HOMEWORK #12 prob 3a solution

EX: Find the inverse Laplace transform for the following expression:

$$F(s) = \frac{4s+11}{s^2+3s+2}$$

SOL'N: We use partial fractions. The first step in using partial fractions is always to factor the denominator into root terms:

$$F(s) = \frac{4s+11}{(s+1)(s+2)}$$

The second step in using partial fractions is to write $F(s)$ in terms of unknown constant coefficients for each root term.

$$F(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

NOTE: To find the root terms for a quadratic denominator, we can use the quadratic formula to find the roots:

$$s_{1,2} = \frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

where the denominator is

$$s^2 + bs + c$$

(If the coefficient of the s^2 term is not equal to one, divide the numerator and denominator by that coefficient.)

Having found the roots, we write the denominator as root terms:

$$s^2 + bs + c = (s - s_1)(s - s_2)$$

NOTE: This discussion assumes all roots are distinct. See other examples for how to treat repeated roots.

The third step is to find A_1 and A_2 by one of several methods. One method is removing the pole and evaluating at the pole value:

$$A_1 = (s+1)F(s)\big|_{s=-1} \quad \text{and} \quad A_2 = (s+2)F(s)\big|_{s=-2}$$

To see why this works we observe what happens to the partial fraction expression when we perform these operations:

$$A_1 = (s+1)F(s)\Big|_{s=-1} = \frac{(s+1)A_1}{(s+1)}\Bigg|_{s=-1} + \frac{(s+1)A_2}{(s+2)}\Bigg|_{s=-1} = A_1 + \frac{0 \cdot A_2}{(-1+2)} = A_1$$

For the present problem, we proceed with the calculations:

$$A_1 = (s+1)F(s)\Big|_{s=-1} = \frac{4s+11}{s+2}\Bigg|_{s=-1} = \frac{4(-1)+11}{-1+2} = \frac{7}{1} = 7$$

$$A_2 = (s+2)F(s)\Big|_{s=-2} = \frac{4s+11}{s+1}\Bigg|_{s=-2} = \frac{4(-2)+11}{-2+1} = \frac{3}{-1} = -3$$

This gives us the partial fraction expansion:

$$F(s) = \frac{7}{s+1} + \frac{-3}{s+2} = 7 \cdot \frac{1}{s+1} + -3 \cdot \frac{1}{s+2}$$

The fourth step is to use the following basic transform pair to invert each of the terms:

$$\mathcal{L}\left\{e^{-at}\right\} = \frac{1}{s+a}$$

This yields our final answer:

$$f(t) = 7e^{-t} - 3e^{-2t}$$

A second or alternative method for finding A_1 and A_2 is to use a common denominator:

$$F(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2} = \frac{A_1(s+2) + A_2(s+1)}{(s+1)(s+2)} = \frac{4s+11}{(s+1)(s+2)}$$

Equating coefficients for each power of s in the numerator yields two equations to be solved:

$$A_1 + A_2 = 4$$

$$A_1 \cdot 2 + A_2 \cdot 1 = 11$$

Solving these two equations yields the same result as before:

$$A_1 = 7 \quad \text{and} \qquad A_2 = -3$$

$$F(s) = \frac{7}{s+1} + \frac{-3}{s+2}$$

$$f(t) = 7e^{-t} - 3e^{-2t}$$

A third method for finding A_1 and A_2 is to substitute convenient values of s in $F(s)$ and find the values of A_1 and A_2 that yield these values of $F(s)$:

$$F(s)\big|_{s=0} = \frac{A_1}{s+1}\bigg|_{s=0} + \frac{A_2}{s+2}\bigg|_{s=0} = A_1 + \frac{A_2}{2} = \frac{4 \cdot 0 + 11}{(0+1)(0+2)} = \frac{11}{2}$$

$$F(s)\big|_{s=1} = \frac{A_1}{s+1}\bigg|_{s=1} + \frac{A_2}{s+2}\bigg|_{s=1} = \frac{A_1}{2} + \frac{A_2}{3} = \frac{4 \cdot 1 + 11}{(1+1)(1+2)} = \frac{15}{6} = \frac{5}{2}$$

NOTE: We are free to use any desired values of s with the following exception: the values of s used must *not* be roots. Here, we must not use $s = -1$ and $s = -2$.

Multiplying our two equations by 2 and 6, respectively, yields two simultaneous equations to be solved for A_1 and A_2:

$$2A_1 + A_2 = 11$$

$$3A_1 + 2A_2 = 15$$

Solving these two equations yields the same result as before:

$$A_1 = 7 \quad \text{and} \qquad A_2 = -3$$

$$F(s) = \frac{7}{s+1} + \frac{-3}{s+2}$$

$$f(t) = 7e^{-t} - 3e^{-2t}$$