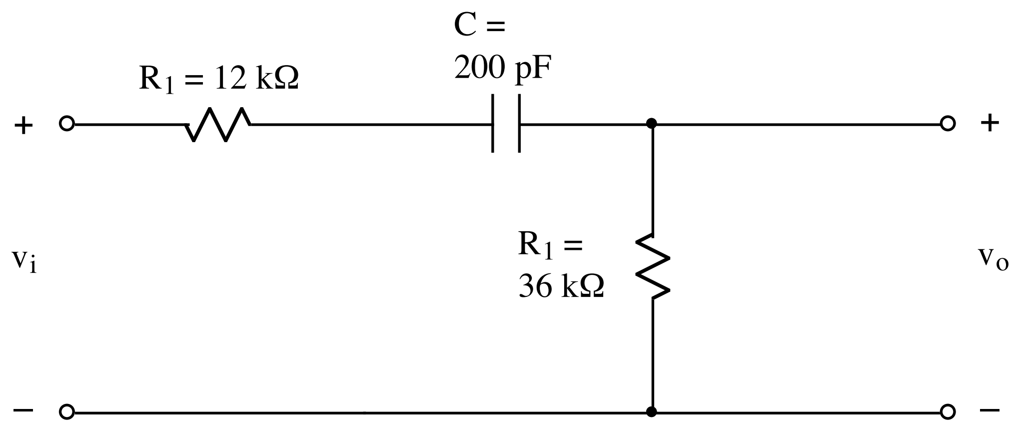
HOMEWORK #18 prob 5 solution

Ex:



Note: vertical resistor should be labeled R_2.

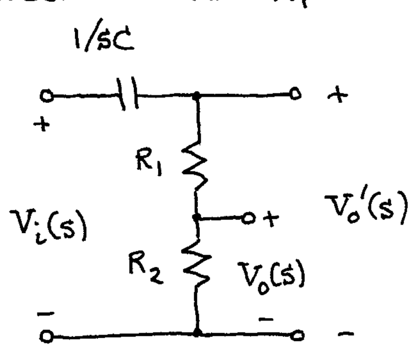
a) Determine the transfer function V_o/V_i. **Hint:** Reverse the order of R_1 and C, and suppose the output were tapped from the point between C and R_1. Then use a voltage divider.

b) Plot $|V_o/V_i|$ versus ω.

c) Find the cutoff frequency, ω_c.

sol'n: a) We can switch the order of R_1 and C without changing $H(s) \equiv V_o(s)/V_i(s)$.

We then consider taking the output, v_o', from between C and R_1.



We observe that $V_o(s) = V_o'(s) \cdot \dfrac{R_2}{R_1 + R_2}$.

Thus, $H(s) \equiv \dfrac{V_o(s)}{V_i(s)} = H'(s) \cdot \dfrac{R_2}{R_1 + R_2}$

where $H'(s) \equiv \dfrac{V_o'(s)}{V_i(s)}$.

This is convenient since $H'(s)$ is the transfer function of an RC filter, with $R = R_1 + R_2$.

$$H'(s) = \frac{V_o'(s)}{V_i(s)} = \frac{V_i(s) \cdot \dfrac{(R_1 + R_2)}{(R_1 + R_2) + 1/sC}}{V_i(s)}$$

$$= \frac{R_1 + R_2}{R_1 + R_2 + \dfrac{1}{sC}}$$

$$= \frac{1}{1 + \dfrac{1}{s(R_1 + R_2)C}}$$

$$= \frac{1}{1 - j\dfrac{1}{\omega(R_1 + R_2)C}}$$

Thus, $H(s) = \dfrac{R_2}{R_1 + R_2} \cdot \dfrac{1}{1 - j\dfrac{1}{\omega(R_1 + R_2)C}}$.

b) $$|H(s)| = \left| \frac{R_2}{R_1+R_2} \cdot \frac{1}{1 - j\,\frac{1}{\omega(R_1+R_2)C}} \right|$$

$$= \frac{R_2}{R_1+R_2} \cdot \frac{1}{\left| 1 - j\,\frac{1}{\omega(R_1+R_2)C} \right|}$$

Since we can write $|a \cdot b| = |a||b|$ and $|a/b| = |a|/|b|$.

$$= \frac{R_2}{R_1+R_2} \cdot \frac{1}{\sqrt{1^2 + \frac{1}{\omega^2 (R_1+R_2)^2 C^2}}}$$

We can sketch $|H(s)|$ by finding a few key values.

