Ex: Laplace transforms are useful for solving linear differential equations that arise in many settings. Consider a linearized model of a motor with a torque control system, for example, where the actual torque of the motor relative to a desired torque is described by the following Laplace transform:

$$H(s) = \frac{\frac{k_P}{L}\left(s + \frac{k_I}{k_P}\right)}{s^2 + \frac{k_P}{L}s + \frac{k_I}{L}}$$

where $L = 1$ mH = inductance of motor windings

k_P = gain for proportional feedback

k_I = gain for integral feedback

The inverse Laplace transform of $H(s)$ contains multiplicative factors of the form e^{-at} that determine how fast time waveforms decay to zero. This decay corresponds to how rapidly the motor torque reaches a desired value. A faster decay is desirable.

Equal roots, (i.e., critical damping), is optimal if vibration (oscillatory solutions) must be eliminated.

a) If $k_P = 3.2$, find the value of k_I that yields critical damping.

b) Find the inverse Laplace transform of $H(s)$ for the values from part (a).

SOL'N: a) When the roots are equal, the denominator must be of the following form:

$$(s + a)^2 = s^2 + 2as + a^2$$

From the given values, we have the following results:

$$a = \frac{k_P}{2L} = \frac{3.2}{2 \cdot 1\text{m}} = 1.6\text{k}$$

$$a^2 = (1.6\text{k})^2 = \frac{k_I}{L} = \frac{k_I}{1\text{m}} \quad \text{or} \quad k_I = 2.56\text{k}$$

NOTE: Because of notational difficulties, units are missing from the above quantities. The s variable is easily confused with "s" used for units of seconds. In addition, because the Laplace integral is with respect to time, quantities in the s-domain may have units such Vs and As, or the extra units may appear as part of the Laplace transform result, (i.e., as something like

$V(s) = \frac{1}{s}$). Consequently, units will be dispensed with in calculations here.

b) Using our symbolic a = 1.6k to express the denominator, and expressing the numerator in terms of a as well, yields the following equation:

$$H(s) = \frac{\frac{k_P}{L}\left(s + \frac{k_I}{k_P}\right)}{s^2 + \frac{k_P}{L}s + \frac{k_I}{L}} = \frac{2a\left(s + \frac{2.56\text{k}}{3.2}\right)}{(s+a)^2}$$

The partial fraction expansion when there is a repeated root requires that we have terms for each power of the root that appears in the denominator. There is only a constant in the numerator of each root term, however.

$$H(s) = \frac{A}{(s+a)^2} + \frac{B}{s+a}$$

To find the coefficient of the highest-power root term, we may use the pole cover-up method slightly modified by multiplying by the highest order root term. As before, we then evaluate at the root value:

$$A = (s+a)^2 H(s)\Big|_{s=-a}$$

or

$$A = (s+a)^2 \frac{2a\left(s + \frac{2.56\text{k}}{3.2}\right)}{(s+a)^2}\Bigg|_{s=-a} = 2a\left(s + \frac{2.56\text{k}}{3.2}\right)\Bigg|_{s=-a}$$

or

$$A = 2(1.6\text{k})\left(-1.6\text{k} + \frac{2.56\text{k}}{3.2}\right) = 3.2\text{k}(-1.6\text{k} + 0.8\text{k})$$

or

$$A = 3.2\text{k}(-0.8\text{k}) = 3.2\text{k}(-800) = -2.56\text{M}$$

To find B an easy approach is to put the partial fraction terms over a common denominator and then match the numerator to the original numerator of $H(s)$:

$$H(s)=\frac{2a\left(s+\frac{2.56\text{k}}{3.2}\right)}{(s+a)^2}=\frac{A}{(s+a)^2}+\frac{B}{s+a}=\frac{A}{(s+a)^2}+\frac{B(s+a)}{(s+a)^2}$$

To match the numerators, the coefficients of the powers of s must match. The coefficient of s is $2a$ on the left, which must equal the coefficient of s on the right, which is B.

$$B = 2a = 2(1.6\text{k}) = 3.2\text{k}$$

The inverse Laplace transform now follows:

$$h(t)=Ate^{-at}+Be^{-at}=-2.56\text{M}te^{-at}+3.2\text{k}e^{-at}$$

Using the value of a in the exponents but expressing it as a time constant, (i.e., as $1/a$), we have the final numerical answer:

$$h(t)=3.2\text{k}\,e^{-t/0.625\text{ms}}\left(1-800t\right)=3.2\text{k}\,e^{-t/0.625\text{ms}}\left(1-t\,/\,0.625\text{ms}\right)$$