Ex: a) Find *f*(*t*) if

$$F(s)=\frac{s+2}{(s+1)^2\,(s+4)}$$

b) Plot the poles and zeros of G(s) in the s plane.

$$G(s)=\frac{12+4s}{(s+2)\left(s^2+25\right)\left(s^2+6s+25\right)}$$

c) Find $\mathcal{L}\left\{t\,u(t-3)\right\}$.

d) i. Find $\lim_{t\to\infty} f(t)$ if

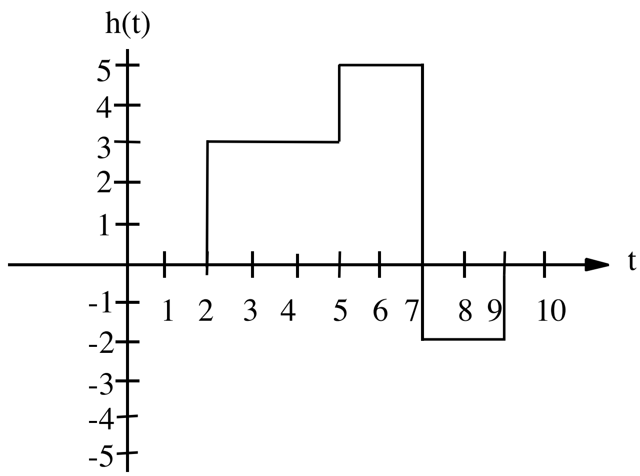
$$F(s)=\frac{2s^4+6s^3+30s^2+25s+120}{s^6+14s^5+112s^4+448s^3+975s^2+625s}$$

ii. Find $\lim_{t\to 0+} f(t)$ if

$$F(s)=\frac{3\left(s^3+7s^2+14s+8\right)}{s^4+14s^3+98s^2+350s+625}$$

(All poles of *F*(*s*) are in the left-half plane.)

e) Write an expression for H(s).



SOL'N: a) Use partial fractions.

$$F(s) = \frac{k_1}{s+4} + \frac{k_2}{(s+1)^2} + \frac{k_3}{s+1}$$

$$k_1 = F(s)(s+4)\Big|_{s=-4} = \frac{s+2}{(s+1)^2}\frac{\cancel{s+4}}{\cancel{s+4}}\Bigg|_{s=-4} = \frac{-4+2}{(-4+1)^2} = -\frac{2}{9}$$

$$k_2 = F(s)(s+1)^2\Big|_{s=-1} = \frac{s+2}{\cancel{(s+1)^2}}\frac{\cancel{(s+1)^2}}{s+4}\Bigg|_{s=-1} = \frac{-1+2}{-1+4} = \frac{1}{3}$$

$$k_3 = \frac{1}{1!}\frac{d}{ds}\left[F(s)(s+1)^2\right]\Big|_{s=-1} = \frac{d}{ds}\frac{s+2}{s+4}\Bigg|_{s=-1}$$

or

$$k_3 = 1\cdot(s+4)^{-1} + (s+2)(-1)(s+4)^{-2}\Big|_{s=-1}$$

or

$$k_3 = \frac{1}{-1+4} + \frac{(-1+2)(-1)}{(-1+4)^2} = \frac{1}{3} + \frac{-1}{9} = \frac{2}{9}$$

Plugging in k_1, k_2, and k_3 gives our expression for $F(s)$:

$$F(s) = \frac{-2/9}{s+4} + \frac{1/3}{(s+1)^2} + \frac{2/9}{s+1}$$

Use inverse Laplace transform for each term to get final answer:

$$\mathcal{L}^{-1}\left\{\frac{k}{s+a}\right\} = ke^{-at} \text{ and } \mathcal{L}^{-1}\left\{\frac{k}{(s+a)^2}\right\} = kte^{-at}$$

