3. (30 points)

a. Derive an expression for \( i_2 \). The expression must not contain more than the circuit parameters \( \beta, V_a, i_a, R_1, \) and \( R_2 \).

\[
\begin{align*}
R_1 &+ \underbrace{R_2}_{\text{v}_2} \quad \text{(1)} \\
\frac{\beta v_1}{R_1} &- i_a - i_2 = v_1 \\
\text{v}_1 &- \underbrace{i_a}_{\text{a}} \quad \text{(2)}
\end{align*}
\]

b. Make at least one consistency check (other than a units check) on your expression. Explain the consistency check clearly.

ans: a) \[
\begin{align*}
i_2 &= \frac{(1-\beta)i_a R_1 + V_a}{(1-\beta)R_1 + R_2}
\end{align*}
\]
b) Many possible answers. See solution below.

sol'n: (a) Use Kirchhoff’s laws to write several equations. Then eliminate unwanted variables.

\[
\begin{align*}
\text{v}_1 &- \underbrace{i_a}_{\text{a}} \quad \text{(1)} \\
\frac{\beta v_1}{R_1} &- i_a - i_2 = v_1 \\
\text{v}_1 &- \underbrace{i_a}_{\text{a}} \quad \text{(2)}
\end{align*}
\]

We sum currents out of top-center node:

\[-i_1 - i_a + i_2 = 0\]

Note that summing currents out of bottom-center node does not give us anything new. By Ohm's law, we also have

\[i_1 = \frac{v_1}{R_1}\]

Now, we sum voltages around a loop. We choose the outer loop because the inner loops have a current source with unknown voltage drop.
\[ \beta v_1 - v_1 + V_a - v_2 = 0 \quad \text{or} \quad (\beta - 1)v_1 + V_a - v_2 = 0 \]

By Ohm's law, we also have
\[ v_2 = i_2 R_2 \]

After the Ohm's law substitutions, we have two equations, and we may eliminate \( v_1 \).

Use the simpler equation first:
\[ \frac{v_1}{R_1} + i_a - i_2 = 0 \quad \text{or} \quad v_1 = (i_2 - i_a) \]

Substitute for \( v_1 \) in the second equation:
\[ (\beta - 1)R_1(i_2 - i_a) + V_a - i_2 R_2 = 0 \]

After some algebra, we get
\[ i_2 = \frac{(1 - \beta)i_a R_1 + V_a}{(1 - \beta)R_1 + R_2} \quad \text{units consistent} \quad \checkmark \]

(b) There are many possible consistency checks.

1) \( i_a = 0 \) and \( R_1 = 0 \). Then \( v_1 = 0 \), \( \beta v_1 = 0 \), and sum v's around outer loop gives \( i_2 = V_a/R_2 \). Our formula also gives \( V_a/R_2 \). \( \checkmark \)

2) Consider \( R_1 = 0 \) and \( R_2 = 0 \). As in (1), \( v_1 = 0 \) and \( \beta v_1 = 0 \). Since \( R_2 = 0 \) we also end up with a short across \( V_a \):

Our formula gives
\[ \lim_{R_2 \to 0} \frac{V_a}{R_2} = \infty \quad \text{for} \quad i_2 \quad \text{from (1)} \quad \checkmark \]
3) Consider \( i_a = 0, \beta = 0 \):

\[
i_2 = \frac{V_a}{R_1 + R_2} \quad \text{by Ohm’s law}
\]

Our formula gives

\[
i_2 = \frac{V_a}{R_1 + R_2}
\]

4) Consider \( R_1 \rightarrow \infty \) (open circuit)

Clearly \( i_a = i_2 \) since the same current flows through elements in series.

Our formula gives:

\[
i_2 = \lim_{R_1 \to \infty} \frac{(1-\beta)i_a R_1 + V_a}{(1-\beta)R_1 + R_2} = \lim_{R_1 \to \infty} \frac{(1-\beta)i_a R_1}{(1-\beta)R_1} = i_a \quad \checkmark
\]

5) Consider \( R_2 \rightarrow \infty \) (open circuit): We have \( i_2 = 0 \) since no current flows through the open circuit. Our formula gives:

\[
i_2 = \lim_{R_1 \to \infty} \frac{(1-\beta)i_a R_1 + V_a}{(1-\beta)R_1 + R_2} = \text{const}_{\infty} = 0 \quad \checkmark
\]

Many more consistency checks are possible.