HOMEWORK #5 prob 4 solution

TRANSFORM PAIR: $\mathcal{L}\{t^n e^{-at}\} = \dfrac{n!}{(s+a)^{n+1}}$

PROOF: We start by finding the Laplace transform of t^n. We may add the e^{-at} at the end, using a convenient identity. Meanwhile, we may derive the transform of t^n by repeatedly applying the identity for multiplication by t, starting with $f(t) = 1 = t^0 = u(t)$.

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

We apply the identity for multiplication by t:

$$\mathcal{L}\left\{tf(t)\right\} = -\frac{d}{ds}F(s)$$

In the present case, we have the following result:

$$\mathcal{L}\left\{tu(t)\right\} = \mathcal{L}\left\{t\right\} = -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2}$$

We apply the identity several more times:

$$\mathcal{L}\left\{t \cdot t\right\} = \mathcal{L}\left\{t^2\right\} = -\frac{d}{ds}\frac{1}{s^2} = \frac{2}{s^3}$$

and

$$\mathcal{L}\left\{t \cdot t^2\right\} = \mathcal{L}\left\{t^3\right\} = -\frac{d}{ds}\frac{2}{s^3} = \frac{2 \cdot 3}{s^4}$$

and

$$\mathcal{L}\left\{t \cdot t^3\right\} = \mathcal{L}\left\{t^4\right\} = -\frac{d}{ds}\frac{3}{s^4} = \frac{2 \cdot 3 \cdot 4}{s^5}$$

At this point the pattern is becoming clear:

$$\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}$$

NOTE: To complete a formal proof by induction, we may apply the multiplication-by-time identity to show the formula holds for t^{n+1} assuming that it holds for t^n.

To complete our proof, we apply the identity for multiplication by e^{-at}:

$$\mathcal{L}\left\{e^{-at} f(t)\right\} = F(s+a)$$

This yields our final result:

$$\mathcal{L}\{t^n e^{-at}\} = \frac{n!}{(s+a)^{n+1}}$$