HOMEWORK #5 prob 1a solution

Ex: Find the Laplace transforms of the following waveform:

$t\sin(t-\tau)u(t-\tau)$ where $\tau > 0$

SOL'N: The $u(t-\tau)$ delays the turn-on of the signal, meaning we will use the delay (or shift) identity:

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s), \qquad a > 0$$

Here, $a = \tau$, and we identify $f(t-\tau)$ as

$$t\sin(t-\tau) = f(t-\tau)$$

To find $f(t)$ for $F(s) = \mathcal{L}\{f(t)\}$ we replace each $t-\tau$ term in $f(t-\tau)$ with t. The problem is that we have t instead of $t-\tau$ in front of the sine function. The solution to this problem is to treat t as $(t-\tau)+\tau$. Thus, we have the following equivalent statement of the problem:

$$\text{Find} \quad \mathcal{L}\{[(t-\tau)+\tau]\sin(t-\tau)u(t-\tau)\}$$

Applying the delay identity, we have

$$f(t-\tau) = [(t-\tau)+\tau]\sin(t-\tau)$$

$$\therefore f(t) = (t+\tau)\sin(t)$$

Note: A shortcut for finding $f(t)$ is to replace each occurance of t with $t+\tau$.

By the identity, our answer will be

$$e^{-\tau s}\mathcal{L}\{(t+\tau)\sin(t)\}.$$

or, if we split the transform into two,

$$e^{-\tau s}\left[\mathcal{L}\{t \sin t\} + \mathcal{L}\{\tau \sin t\}\right]$$

Moving the τ outside of the second transform, since τ is a constant, yields

$$\mathcal{L}\{\tau \sin t\} = \tau\,\mathcal{L}\{\sin t\} = \tau \cdot \frac{1}{s^2+1^2}$$

For the first transform, we use the identity for multiplication by t:

$$\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}\,F(s)$$

In this case, $F(s) = \mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1^2}$.

Thus, $\mathcal{L}\{t \sin t\} = -\dfrac{d}{ds}\,\dfrac{1}{s^2+1} = -\dfrac{-2s}{(s^2+1)^2}$.

Our final result:

$$\mathcal{L}\{t \sin(t-\tau)\,u(t-\tau)\} = \frac{2s}{(s^2+1)^2} + \frac{\tau}{s^2+1}$$