Homework 12

1. Find the Laplace transforms of the following waveforms:

a) $(t^2 - 1)u(t-1)$

b) $\dfrac{d}{dt}\left[e^{-at}\sin(\omega t)\right]$

c) $\dfrac{e^{-2t}}{t}$

d) $\int_0^t te^{-at}\,dt$

2. Show that the following identity is valid:

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$$

3. Find the inverse Laplace transform for each of the following expressions:

a) $F(s) = \dfrac{4s+11}{s^2+3s+2}$

b) $F(s) = \dfrac{7}{(s^2+6s+58)}$

c) $F(s) = -\dfrac{3s^2+3}{s^4}$

d) $F(s) = \dfrac{6s^2+36s+198}{(s+3)(s^2+6s+45)}$

4. Laplace transforms are useful for solving linear differential equations that arise in many settings. Consider a linearized model of a motor with a torque control system, for example, where the actual torque of the motor relative to a desired torque is described by the following Laplace transform:

$$H(s) = \frac{\dfrac{k_P}{L}\left(s + \dfrac{k_I}{k_P}\right)}{s^2 + \dfrac{k_P}{L}s + \dfrac{k_I}{L}}$$

where $L = 1$ mH = inductance of motor windings

k_P = gain for proportional feedback

k_I = gain for integral feedback

The inverse Laplace transform of $H(s)$ contains multiplicative factors of the form e^{-at} that determine how fast time waveforms decay to zero. This decay corresponds to how rapidly the motor torque reaches a desired value. A faster decay is desirable.

Equal roots, (i.e., critical damping), is optimal if vibration (oscillatory solutions) must be eliminated.

a) If $k_P = 3.2$, find the value of k_I that yields critical damping.

b) Find the inverse Laplace transform of $H(s)$ for the values from part (a).

5. Find the inverse Laplace transform of $\dfrac{1}{(s+a)^n}$.

Answers:

1.a) $2e^{-s}\left(\dfrac{s+1}{s^3}\right)$ b) $\dfrac{s\omega}{(s+a)^2+\omega^2}$ c) ∞ d) $\dfrac{1}{s(s+a)^2}$

2. Hint: transform the right side into the left side.

3.a) $f(t) = 7e^{-t} - 3e^{-2t}$ b) $f(t) = e^{-3t}\sin(7t)u(t)$

c) $f(t) = -\dfrac{t^3}{2} - 3t$ d) $\left[4e^{-3t} + 2e^{-3t}\cos(6t)\right]u(t)$

4.a) $k_I = 2.56\text{k}$ b) $h(t) = 3.2\text{k}\,e^{-t/0.625\text{ms}}(1-800t) = 3.2\text{k}\,e^{-t/0.625\text{ms}}(1-t/0.625\text{ms})$

5. Hint: try to deduce the identity, then to prove it is always correct, assume the identity works for n and show it works for $n+1$ (induction proof).