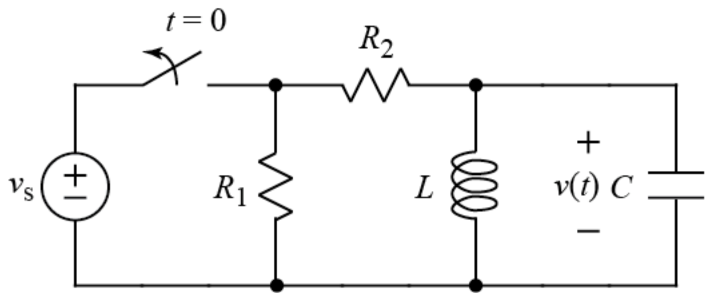
HOMEWORK #1 prob 3 solution

Ex:



After being closed for a long time, the switch opens at $t = 0$.

$L = 10$ nH $C = 250$ nF $R_1 = 0.1\ \Omega$ $R_2 = 0.025\ \Omega$

If $v_s = 10$ V, find $v(t)$ for $t > 0$.

SOL'N: We may perform the following initial steps in any order:

1) Find characteristic roots for the parallel RLC circuit (for $t > 0$)
2) Find the final value of $v(t)$ as $t \to \infty$, which is the A_3 (constant) term in the solution.
3) Find the initial values of energy variables: $i_L(0^+) = i_L(0^-)$ and $v_C(0^+) = v_C(0^-)$

We will perform the steps in the order listed. First, we find the characteristic roots.

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$$

After time $t = 0$, the voltage source and switch will be disconnected from the circuit and may be ignored. The two resistors in series then sum to give $R = R_1 + R_2 = 0.1\ \Omega + 0.025\ \Omega = 0.125\ \Omega$. We have a parallel RLC circuit for which the value of α is one-half the inverse RC time constant:

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \cdot (0.125)\Omega \cdot 250\text{nF}} = 16\text{ M/s}$$

For both parallel and series RLC circuits, the resonant frequency, ω_o, is the inverse of the square root of the product of L and C:

$$\omega_o^2 = \frac{1}{LC} = \frac{1}{10\text{ nH} \cdot 250\text{ nF}} = (20\text{M})^2(\text{r/s})^2$$

Substituting into the equation for the roots yields the following:

$$s_{1,2} = -16 \text{ Mr/s} \pm \sqrt{16^2 - 20^2} \text{ Mr/s} = -16 \text{ Mr/s} \pm j12 \text{ Mr/s}$$

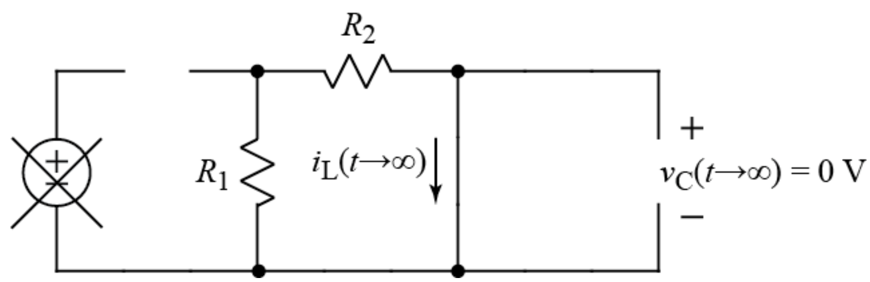
The roots are complex, and the imaginary part of the roots is the damping frequency:

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2} = 12 \text{ Mr/s}$$

For complex roots, we use the following form of solution (that allows us to avoid using complex numbers in our calculations):

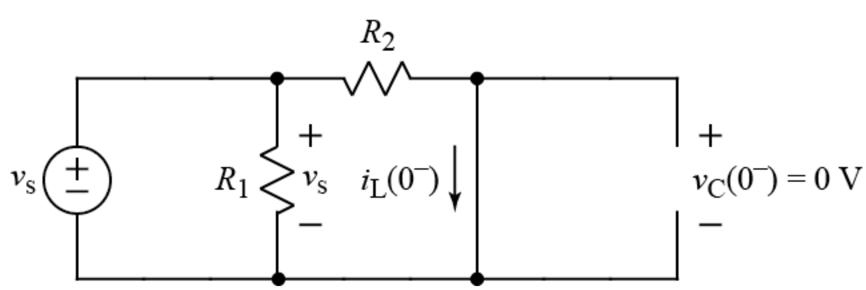
$$v(t) = A_1 e^{-\alpha t}\cos(\omega_d t) + A_2 e^{-\alpha t}\sin(\omega_d t) + A_3$$

Now we proceed to step 2, where we find the A_3 value from $v(t)$ as t->∞. (The exponential terms decay as t->∞, leaving only A_3.) We assume the circuit values become constant as t->∞, causing the L to act like a wire and the C to act like an open circuit. The switch is also open, detaching the voltage source and resistor on the left, as shown in the diagram below.