Thus, our answer is

$$f(t) = -\frac{2}{9}e^{-4t} + \frac{1}{3}te^{-t} + \frac{2}{9}e^{-t}.$$

b) $$G(s) = \frac{4(3+s)}{(s+2)(s+j5)(s-j5)(s+3+j4)(s+3-j4)}$$

The zeros are the roots of the numerator, (i.e., the values of s where G(s) goes to zero:

$$4(3+s)=0 \ \Rightarrow \ 3+s=0 \ \Rightarrow \ s=-3$$

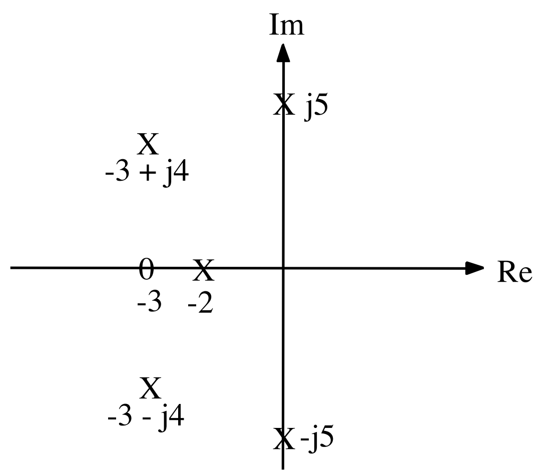
We plot zeros as 0's in s-plane. (See answer plot below.)

The poles are the roots of the denominator, (i.e., the values of s where G(s) goes to infinity).

The root for a factor of form s + a is s = -a.

Therefore, poles are at s = –2, –j5, j5, –3 – j4, and –3 + j4.

We plot poles as x's in s-plane.



c) One way to solve this problem is to make $t\ u(t-3)$ look like $v(t-3)u(t-3)$. To do so, we write t as $(t-3)+3$:

$$\mathcal{L}\{tu(t-3)\} = \mathcal{L}\{[(t-3)+3]u(t-3)\}$$

So $v(t-3) = (t-3) + 3$ and we use the delay identity:

$$\mathcal{L}\{v(t-a)u(t-a\} = e^{-as}\mathcal{L}\{v(t)\} \quad \text{when } a > 0$$

We have $v(t) = t + 3$ when we replace $t - 3$ with t.

$$\mathcal{L}\{t+3\} = \frac{1}{s^2} + \frac{3}{s}$$

So our answer is

$$\mathcal{L}\{tu(t-3)\} = e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$$

Another approach would be to use the identity for multiplication by *t:*

$$\mathcal{L}\{tv(t)\} = -\frac{d}{ds}\mathcal{L}\{v(t)\}$$

So we have $v(t) = u(t-3) = 1 \cdot u(t-3)$, and we use the delay identity with $v(t-3) = 1$, which means $v(t) = 1$. (When we change from $v(t-a)$ to $v(t)$, we shift the function $v(t-a)$ to the left by a. If we have a constant function, shifting it left has no effect—it is still a horizontal line.

$$\mathcal{L}\{1 \cdot u(t-3)\} = e^{-3s}\mathcal{L}\{1\} = \frac{e^{-3s}}{s}$$

Now we use the identity for multiplication by *t*.

$$\mathcal{L}\{t \cdot u(t-3)\} = -\frac{d}{ds}\frac{e^{-3s}}{s} = -(-3)\frac{e^{-3s}}{s} - -\frac{e^{-3s}}{s^2}$$

or

$$\mathcal{L}\{tu(t-3)\} = e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$$

This is the same answer as before.

d) i. Final value theorem:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$= \lim_{s\to 0} \frac{s \cdot (2s^4 + 6s^3 + 30s^2 + 25s + 120)}{s^6 + 14s^5 + 112s^4 + 448s^3 + 975s^2 + 625s}$$

We factor out the highest power of s that is common to every term in the numerator and refer to this as s^n. Since we multiply F(s) by s, we always can factor out s^1 from the numerator of sF(s). Here, s^1 is the highest power of s we can factor out from the numerator of sF(s).