For $\omega = 0$ we have $|H| = \frac{R_2}{R_1+R_2} \cdot \frac{1}{\sqrt{1+\frac{1}{0}}}$

$$= \frac{R_2}{R_1+R_2} \cdot \frac{1}{\sqrt{1+\infty}}$$

$$= \frac{R_2}{R_1+R_2} \cdot \frac{1}{\infty}$$

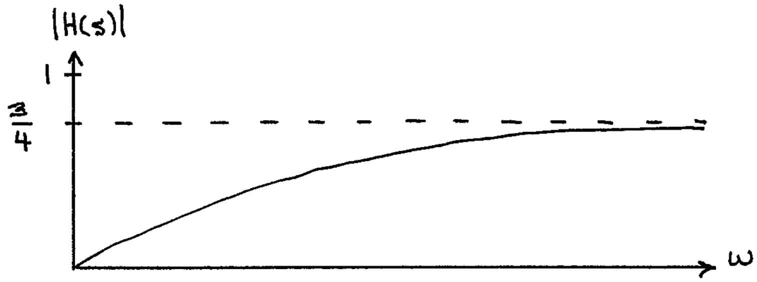
$$= 0$$

For $\omega \to \infty$ we have $|H| = \frac{R_2}{R_1+R_2} \cdot \frac{1}{\sqrt{1+\frac{1}{\infty}}}$

$$= \frac{R_2}{R_1 + R_2} \frac{1}{\sqrt{1+0}}$$

$$= \frac{R_2}{R_1 + R_2} \cdot 1$$

Here, $\frac{R_2}{R_1+R_2} = \frac{36\,k\Omega}{12\,k\Omega + 36\,k\Omega} = \frac{3}{4}.$



c) The cutoff frequency, ω_c, is the frequency where $|H(s)| = \frac{1}{\sqrt{2}} \max_{\omega} |H(s)| = \frac{1}{\sqrt{2}} \cdot \frac{3}{4}.$

We could solve the problem this way, but it is helpful to observe that we get the same ω_c for $H'(s)$ because the factor of $\frac{3}{4}$ in $H(s)$ cancels out

the $\frac{3}{4}$ in $\frac{1}{\sqrt{2}} \cdot \frac{3}{4}.$

For $H'(s)$ we solve for ω_c where $|H'(s)| = \frac{1}{\sqrt{2}}.$

$$|H'(s)| = \frac{1}{\left| 1 - j\frac{1}{\omega_c(R_1+R_2)C} \right|} = \frac{1}{\sqrt{2}}$$

This is equivalent to

$$\left| 1 - j \frac{1}{\omega_c (R_1+R_2)C} \right| = \sqrt{2}$$

Since $\sqrt{1+ja} = \sqrt{1^2+a^2} = \sqrt{1+a^2} = \sqrt{2}$ is solved by $a = \pm 1$, we must have

$$\frac{1}{\omega_c (R_1+R_2)C} = \pm 1$$

$\omega_c > 0$ always, so we use $+1$ on the right:

$$\omega_c (R_1+R_2)C = 1$$

or

$$\omega_c = \frac{1}{(R_1+R_2)C} = \frac{1}{(12k+36k)200p} \text{ r/s}$$

$$= \frac{1}{48k \cdot 200p} \text{ r/s} = \frac{1}{9600\,n} \text{ r/s}$$

$$= \frac{1}{9.6\,\mu} \text{ r/s} = \frac{1M}{9.6} \text{ r/s}$$

$$\omega_c \doteq 104 \text{ kr/s}$$