Since the L acts like a wire that shorts out the capacitor, we have $A_3 = v(t\text{->}\infty) = 0\text{V}$.

Third, we find the initial values of energy variables: $i_L(0^+) = i_L(0^-)$ and $v_C(0^+) = v_C(0^-)$. At $t = 0^-$, we assume circuit values are constant, causing the L to act like a wire and the C to act like an open circuit. The switch is closed, connecting the parallel RLC to the voltage source on the left.



The L, acting like a wire, shorts out the C. Thus, the initial C voltage is zero:

$$v_C(0^+) = v_C(0^-) = 0\text{V}$$

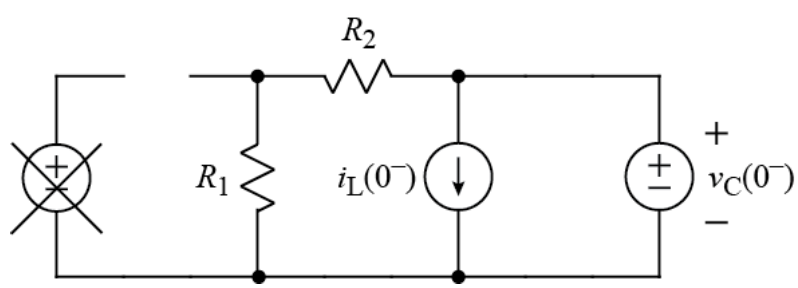
Also, the two R's are in parallel across the voltage source. The current flowing through the L will equal the current flowing through R_2, (since the current flowing through R_2 cannot flow through C). Since v_s is across R_2, we use Ohm's law to find the current in R_2 and L:

$$i_L(0^-) = \frac{v_s}{R_2} = \frac{10\text{ V}}{0.025\ \Omega} = 400\text{ A}$$

The current in L at time 0^+ is the same as at time 0^-, since the energy in the L cannot change instantly:

$$i_L(0^+) = i_L(0^-) = 400\text{ A}$$

At time $t = 0^+$, we treat the C as a voltage source of 0V and the L as a current source of 400 A. (The switch is also open, removing the v_s source from the circuit.) We can solve the circuit for any voltage or current at $t = 0^+$. Using Kirchhoff's laws for voltage loops and current sums at nodes, we have the following results, (with voltages measured with plus on top, and currents measured flowing in the down direction):



$$v_L(0^+) = v_C(0^+) = v_C(0^-) = 0\text{ V}$$

$$i_{R1}(0^+) = i_{R2}(0^+) = \frac{v_C(0^+)}{R_1 + R_2} = \frac{0\text{ V}}{125\text{ m}\Omega} = 0\text{ A}$$

No current flows through the R's, so all the current in the L must flow through the C (but in the negative direction, since we are measuring currents in the down direction):

$$i_C(0^+) + i_L(0^+) + i_{R2}(0^+) = i_C(0^+) + 400\,\text{A} + 0\ \text{A} \Rightarrow i_C(0^+) = -400\,\text{A}$$

Now we are ready to find A_1 and A_2 by matching our symbolic solution to circuit values for $v(0^+)$ and $\left.\frac{dv(t)}{dt}\right|_{t=0^+}$. We have already found $v(0^+)$ to be 0 V. Matching this to the symbolic solution for $t = 0^+$, which is A_1, we have the following:

$$A_1 = 0\text{V}$$

For the value of the derivative in the circuit, we always try to first write our variable, (v in this case), in terms of energy (or state) variables, i_L and/or v_C. Here, this is a simple matter:

$$v(t) = v_C(t)$$

Then we differentiate, and use the component equations involving d/dt for L and/or C:

$$\left.\frac{d}{dt}v(t)\right|_{t=0^+} = \left.\frac{d}{dt}v_C(t)\right|_{t=0^+} = \frac{i_C(0^+)}{C} = \frac{-400\ \text{A}}{250\ \text{nF}} = -16\ \text{GV/s}$$

We equate this to the symbolic derivative:

$$\left.\frac{d}{dt}v(t)\right|_{t=0^+} = -\alpha A_1 + \omega_d A_2 = \omega_d A_2 = -16\ \text{GV/s}$$

or

$$A_2 = \frac{-1.6\ \text{GV/s}}{\omega_d} = \frac{-1.6\ \text{GV/s}}{12\,\text{Mr/s}} = -133.3\text{V}$$

Plugging in values gives the solution for $v(t > 0)$:

$$v(t) = -133.3e^{-16\ \text{M/s}\cdot t}\sin(12\,\text{Mr/s}\cdot t)\,\text{V}$$