{LC} \quad \Rightarrow \quad 5\alpha^2 = \frac{1}{LC}$$

$$\text{or} \quad 5\left(\frac{1}{2R_1C}\right)^2 = \frac{1}{LC}$$

or $\quad \dfrac{5}{4R_1^2C^2} = \dfrac{1}{LC}$

or

$$\frac{4}{5} R_1^2 C = L$$

or $\quad R_1^2 = \dfrac{5L}{4C}$

or $\quad R_1 = \sqrt{\dfrac{5L}{4C}} = \sqrt{\dfrac{5\left(\frac{5}{4}\right)}{4 \cdot 200n}}\ \Omega$

or $\quad R_1 = \dfrac{5}{4}\sqrt{5M}\ \Omega$

$$R_1 = \frac{5}{4}\sqrt{5}\ k\Omega \doteq 2.8\ k\Omega$$