NOTE: If the numerator of F(s) ends with a term such as 3s, (for example), then we can factor out s^2 from the numerator of sF(s).

We also factor out the highest power of s that is common to every term in the denominator of sF(s) and refer to this as s^{m}. Here, s^1 is the highest power of s we can factor out from the denominator of sF(s).

We now write

$$sF(s) = \frac{s^n}{s^m}\frac{p(s)}{q(s)}$$

where p(s) and q(s) are polynomials with nonzero constant terms.

In the limit as $s \to 0$, we have $p(s) = p(0)$ and $q(s) = q(0)$. We also have $s^{\text{n}}/s^{\text{m}} = 1/s^{\text{n}-\text{m}}$, and we can easily determine the behavior of this term as $s \to 0$. The following equation encapsulates these results:

$$\lim_{s\to 0} sF(s) = \begin{cases} 0 & n > m \\ \infty & n < m \\ \dfrac{p(0) = \text{constant term of } p(s)}{q(0) = \text{constant term of } q(s)} & \text{n = m} \end{cases}$$

Here,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} \frac{s^1}{s^1} \frac{\overbrace{2s^4 + 6s^3 + 30s^2 + 25s + 120}^{p(s)}}{\underbrace{s^5 + 14s^4 + 112s^3 + 448s^2 + 975s + 625}_{q(s)}}$$

This reduces to

$$\lim_{t\to\infty} f(t) = \frac{p(0)}{q(0)} = \frac{120}{625} = \frac{24}{125} = 0.192.$$

NOTE: $\text{s}^{\text{n}}/\text{s}^{\text{m}}$ term gives m - n = # poles at origin (net).

$$\lim_{t\to\infty} f(t) = \infty$$

if $F(s)$ has two more poles than zeros at origin.

NOTE: We must have all poles in the left-half plane or at the origin. Otherwise, our time-domain solution will contain a term of form e^{at}, $a \geq 0$, in $f(t)$. This is a nondecaying oscillation if $a = 0$ and an exponentially growing solution if $a > 0$.

Thus, the first step in applying the final value theorem is to verify that poles are in the left-half plane or at the origin, (i.e. a system that stabilizes).

d) ii. Initial value theorem.

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

or

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty}\left[sF(s) = \frac{s\cdot 3\left(s^3+7s^2+14s+8\right)}{s^4+14s^3+98s^2+350s+635}\right]$$

The highest power of s in numerator and denominator dominates as s becomes large. In other words, s^2 becomes much larger than s or a constant term as s approaches infinity. Thus, for terms that are summed, we need only consider the term with the highest power of s.

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} \frac{\not{s}\cdot 3\cdot \not{s}^3}{\not{s}^4} = 3$$

$$\therefore \quad f(t=0^+) = 3$$

e) Use delayed step-functions to create windows for piecewise definition of h(t).

$$h(t) = 3[u(t-2) - u(t-5)]$$

$$+5[u(t-5) - u(t-7)]$$

$$-2[u(t-7) - u(t-9)]$$

Gather coefficients for each step-function:

$$h(t) = 3u(t-2) + (5-3)u(t-5) - (2+5)u(t-7) + 2u(t-9)$$

or

$$h(t) = 3u(t-2) + 2u(t-5) - 7u(t-7) + 2u(t-9)$$

Now use the identity for delayed functions:

$$\mathcal{L}\{f(t-a)u(t-a),\ a>0\} = e^{-as}\ F(s)$$

where

$$f(t) = u(t), \quad \mathcal{L}\{f(t) = u(t)\} = \frac{1}{s}.$$

Plugging the various values of delay for *a*, we get our final answer:

$$H(s) = \frac{3e^{-2s} + 2e^{-5s} - 7e^{-7s} + 2e^{-9s}}{s}$$