Homework #5 prob 3.b solution

EX: Find the inverse Laplace transform for the following expression:

$$F(s) = \frac{2s - 26}{s^2 + 10s + 169}$$

SOL'N: Our first step in finding the inverse transform is to express the denominator in terms of roots. For a quadratic polynomial, if the square of half the coefficient of s is less than the constant coefficient, the roots are complex. In that case, we can write the denominator in terms of the real and imaginary parts of the roots:

$$s^2 + 10s + 169 = (s + a)^2 + \omega^2 = s^2 + 2as + a^2 + \omega^2$$

In this expression, a is the real part of the root and ω is the imaginary part of the root. From this expression, we see that a equals half the middle coefficient:

$$a = \frac{10}{2} = 5$$

To find ω, we use the value of a and the constant term in the denominator:

$$a^2 + \omega^2 = 169$$

$$5^2 + \omega^2 = 169$$

$$\omega = \pm\sqrt{169 - 5^2} = \pm\sqrt{144} = \pm 12$$

Our roots are complex conjugates:

$$s_{1,2} = -a \pm j\omega = 5 \pm j12$$

We can express the denominator in several ways:

$$s^2 + 10s + 169 = (s - (-a + j\omega))(s - (-a - j\omega)) = (s + a - j\omega)(s + a + j\omega)$$

$$s^2 + 10s + 169 = (s + 5 - j12)(s + 5 + j12)$$

or

$$s^2 + 10s + 169 = (s + a)^2 + \omega^2$$

$$s^2 + 10s + 169 = (s + 5)^2 + 12^2$$

If we choose the first form for the denominator, we express $F(s)$ as partial fractions:

$$F(s) = \frac{A_1}{s+5-j12} + \frac{A_1^*}{s+5+j12}$$

NOTE: Because the roots are conjugates and all the coefficients in $F(s)$ are real, the coefficients of the partial fractions are always complex conjugates of each other.

We find A_1 by multiplying $F(s)$ by the root term and evaluating at the value of the root.

$$A_1 = (s+5-j12)F(s)\Big|_{s=-(5-j12)} = \frac{2s-26}{s+5+j12}\Bigg|_{s=-(5-j12)}$$

or

$$A_1 = \frac{2[-(5-j12)]-26}{-(5-j12)+5+j12} = \frac{-36+j24}{j24} = \frac{(-j)(-36+j24)}{24} = 1+j\frac{3}{2}$$

NOTE: The value in the denominator will always be two times the imaginary part of the root we are evaluating, as the real parts will cancel out.

If we use a common denominator, we can identify terms for a decaying cosine and sine.

$$F(s) = \frac{1+j\frac{3}{2}}{s+5-j12} + \frac{1-j\frac{3}{2}}{s+5+j12}$$

or

$$F(s) = \frac{\left(1+j\frac{3}{2}\right)(s+5+j12)+\left(1-j\frac{3}{2}\right)(s+5-j12)}{(s+5-j12)(s+5+j12)}$$

or

$$F(s) = \frac{2s+2\cdot 5-2\cdot\frac{3}{2}\cdot 12}{s^2+10s+169} = \frac{2s-26}{s^2+10s+169}$$

Symbolically, if we write A_1 as a complex number, we can express our results in generic form.

$$A_1 \equiv c + jd$$

$$F(s) = \frac{c + jd}{s + a - j\omega} + \frac{c - jd}{s + a + j\omega} = \frac{2cs + 2ca - 2d\omega}{(s + a)^2 + \omega^2}$$

Although it appears we have simply come full circle back to our original expression for $F(s)$, we can ultimately express our results in terms of A_1.

We observe that the denominator is the denominator of a decaying cosine or sine. We now represent $F(s)$ as a sum of transforms for a decaying cosine and sine:

$$F(s) = \frac{K_1(s + a)}{(s + a)^2 + \omega^2} + \frac{K_2\omega}{(s + a)^2 + \omega^2}$$

Equating the numerators with the numerator of the previous expression yields expression for K_1 and K_2:

$$2cs + 2ca - 2d\omega = K_1(s + a) + K_2\omega$$

Matching the coefficient for the highest power of s first yields our result in terms of the real and complex parts of A_1:

$$K_1 = 2c \quad \text{and} \quad K_2 = -2d$$

NOTE: We can also bypass the steps of finding A_1 and equate the numerator of $F(s)$ directly with $K_1(s+a)+K_2\omega$. We see that K_1 is the coefficient of s. Once we find K_1, we solve for K_2:

$$K_2 = \frac{\text{constant term } - K_1 a}{\omega}$$

Here, we will have $K_1 = 2$ and $K_2 = -3$.

This approach is possible, however, only if we have an expression that has only two roots. If there are more roots, we must find A_1.

Now we take the inverse transform:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{K_1(s+a)}{(s+a)^2+\omega^2}\right\} + \mathcal{L}^{-1}\left\{\frac{K_2\omega}{(s+a)^2+\omega^2}\right\}$$

or

$$f(t) = K_1 e^{-at}\cos(\omega t) + K_2 e^{-at}\sin(\omega t)$$

or

$$f(t) = 2ce^{-at}\cos(\omega t) - 2de^{-at}\sin(\omega t)$$

or

$$f(t) = 2\,\mathrm{Re}[A_1]e^{-at}\cos(\omega t) - 2\,\mathrm{Im}[A_1]e^{-at}\sin(\omega t)$$

NOTE: A_1 is the coefficient of the root term in the denominator that has a minus sign in it. If we find the coefficient of the root term in the denominator that has a plus sign in it, then we must use the conjugate of A_1 in the above expression. This changes the sign of the decaying sine term. (The cosine term is unaffected.)

Here, these formulas give our final result:

$$f(t) = 2e^{-5t}\cos(12t) - 3e^{-5t}\sin(12t)$$